

# Complexity of Recursive Normal Default Logic

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To the memory of Cecylia Rauszer

**Abstract.** Normal default logic, the fragment of default logic obtained by restricting defaults to rules of the form  $\frac{\alpha:M\beta}{\beta}$ , is the most important and widely studied part of default logic. In [MNR94], we proved a basis theorem for extensions of recursive propositional logic normal default theories and hence for finite predicate logic normal default theories. That is, we proved that every recursive propositional normal default theory possesses an extension which is r.e. in  $0'$ . Here we show that this bound is tight. Specifically, we show that for every r.e. set  $A$  and every set  $B$  r.e. in  $A$  there is a recursive normal default theory  $\langle D, W \rangle$  with a unique extension which is Turing-equivalent to  $A \oplus B$ . A similar result holds for finite predicate logic normal default theories.

## 1. Introduction

In this paper we investigate the complexity of extensions of normal default theories and for other closely related modes of nonmonotonic reasoning. In the past number of years nonmonotonic reasoning [Rei80, McC80, MD80, McD82, Moo85, VRS91] has been formalized and well-understood [MNR90, MNR92b, ABW88, Van88, Prz89b, Lif85, MST90, Prz89a, Fit91]. These advances were followed by analysis of complexity issues. In case of finite

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propositional logic nonmonotonic formalisms, the basic results are found in [BF91, MT91, Got92, Van89]. In case of formalisms admitting variables and, more generally, infinite recursive propositional nonmonotonic formalisms, a number of results has been found. These include basic complexity results for Horn logic [Smu68, AN78], the complexity of the perfect model semantics of stratified logic programs [AB90], the complexity of the perfect model semantics of locally stratified logic programs [BMS91], the complexity of the stable model semantics of logic programs [MNR92a], and the complexity of the well-founded model semantics for logic programs [Sch94]. The complexity of (predicate) circumscription has been established [Sch87]. A topological characterization of stable, supported and minimal model semantics of logic programs has been recently studied in [Fer94].

The complexity of finite versions of nonmonotonic formalisms do not always correspond to the complexity of the corresponding infinitary problems. For instance, the existence problem for the stable model semantics of logic programs is NP-complete in the propositional case [BF91, MT91] whereas the corresponding problem for finite predicate programs is  $\Sigma_1^1$ -complete. On the other hand the the well-founded model for finite propositional logic programs can be computed in polynomial time [Van89] whereas the well founded model of a finite predicate logic program may be a complete  $\Pi_1^1$  set. Even bigger discrepancies appear in the case of circumscription, see [Prz89a, Sch87].

In this paper we study the complexity of the extensions of normal default theories and establish a precise bounds the Turing degrees of such extensions. First of all, recall that by the fundamental result of Reiter [Rei80] normal default theories always possess an extension. An extension of  $\langle D, W \rangle$  can be computed by a kind of forward-chaining procedure in which one constructs an extension in stages where at each stage one selects the first (in some well-ordering) applicable rule and add its conclusion to the extension, see details in [MT93, MNR93]. In [MNR94], we showed that when this forward chaining procedure is properly formulated one can prove a basis result for the set of extensions of a recursive normal propositional logic default theory, namely that a recursive normal default theory always possesses an extension that is r.e. in  $0'$  and hence recursive in  $0''$ .

The main result of this paper is to show that this basis result is sharp. Specifically, we show that given an arbitrary r.e. set  $A$  and a set  $B$  which is recursively enumerable in  $A$ , we can find a  $FC$ -normal nonmonotonic rule system with a unique extension  $E$  such that  $E \equiv_T A \oplus B$ . In fact, we will construct a nonmonotonic rule system [MNR90]  $\mathcal{S}_{A,B}$  which is stratified, with two strata, such that its unique extension is Turing-equivalent to  $A \oplus B$ . The system  $\mathcal{S}_{A,B}$  uses a form of closed world assumption. One can then use various known

translations, [GL90, MT93, MNR93], to show that this nonmonotonic rule system can also be represented as recursive propositional or a finite predicate logic program with classical negation and a recursive propositional or finite predicate logic normal default theory.

We note in passing that the technique used in this paper can be used to find an alternative and much simpler argument for the main result of [BMS91] where it is shown that any given hyperarithmetical set is computed by a suitably chosen finite, locally stratified, logic program.

## 2. Preliminaries

### 2.1. Nonmonotonic rule systems

A nonmonotonic rule system is a pair  $\mathcal{S} = \langle U, N \rangle$ , where  $U$  is any set, and  $N$  is a set of nonmonotonic rules over  $U$ , that is, entities of the form:

$$r = \frac{a_1, \dots, a_m : b_1, \dots, b_n}{c} \quad (1)$$

where  $a_1, \dots, a_m, b_1, \dots, b_n \in U$ . Here  $m, n$ , or both can be equal to 0.  $N$  splits in a natural fashion into two sets of rules. Those in  $Mon(\mathcal{S})$  have  $n$  equal to 0 and those in  $Nmon(\mathcal{S})$  have  $n > 0$ . The rules in  $Mon(\mathcal{S})$  are called *monotonic* whereas those in  $Nmon(\mathcal{S})$  are called *nonmonotonic*.

The intended meaning of a rule of the form (1) is as follows: “If for a given context  $C$ , all  $a_i$ ’s are derived, and all  $b_j$ ’s are not in  $C$ , then derive  $c$ .” More formally, we introduce the notion of  $S$ -derivation (for  $S \subseteq U$ ). This is a sequence  $\langle c_1, \dots, c_s \rangle$  such that there is a rule of the form

$$(1) \quad \frac{: b_1, \dots, b_n}{c_1} \in N$$

where  $b_1, \dots, b_n \notin S$  ( $n$  can be 0) and for every  $1 < j \leq s$ , there is a rule of the form

$$(2) \quad \frac{c_{i_1}, \dots, c_{i_r} : b_1, \dots, b_n}{c_j} \in N$$

where  $i_1, \dots, i_r < j$ , and  $b_1, \dots, b_n \notin S$  ( $n$  can again be 0).

Thus, an  $S$ -derivation is like a usual derivation except that the applicability of rules is controlled by  $S$ , called a *context*. The set of  $S$ -derivable elements of  $U$  is denoted by  $C_S$ . We then introduce the notion of an *extension* of a system  $\mathcal{S}$  by saying that  $S$  is an extension of  $\mathcal{S}$  if  $S = C_S$ . The notion of extension generalizes the notions of an extension of a default theory [Rei80], an extension of a truth maintenance system [Doy79], a stable model of logic program [GL88], an answer set of logic program with classical negation [GL90], an

$S$ -expansion of a modal theory [MD80, McD82] and a stable expansion of modal theory [Moo85]. The precise connection of nonmonotonic rule systems and these formalisms is discussed in [MNR90, MNR92b].

In [MNR94] we studied conditions that guarantees the existence of extensions since that is a minimal condition to have a viable theory of belief revision in many applications. There are several such conditions in the published literature. Some of these will be used below. These include the notion of *stratification* [ABW88] and its generalization, *local stratification* [Prz88]. These conditions guarantee (when described in the language of nonmonotonic rule systems) the existence of a *unique* extension (this is, of course, the *perfect* model of the program encoding the system). In the case of default logic, Reiter introduced the notion of a *normal* default theory and proved that any normal default theory always possesses an extension although, in general, it is not unique.

In [MT93] it has been shown how to construct Reiter extensions of a normal default theory via a kind of forward chaining construction. The key to generalizing this construction to nonmonotonic rule systems was to realize that the form of the rules in a normal default theory insured that the forward chaining construction maintained consistency. In [MNR94], we generalized normal default theories to what we called Forward Chaining (*FC*) normal nonmonotonic rule systems. We briefly recall the relevant notions here.

Given a set  $U$ , a family of subsets of  $U$ ,  $Cons$  is called a *consistency property* ([Sco82]) if the following conditions are met:

1.  $\emptyset \in Cons$
2.  $X \in Cons$  and  $Y \subseteq X$  implies  $Y \in Cons$
3. If  $\mathcal{A} \subseteq Cons$  is directed (i.e.,  $X, Y \in \mathcal{A}$  implies there is  $Z \in \mathcal{A}$  such that  $X \subseteq Z, Y \subseteq Z$ ), then  $\bigcup \mathcal{A} \in Cons$ .

When  $\mathcal{S} = \langle U, N \rangle$  is a nonmonotonic rule system and  $Cons$  is a consistency property, then we say that  $\mathcal{S}$  is *FC-normal* with respect to  $Cons$  if these additional two properties are met:

4. If  $A \in Cons$  then  $Cl_{mon}(A) \in Cons$  (where  $Cl_{mon}(A)$  is the closure of  $A$  under all rules in  $Mon(\mathcal{S})$ ).
5. All rules in  $Nmon(N)$  are *FC-normal* with respect to  $Cons$ , i.e. if

$$r = \frac{a_1, \dots, a_m : b_1, \dots, b_n}{c}$$

belongs to  $N$  and  $A \subseteq U$  are such that  $Cl_{mon}(A) = A$ ,  $a_1, \dots, a_m \in A$ ,  $b_1, \dots, b_n \notin A$ , and  $c \notin A$  then (a)  $A \cup \{c\} \in Cons$  and (b)  $A \cup \{b_j, c\} \notin Cons$  for every  $1 \leq j \leq n$ .

We say that  $\mathcal{S}$  is *FC-normal* if it is *FC*-normal with respect to some consistency property.

The idea here is that we build our extension using forward chaining (hence *FC* in the name) maintaining the ‘‘approximations’’ to the constructed extension within the family *Cons*. The crucial condition (5) makes sure that the application of a rule will never be invalidated in the future. Indeed, once a rule has been applied and its conclusion accepted, we maintain consistency (condition (5.a)) but we will never will be able to add any of the restraints  $b_i$  while maintaining consistency (condition (5.b)).

The following results generalize the results normal default theories of Reiter [Rei80] are proved in [MNR94].

First, an *FC*-normal nonmonotonic rule system  $\mathcal{S}$  always possesses at least one extension. Also, all extensions of  $\mathcal{S}$  are consistent with respect to any consistency property *Cons* for which  $\mathcal{S}$  is *FC*-normal.

Second, the forward chaining algorithm for constructing extensions (as introduced in [MNR93]) produces extensions (and not only partial extensions) for all the well orderings of  $Nmon(\mathcal{S})$ .

Third, all extensions of  $\mathcal{S}$  are obtained by the forward chaining procedure.

Forth, different extensions of  $\mathcal{S}$  are orthogonal with respect to *Cons*, i.e. their union does not belong to *Cons*.

Other properties exhibited in Reiter’s paper [Rei80] generalize to the context of normal nonmonotonic rule systems as well. Moreover, we show in [MNR94] that the normal default theories can, indeed, be represented as *FC*-normal nonmonotonic rule systems.

## 2.2. Default Logic and its fragment, Normal Default Logic

A default rule is a nonmonotonic derivation rule

$$d = \frac{\alpha : M\beta_1, \dots, M\beta_k}{\gamma}$$

where  $\alpha, \beta_1, \dots, \beta_k, \gamma$  are formulas of propositional language. A default theory is a pair  $\langle D, W \rangle$  such that  $D$  is a set of defaults, and  $W$  is a set of formulas of the propositional language.

Given a default theory  $\langle D, W \rangle$  introduce the operator  $\Gamma : \mathcal{L} \rightarrow \mathcal{L}$  as follows. Given  $S$ , let  $\Gamma(S)$  be the least set of formulas  $U$  such that  $W \subseteq U$ ,  $U$  is closed under logical consequences, i.e.  $Cn(U) = U$ , and for all defaults  $d$  in  $D$ , if  $\alpha \in S$  and  $\neg\beta_1 \notin S, \dots, \neg\beta_k \notin S$  then  $\gamma \in U$ .

It is easy to see that such  $U$  always exists. The operator  $\Gamma$  is antimonotonic. Fixed points of  $\Gamma$  are called extensions of  $\langle D, W \rangle$ . Extensions of default theories and their properties has been intensely studied by researchers in logical foundations of Artificial Intelligence.

Of particular importance for applications are default rules of the form  $\frac{\alpha:M\beta}{\beta}$ . These are called normal default rules. When  $D$  consists of normal default rules, then  $\langle D, W \rangle$  is called normal default theory. Normal default theories always possess extensions. In [MNR94] we proved that a recursive normal default theory possesses an extension which is r.e. in  $0'$ . It is natural to ask how sharp is this bound. Can it be lowered? It is our goal to show that it is not possible.

### 3. Nonmonotonic rule systems with a unique extension coding a desired set

Let  $\phi_0, \phi_1, \dots$  be a list of all partial recursive functions and  $\phi_0^A, \phi_1^A, \dots$  be a list of all partial recursive function relative to an oracle  $A$ . A subset  $X \subseteq \omega$  is called r.e. (r.e. in  $A$ ) if  $X$  equals the domain of some  $\phi_e$  ( $\phi_e^A$ ). Given a pair of sets,  $A, B \subseteq \omega$ ,  $A \oplus B = \{2x : x \in A\} \cup \{2x + 1 : x \in B\}$ .

Let  $M_e$  be the  $e^{\text{th}}$  Turing oracle machine. Then there is a predicate  $\varphi_{e,x,n}^{C,D}$  which holds precisely if the machine  $M_e$  with the input  $x$  converges in exactly  $n$  steps and during this computation the elements about which the oracle receives queries are precisely the elements of  $C \cup D$  and moreover the oracle answers YES to the queries in  $C$  and NO to queries in  $D$ .

Let  $A$  be an r.e. set, and let  $B$  be a set r.e. in  $A$ . In this section we describe a construction of a recursive  $FC$ -normal nonmonotonic rule system  $\mathcal{S} = \mathcal{S}_{A,B} = \langle U, n \rangle$  such that  $\mathcal{S}$  possesses a unique extension,  $E$ , and  $E \equiv_T A \oplus B$ .

Let  $e$  be such that  $B = \{x : \phi_e^A(x) \downarrow\}$ . Let  $f$  be a one-one recursive function whose range is  $A$  and let  $A^S = \{f(0), \dots, f(s)\}$ .

The set  $U$  is the union of three disjoint sets of elements:

1.  $\{a_i^s : i, s \in \omega\}$ . These describe the enumeration of the set  $A$ .
2.  $\{a_{i,x}^n : i, n, x \in \omega\} \cup \{\bar{a}_{i,x}^n : i, n, x \in \omega\}$ . These are the elements used to describe the elements of used in the computation  $\phi_e^A(x)$ .
3.  $\{b_x : x \in \omega\}$ . These elements will be used to construct the set  $B$ .

Next we describe the set of rules  $N$ . The set  $N$  consists of two groups of rules: monotonic rules  $Mon(\mathcal{S})$  and nonmonotonic rules  $Nmon(\mathcal{S})$ .

(a)  $Mon(\mathcal{S})$  consists of two groups of rules:

- (i)  $\{\frac{\dot{\cdot}}{a_i^s} : s \in \omega, i = f(s)\}$ . These rules reflect the enumeration of  $A$  in stages.
- (ii)  $\{\frac{a_i^s}{a_{i,x}^n} : i, n, s, x \in \omega\}$ . These rules do two things for us. First, they provide us copies of the elements of  $A$  to code the computation  $\phi_e^A(x)$ . Second, they will be used to enforce normality of our system.

(b)  $Nmon(\mathcal{S})$  consists again of two groups of rules:

- (i)  $\{\frac{a_{i,x}^n}{\bar{a}_{i,x}^n} : i, n, x \in \omega\}$ . These are nothing else but the closed world assumption with respect to the constants  $a_{i,x}^n$ . It will be crucial in our proof of normality of  $\mathcal{S}$ .
- (ii) Here we have an infinite collection of rules:  

$$\frac{a_{i_1,x}^n \dots a_{i_k,x}^n \bar{a}_{j_1,x}^n \dots \bar{a}_{j_l,x}^n}{b_x} : \varphi_{e,x,n}^{C,D} \text{ holds and } C = \{i_1, \dots, i_k\},$$

$$D = \{j_1, \dots, j_l\}$$

These are the rules which we use to compute  $B$ .

The required consistency property,  $Cons$ , is defined as follows.

$$X \in Cons \Leftrightarrow \neg \exists_{i,n,x} \{a_{i,x}^n, \bar{a}_{i,x}^n\} \subseteq X$$

Our rules ensure that  $a_{i,x}^n$  will be in the extension if  $i \in A$  and that  $\bar{a}_{i,x}^n$  will be in the extension if  $i \notin A$ . Thus,  $X$  is “consistent” if it does not contain a pair of “contradictory” elements  $a_{i,x}^n, \bar{a}_{i,x}^n$ .

Our main result is the following theorem.

**Theorem 3.1.**  $\mathcal{S} = \langle U, N \rangle$  is a recursive  $FC$ -normal nonmonotonic rule system with respect to the consistency property  $Cons$  above. Moreover  $\mathcal{S}$  has a unique extension  $E = \{a_i^s : f(s) = i\} \cup \{a_{i,x}^n : i \in A\} \cup \{\bar{a}_{i,x}^n : i \notin A\} \cup \{b_x : x \in B\}$  which is Turing equivalent to  $A \oplus B$ .

Proof: We will prove four facts:

1.  $\mathcal{S}$  is a recursive nonmonotonic rule system
2.  $\mathcal{S}$  is  $FC$ -normal with respect to the consistency property  $Cons$  as defined above
3.  $\mathcal{S}$  is stratified, with two strata (this implies uniqueness of extension)
4. The unique extension  $E$  of  $\mathcal{S}$  is Turing equivalent to  $A \oplus B$ .

Clearly,  $U$  is a recursive set, since it is easy to see that each of its three components is recursive.

Next, we look at the rules of  $N$ . Since the family  $\langle A^s \rangle_{s \in \omega}$  is a recursive decomposition of  $A$ , the group (a)(i) is recursive. It should be entirely obvious that the group (a)(ii) is recursive too. The group (b)(i) is clearly recursive – it encodes the Closed World Assumption with respect to the set of elements  $a_{i,x}^n$  which is a recursive set. Finally, the group (b)(ii) is again recursive since there is an effective algorithm to see if  $\varphi_{x,n}^{C,D}$  holds. This shows that  $N$  is a recursive set of rules. Thus  $\mathcal{S}$  is a recursive nonmonotonic rule system.

In order to prove *FC*-normality of  $\mathcal{S}$ , we investigate the form of families of sets closed under the monotonic rules. Clearly, if  $S$  is such set then  $S$  contains all the elements  $a_i^s$  for all  $i \in A^s \setminus a^{s-1}$ . Moreover, whenever  $a_i^s \in S$  then  $a_{i,x}^n \in S$ . From the form of our rules it follows that the two above conditions determine all sets closed under the monotonic rules. Notice that no *monotonic* rule allows derivation of any  $\bar{a}_{i,x}^n$  or  $b_x$ .

Now, we show that every nonmonotonic rule in  $N$  is normal with respect to the property *Cons*. So assume that  $S$  is consistent with respect to *Cons* and closed under monotonic rules. We need to consider two cases.

Case of the rules of the type (b)(i). Here

$$r = \frac{a_{i,x}^n}{\bar{a}_{i,x}^n}$$

Since  $\bar{a}_{i,x}^n$  is not a premise of any monotonic rule, the monotonic closure of  $S \cup \{\bar{a}_{i,x}^n\}$  coincides with  $S \cup \{\bar{a}_{i,x}^n\}$ . Moreover  $S \cup \{\bar{a}_{i,x}^n, a_{i,x}^n\}$  is inconsistent. Thus all these rules are normal.

Case of the rules of the type (b)(ii). Here the first condition of normality follows in the fashion similar to the previous case. If  $S$  is closed and consistent w.r.t *Cons* then so is  $S \cup \{b_x\}$ . On the other hand, adding to that set  $\bar{a}_{i,x}^n$  (where  $\bar{a}_{i,x}^n$  appears in the constraint of the rule  $r$ ) must result in inconsistency, for since  $x$  does not belong to the complement of  $A$ ,  $x$  belongs to  $A$  and so the element  $a_{i,x}^n \in S$ . This shows that our system  $\langle U, N \rangle$  is normal.

We are now showing that the system  $\langle U, N \rangle$  is stratified for then we can use the argument of [ABW88]. Actually, it is very easy to see that the following is a stratification of the length two.

The objects  $a_i^s, a_{i,x}^n$  are in stratum 0.

The objects  $\bar{a}_{i,x}^n$  are in stratum 1.

The objects  $b_x$  are also in stratum 1.

The fact that  $\langle U, N \rangle$  is stratified is immediate – by the form of rules in  $N$ . This implies that



$\langle U, N \rangle$  has a unique extension.

Finally, in order to show that the unique extension  $E$  of  $\mathcal{S}$  is Turing equivalent to  $A \oplus B$ , we establish four equivalences which together imply our statement.

- (a)  $a_i^s \in E$  if and only if  $i \in A^s \setminus A^{s-1}$
- (b)  $a_{n,x}^i \in E$  if and only if  $i \in A$
- (c)  $\bar{a}_{n,x}^i \in E$  if and only if  $i \notin A$
- (d)  $b_x \in E$  if and only if  $x \in B$

Both (a) and (b) are obvious. Thus we know that

$$E \cap (\{a_i^s : s, i \in \omega\} \cup \{a_{i,x}^n : i, n, x \in \omega\}) = \{a_i^s \exists_s i \in A^s \setminus A^{s-1}\} \cup \{a_{i,x}^n : i \in A\}$$

This immediately implies (c) by the fact that we have a Closed World Assumption on the constants  $a_{n,x}^i, \bar{a}_{n,x}^i$ . (d) follows from the description of how  $B$  is computed from  $A$ . Thus  $E$  is Turing reducible to  $A \oplus B$ . The converse reducibility is obvious from (b) and (d).  $\square$

In [MNR94] we proved the following result

**Theorem 3.2.** ([MNR94]) Suppose that  $\mathcal{S} = \langle U, N \rangle$  is a recursive nonmonotonic rule system and  $\mathcal{S}$  is *FC-normal*. Then  $\mathcal{S}$  has an extension  $E$  such that  $E$  is r.e. in  $0'$  and hence  $E \leq_T 0''$ .

Since our system  $\mathcal{S}_{A,B}$  possesses a unique extension  $E$  which is Turing-equivalent to  $A \oplus B$ , the estimate of Proposition 3.2 cannot be improved.

We will show now how our results on the complexity of nonmonotonic rule systems can now be used to show that the bound on the complexity of extensions of normal default theories cannot be lowered. To this end we again rewrite the nonmonotonic rule system  $\mathcal{S}_{A,B}$  as a logic program  $P^*$ .

- (0)  $a_x^s \leftarrow$  for  $x \in A^s \setminus A^{s-1}$ .
- (1)  $a_{i,x}^n \leftarrow a_i^s$  for all  $i, n, s, x \in \omega$ .
- (2)  $\bar{a}_{i,x}^n \leftarrow \text{not}(a_{i,x}^n)$  for all  $i, n, x \in \omega$ .
- (3)  $b_x \leftarrow a_{i_1,x}^n, \dots, a_{i_k,x}^n, \bar{a}_{j_1,x}^n, \dots, \bar{a}_{j_l,x}^n$  whenever  $\varphi_{x,n}^{C,D}$  holds and  $C = \{i_1, \dots, i_k\}, D = \{j_1, \dots, j_l\}$ .

This logic program is then translated into default logic in the manner of [MT89, GL90]. One additional element is that we must make sure that under this logical interpretation  $\bar{a}_{i,x}^n$  is equivalent to  $\neg a_{i,x}^n$ . We are then getting the following default theory  $\langle D, W \rangle$ , where  $W$  is:

$a_x^s$  for  $x \in A^s \setminus A^{s-1}$ .

$a_i^s \supset a_{i,x}^n$  for all  $i, n, s, x \in \omega$ .

$a_{i_1,x}^n, \dots, a_{i_k,x}^n, \bar{a}_{j_1,x}^n, \dots, \bar{a}_{j_l,x}^n \supset b_x$  whenever  $\varphi_{x,n}^{C,D}$  holds and

$C = \{i_1, \dots, i_k\}, D = \{j_1, \dots, j_l\}$ .

$\bar{a}_{i,x}^n \Leftrightarrow \neg a_{i,x}^n$

Default part of our theory is:

$$\left\{ \frac{:M\neg a_{i,x}^n}{\neg a_{i,x}^n} : i, n, x \in \omega \right\}.$$

It is obvious that the default theory  $\langle D, W \rangle$  is normal. We use now the following fact [BF91, MT89, GL90]:

**Proposition 3.1.**  $M$  is an extension of the program  $P^*$  if and only if  $Cn(M)$  is a default extension of  $\langle D, W \rangle$ .

It is obvious that  $M$  and  $Cn(M)$  are of the same Turing degree. Since  $M$  is Turing equivalent to  $A \oplus B$ , so is  $Cn(M)$  which is a unique extension of the default theory  $\langle D, W \rangle$ . This completes our discussion of the complexity estimate of default extensions of normal default theories. We formulate it explicitly as a corollary.

**Corollary 3.1.** Let  $A$  be an r.e. set and let  $B$  be r.e. in  $A$ . Then there exists a recursive normal default theory  $\langle D, W \rangle$  such that  $\langle D, W \rangle$  possesses a unique extension  $S$  and  $S \equiv_T A \oplus B$ .

We do not have the space in this abstract to show how we can write a finite logic program and a finite predicate logic normal default theory which are equivalent respectively to the recursive logic program and the recursive normal default theory described above. However one can use the techniques in [MNR92a] to accomplish this.

We end this abstract with one last remark. One way of looking at the nonmonotonic rule system  $\mathcal{S}_{A,B}$  is that we have constructed a stratified nonmonotonic rule system with two strata whose unique extension which allows us to code up to  $0''$ . This process can be extended. For example, suppose that  $C$  is r.e. in  $B$ , say  $C = \{x : \phi_f^B(x) \downarrow\}$ . Then we can add a new strata of rules to  $\mathcal{S}_{A,B}$  to form a stratified nonmonotonic rule system  $\mathcal{S}_{A,B,C}$  with 3 strata whose unique extension is Turing equivalent to  $(A \oplus B) \oplus C$ . That is, we simply add the following rules.

$$(i) \left\{ \frac{b_i}{b_{i,x}^n} : i, n, s, x \in \omega \right\}.$$

- (ii)  $\{\frac{b_{i,x}^n}{b_{i,x}^n} : i, n, x \in \omega\}$ .
- (iii) Here we have an infinite collection of rules:  
 $\{\frac{b_{i_1,x}^n, \dots, b_{i_k,x}^n, \bar{b}_{j_1,x}^n, \dots, \bar{b}_{j_l,x}^n}{b_x} : \varphi_{f,x,n}^{H,K}$  holds and  $H = \{i_1, \dots, i_k\}$ ,  
 $K = \{j_1, \dots, j_l\}\}$

We note, however, that  $\mathcal{S}_{A,B,C}$  will no longer be  $FC$ -normal by Theorem 3.2.

This process can easily be continued to produce a stratified nonmonotonic rule system with  $n$  strata, for any  $n \in \omega$ , whose unique extension codes up the  $n$ -th jump of the recursive sets,  $O^{(n)}$  which implies the main result in [AB90]. Indeed, the process can easily be extended to the transfinite, to produce a recursive locally stratified nonmonotonic rules system whose unique extension will code  $0^{(\alpha)}$  for any recursive ordinal  $\alpha$  which implies the main result of [BMS91]. In both cases, we can replace the recursive nonmonotonic rule systems by finite predicate logic programs.

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