Computing intersection of autoepistemic expansions

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Abstract

In this paper, we consider the question of skeptical reasoning for an important nonmonotonic reasoning system — the autoepistemic logic of Moore. Autoepistemic logic is a method of reasoning which assigns to a set of formulas the collection of theories called *stable expansions*. A naive method to perform skeptical autoepistemic reasoning — deciding whether a given formula φ belongs to all expansions of a theory — is to compute first all expansions and then check whether φ belongs to each of them. This approach to skeptical autoepistemic reasoning is however prohibitively inefficient.

The goal of this paper is to propose a different approach to computing intersection of all expansions of a theory. Our approach does not require us to compute any expansion of a theory. It reduces the question of membership in the intersection of all expansions to the question of propositional provability. More precisely, we describe a method that assigns to a modal theory I a propositional theory P_I and to a modal-free formula φ another formula φ' in such a manner that φ is in the intersection of all expansions of I if and only if $P_I \vdash \varphi'$.

In general, the theory P_I is much larger than the original theory I. We have found, however, several cases when it is not so and the size of the theory P_I is a polynomial in the size of I. These classes of theories are closely related to logic programs and disjunctive logic programs. Consequently, we obtain methods to check whether an atom is in the intersection of all supported (or stable) models of a (disjunctive) logic program, as well as numerous complexity results.

1 Introduction

In nonmonotonic reasoning, there are two main ways of defining consequences of a set of assumptions. A general scheme is that to every set I of "initial assumptions", we assign a number of belief sets $B_1, B_2, \ldots$. Then, we either accept as the agent's set of consequences the intersection of all B_is — this mode of reasoning is called *skeptical* — or, we select one particular belief set B_i — this mode of reasoning is called *brave*. Of course, both modes can be used in the same time.

From this point of view, classical logic reasoning can be viewed as skeptical — we take as B_i's all complete theories containing I and then accept as the set of theorems their intersection. Similarly, a well-founded semantics for logic programs [VRS88, Prz89] can be regarded as an example of skeptical reasoning. Skeptical reasoning was also considered in the context of disjunctive logic programming [RLM89] and for inheritance nets with exceptions [HTT87].

One might expect that skeptical reasoning is much more costly than brave one. At first glance it seems that to perform skeptical reasoning one has to know all possible belief sets B_i and only then compute their intersection. Fortunately, in many important cases it is not so. For example, in the case of classical propositional logic we can decide whether a formula φ follows from a theory I without constructing all complete theories containing I (models of I). The same is true for important fragments of first-order logic, where in some case the decision can be reached in polynomial time (like in the case of Horn theories)

In this paper, we consider the question of skeptical reasoning for an important nonmonotonic reasoning system — the autoepistemic logic of Moore [Moo85]. This logic serves as a formalization of nonmonotonic reasoning about beliefs and was thoroughly investigated [Moo85, MT91, Kon88, Shv88, Shv90]. Autoepistemic logic is a method of reasoning which assigns to a set of formulas $I \subseteq \mathcal{L}_L$, the collection of *stable expansions* of I. A theory T is a *stable expansion* of I if T satisfies the equation:

$$T = Cn(I \cup \{L\varphi \colon \varphi \in T\} \cup \{\neg L\varphi \colon \varphi \notin T\}). \tag{1}$$

Since T appears on both sides of the equation (1), a theory I may have none, one or many expansions. The possibility of existence of many expansions implies that autoepistemic logic admits both skeptical and brave modes of reasoning.

A naive method to perform skeptical autoepistemic reasoning — deciding whether a given formula φ belongs to all expansions of a theory — is to compute first all expansions and then check whether φ belongs to each of them. This approach was investigated in many papers [Nie88, MT91, Shv88, JK90]. The method of computing all expansions involves, in particular, testing 2^m candidate theories, where m is the size of I, and this estimate is valid only when I is already in the so called Moore's normal form. Each such test requires numerous calls to a procedure for testing whether a formula is a propositional consequence of a theory. Thus, this approach to skeptical autoepistemic reasoning is prohibitively expensive.

The goal of this paper is to propose a different approach to computing intersection of all expansions of a theory. Our approach does not require us to compute all expansions of a theory. It reduces the question of membership in the intersection of all expansions to the question of propositional provability.

More precisely, we describe a method that assigns to a modal theory I a propositional theory P_I and to a modal-free formula φ another formula φ' in such a manner that φ is in the intersection of all expansions of I if and only if $P_I \vdash \varphi'$. In addition, φ belongs to some expansion if and only if $P_I \cup \{\varphi'\}$ is consistent.

We have to comment, though, that so far we were successful in applying our technique only to compute the intersection of modal-free parts of the expansions of a theory I. While even our restricted results have useful applications in logic programming, the general question of how to avoid explicit computation of all expansions in skeptical autoepistemic reasoning involving nested beliefs still needs to be investigated.

In general the size of the theory P_I is much bigger than I. This is to be expected. After all, one of the main objectives of nonmonotonic reasoning is to allow concise representation of vast amounts of information — large parts of knowledge, especially negative information, is only implicitly represented (by the fact of its absence). A somewhat surprising result of our research is that in some important cases theory P_I consists of propositional clauses and is of roughly the same size as the original theory I. Among classes of theories I with such property are the classes of epistemic programs and disjunctive epistemic programs, which correspond naturally to the classes of logic programs and disjunctive logic programs.

We discuss these two important cases here. As an additional application of our approach, we show how to compute the intersection of all supported and all stable models of finite propositonal logic program. (A similar result for stable models of arbitrary logic programs, but with $\mathcal{L}_{\omega_1,\omega}$ theories, has been obtained in [MNR90]). The last result is not surprising. Supported models are models of Clark's completion, and an epistemic interpretation of programs corresponding to Clark's completion has been given in [MS89].

An important technique used to ensure that the theories we built consist of clauses (and so classical complexity results apply) is to introduce new propositional variables which play a role of abbreviations. We borrowed this technique from proof theory.

The paper is organized as follows. In the next section we review autoepistemic logic. In Section 3, we consider the special, and most elegant, case of epistemic programs. In Section 4, we consider the general case and, in Section 5, we give a number of applications of our technique. In Section 6 we give a proof of Theorem 4.1, the main result of the paper.

2 Review of Autoepistemic Logic

In the paper, we restrict ourselves to the propositonal case only. We consider a fixed language $\mathcal{L}$ of propositional calculus and its modal extension $\mathcal{L}_L$ by one modal operator L. The set of atoms of $\mathcal{L}$ is denoted by At.

Definition 2.1 *A theory $T \subseteq \mathcal{L}_L$, is* stable *if T satisfies these conditions:*

1. *T is closed under propositional consequence.*

2. *For every formula $\varphi \in \mathcal{L}_L$, if $\varphi \in T$ then $L\varphi \in T$.*

3. *For every formula $\varphi \in \mathcal{L}_L$, if $\varphi \notin T$ then $\neg L\varphi \in T$.*

A stable theory T is uniquely determined by its objective (or modal-free) part, that is $T \cap \mathcal{L}$ ([Moo85, Kon88]). In fact there is an operator E such that for every modal-free theory S, $E(S)$ is the unique stable theory T such that $T \cap \mathcal{L} = Cn(S)$ ([Mar89]).

It follows immediately from the definition of a stable expansion (given by equation (1)) that theories I_1 and I_2 such that $Cn(I_1) = Cn(I_2)$ have precisely the same expansions. This, in turn, implies that for every theory I_1 there is a theory I_2 consisting of *epistemic clauses* that is formulas of the form

$$L\alpha_1 \wedge \ldots \wedge L\alpha_k \wedge \neg L\beta_1 \wedge \ldots \wedge \neg L\beta_m \Rightarrow \gamma \tag{2}$$

with $\gamma \in \mathcal{L}$, and such that I_1 and I_2 have precisely the same expansions. We can even assume that γ is a propositional clause. Actually, although it does not follow directly from the definition of expansions, we can assume that α_i, and β_i belong to $\mathcal{L}$ as well [MT91, Kon88].

For a theory I consisting of epistemic clauses, we define $H(I)$ to consist of all formulas $\gamma \in \mathcal{L}$ such that for some α_i and β_i, the clause

$$L\alpha_1 \wedge \ldots \wedge L\alpha_k \wedge \neg L\beta_1 \wedge \ldots \wedge \neg L\beta_m \Rightarrow \gamma$$

is in I. The crucial role in our considerations is played by two results characterizing expansions of theories consisting of epistemic clauses. The first was obtained in [MT91], the second is an easy consequence of the first one. To formulate them, we need one more definition.

Definition 2.2 *Let I be an epistemic theory and let T be stable. We say that a formula $\varphi \in H(I)$ has an I-*support *in T if for some clause*

$$L\alpha_1 \wedge \ldots \wedge L\alpha_k \wedge \neg L\beta_1 \wedge \ldots \wedge \neg L\beta_m \Rightarrow \varphi$$

from I, $\alpha_i \in T$, $1 \leq i \leq k$, and $\beta_i \notin T$, $1 \leq i \leq m$.

The two characterizations mentioned earlier are gathered in the theorem below.

Theorem 2.3 *Let $I \subseteq \mathcal{L}_L$ consist of epistemic clauses.*
(a) *A theory T is an expansion of I if and only if $I \subseteq T$ and for some set $S \subseteq H(I)$, such that each $\gamma \in S$ has an I-support in T, we have $T = E(S)$.*
(b) *A theory $E(S)$ is an expansion of I if and only if $I \subseteq E(S)$ and for every $\varphi \in S$, there is $\Gamma \subseteq H(I)$, such that $\Gamma \vdash \varphi$ and each $\gamma \in \Gamma$ has a support in $E(S)$.*

3 Case of epistemic programs

We start the presentation of our results with the discussion of computing skeptical autoepistemic reasoning in the case of particularly simple theories — *epistemic programs.* In this simple case we get elegant results and we are able to introduce main concepts of our approach.

An *epistemic program clause* is a clause of the form

$$La_1 \wedge \ldots La_n \wedge \neg Lb_1 \wedge \ldots \neg Lb_m \Rightarrow c \tag{3}$$

where all a_i's, $b'j$, and c are atoms of $\mathcal{L}$. An *epistemic program* is a collection of epistemic program clauses.

Let I be an epistemic program. A *propositional transform* of the theory I, in symbols T_I, is a theory in a propositional language $\mathcal{L}'$ which is generated by the atoms of $\mathcal{L}$ and, in addition, by some other atoms. Namely, for each clause $C \in I$ we add a new atom d_C.

We describe now the construction of the transform T_I of I. First, for each clause C of the form (3) we add to T_I two formulas

$$\Phi_C := \quad d_C \Rightarrow c, \text{ and}$$

$$\Psi_C := \quad d_C \Leftrightarrow a_1 \wedge \ldots \wedge a_n \wedge \neg b_1 \wedge \ldots \wedge \neg b_m.$$

Next, for *each* atom s of $\mathcal{L}$ add to T_I the formula

$$\Sigma_s := \quad s \Rightarrow \bigvee\{d_C \colon s \text{ is a head of } C\}.$$

This last clause has as an effect that if s is not the head of a clause in I, then $\neg s$ is added to the transform.

It should be clear that the transform, T_I is equivalent to a set of clauses of at most twice the size of I. It is also important to note that the size of T_I is of the same order as the size of I. Another important fact that needs to be realized is that the transform T_I is *not* monotonic in the argument I.

Given a theory $T \subseteq \mathcal{L}'$, by a model of T we mean any assignment (valuation) V of atoms of $\mathcal{L}'$ into $\{0, 1\}$ such that for every $\varphi \in T$, $V(\varphi) = 1$.

Informally, the idea behind the construction of T_I is as follows: Formulas Φ_C and Ψ_C ensure that a valuation V of $\mathcal{L}'$ is a model of T_I if and only if I is a subset of a stable set generated by the atoms of $\mathcal{L}$ satisfied by V. Formulas Σ_s guarantee that atoms satisfied by V have an I-support in the expansion corresponding to V.

Example: Consider the following epistemic program I:

$$Lp \wedge \neg Ls \Rightarrow r$$
$$Lr \Rightarrow p.$$

Its transform T_I consists of the following formulas (here the two new atoms, one for each clause, are denoted by d_1 and d_2:

$$d_1 \Rightarrow r,$$
$$d_2 \Rightarrow p,$$

$d_1 \Leftrightarrow p \wedge \neg s,$
$d_2 \Leftrightarrow r,$
$r \Rightarrow d_1,$
$p \Rightarrow d_2,$
$\neg s.$

It is easy to see that I has two expansions: $E(\emptyset)$ and $E(\{p, r\})$. An examination of the transform shows that there are exactly two valuations that satisfy T_I: in one each original atom is assigned 0, in the other, p and r are assigned 1 and s is assigned 0. As we will see below, this is not a coincidence.

Let us also, observe that if $I \subseteq I'$ then we do not have, in general, that $T_I \subseteq T_{I'}$. For example if $I' = I \cup \{s\}$, then $T_{I'}$ no longer contains $\neg s$. Such behavior is to be expected, as autoepistemic logic is nonmonotonic. $\Box$

Making these intuitions precise and using Theorem 2.3(a) gives a theorem establishing a one-to-one correspondence between the models of the theory T_I and expansions of I.

Theorem 3.1 *Let I be an epistemic program, and T_I its transform. Then a valuation V is a model of T_I if and only if $E(\{s \in \mathcal{L}\colon V(s) = 1\})$ is an expansion of I.*

The proof of Theorem 3.1 is a simplified version of the proof of Theorem 4.1, and is omitted.

Let us briefly comment on the role of atoms d_C in our theorem. These atoms are uniquely determined by their defining formulas Ψ_C. Therefore we can find a theory t_I *not involving* any extra atoms (that is, a theory in $\mathcal{L}$) which has the property that its models are in one-to-one correspondence with expansions of I. But if we do so, the resulting theory t_I does not consist of clauses, nor can it be transformed to an equivalent theory consisting of clauses only without exponential growth in size. Thus, our introduction of new atoms d_C may be viewed as a device to obtain a clausal form at a modest cost.

Theorem 3.1 implies a result which, in turn, determines the complexity of membership problem for literals in the intersection (and union) of expansions of an epistemic program.

Proposition 3.2 *Let I be an epistemic program, and let a be an atom.*
(i) *a belongs to all expansions of I if and only if $T_I \vdash a$.*
(ii) *a belongs to no expansion of I if and only if $T_I \vdash \neg a$.*
(iii) *a belongs to some expansion of I if and only if $T_I \cup \{a\}$ is consistent.*

When we look closely on the size of the theory T_I as a function of the size of theory I (they are, as was said earlier, of the same order) we get, as a corollary of Proposition 3.2, the following complexity results.

Proposition 3.3 **(i)** *The problem of membership of an atom in all expansions of an epistemic program is co-NP complete.*

(ii) *The problem of membership of an atom in some expansion of an epistemic program is NP complete.*

Proposition 3.3 gives a complete picture of the complexity of the problems for both intersection and union of all expansions of epistemic programs.

4 The case of arbitrary clauses of L-depth 1

In this section we will consider a more general situation. We are interested here in the class of theories consisting of clauses of the form (2), with α_i, β_i and γ being modal-free. Such clauses will be referred to as *1-epistemic clauses* and collections of 1-epistemic clauses are called *1-epistemic theories*.

Our goal is to reduce the problem of computing the intersection of the modal-free parts of expansions of an epistemic theory to the problem of provability in propositional calculus. Our approach is based on the ideas illustrated earlier in the special case of epistemic programs.

The crucial role in our considerations in this section is played by Theorem 2.3(b). Let I be a 1-epistemic theory. We will construct now a propositional theory P_I (we use a different symbol than before because our construction here differs slightly form that applied in the case of epistemic programs) with the property that expansions of I correspond to models of P_I. This will allow us to reduce the problem of the membership of a modal-free formula in the intersection of expansions of I to the question whether some other formula is provable from P_I. To describe P_I we need some additional terminology.

First, by $U(I)$ we denote the set of all formulas $\varphi \in \mathcal{L}$ such that $L\varphi$ occurs in a clause from I or such that $\varphi \in H(I)$. For each $\varphi \in U(I)$, we introduce a new propositional variable $\overline{\varphi}$. Now, for a 1-epistemic clause C of I, given by (2), we define

$$s_C = \overline{\alpha}_1 \wedge \ldots \overline{\alpha}_k \wedge \neg\overline{\beta}_1 \wedge \ldots \wedge \neg\overline{\beta}_m.$$

Next, for a formula $\varphi \in U(I)$ we define

$$S_\varphi = \bigvee\{s_C : \varphi \text{ is the head of } C\}.$$

Note that if φ is not the head of any clause from I, then $S_\varphi = \bot$. Finally, for a set of formulas $\Gamma \subseteq U(I)$, we define

$$S_\Gamma = \bigwedge\{S_\varphi : \varphi \in \Gamma\}.$$

The intuition behind this notation is as follows: we will design P_I so that models of P_I can be associated with expansions of I. More precisely, we will have that formula S_φ is true in a model if and only if φ has an I-support in the corresponding expansion T, and formula S_Γ is true in the model if and only if all formulas of Γ have I-supports in T.

Now, we formally define the theory P_I. For each 1-epistemic clause (2) of I, we include in P_I a formula

$$s_C \Rightarrow \overline{\gamma}. \tag{4}$$

In addition, for each $\varphi \in U(I)$ we include in P_I the formula

$$\overline{\varphi} \Leftrightarrow \bigvee\{S_\Gamma : \Gamma \subseteq H(I) \text{ is minimal such that } \Gamma \vdash \varphi\}. \tag{5}$$

(Let us mention that this last formula could be repalced by the formula $\overline{\varphi} \Leftrightarrow \bigvee\{S_\Gamma : \Gamma \vdash \varphi\}$, but the formula we have chosen is, in general, shorter.)

Informally, formulas of the first type correspond to the requirement of Theorem 2.3(b) that $I \subseteq E(S)$ and formulas of the second type correspond to the requirement that each element in S be provable from elements having I-supports.

Theorem 4.1 **(a)** *Let a valuation V of atoms $\overline{\varphi}$ be a model of P_I. Then*

$$E(\{\varphi : V(\overline{\varphi}) = 1\})$$

is an expansion of I.
(b) *Let E be an expansion of I. Then the valuation V defined by*

$$V(\overline{\varphi}) = 1 \quad \textit{if and only if} \quad \varphi \in E$$

is a model of P_I.

Theorem 4.1 establishes a *one-to-one* correspondence between expansions of I and models of P_I. This correspondence yields a method to check whether a formula $\varphi \in U(I)$ belongs to the intersection of all expansions of I. The method can easily be obtained from the following corollary to Theorem 4.1.

Corollary 4.2 *Let I be an epistemic theory and let $\varphi \in U(I)$. Then, φ is in the intersection of all expansions of I if and only if $P_I \vdash \overline{\varphi}$.*

For the case of an arbitrary formula $\varphi \in \mathcal{L}$, we have the following result.

Corollary 4.3 *Let I be a finite epistemic theory and let $\varphi \in \mathcal{L}$. Let $\Gamma_1, \ldots, \Gamma_n$ be all minimal subsets of $H(I)$ proving φ. Then, φ is in the intersection of all expansions of I if and only if $P_I \vdash \bigvee_{i=1}^{n} \bigwedge\{\overline{\gamma}: \gamma \in \Gamma_i\}$.*

Example. Consider the following 1-epistemic theory I consisting of the formulas:

$$L(p \vee r) \Rightarrow p \vee \neg q, \qquad L(p \vee r) \Rightarrow q \vee r.$$

We will slightly change notation in this example. Instead of denoting the new atom for $p \vee r$ by $\overline{p \vee r}$, we denote this atom by a. Similarly, the new atom for $p \vee \neg q$ is denoted by b and the new atom for $q \vee r$ is denoted by c. Now, the theory P_I consists of the following formulas:

$$a \Rightarrow b,$$

$$\begin{aligned}&a \Rightarrow c,\\&a \Rightarrow a \wedge a,\\&b \Leftrightarrow a,\\&c \Leftrightarrow a.\end{aligned}$$

Clearly, theory P_I is propositionally equivalent to $P_I' = \{b \Leftrightarrow a, c \Leftrightarrow a\}$. It can be verified that I has two expansions $E(\emptyset)$ and $E(\{p \vee \neg q, q \vee r\})$ (the latter contains $p \vee r$, as well), and P_I' (and hence P_I, as well) is satisfied by two valuations: one that makes atoms a, b and c false, the other which makes them true. □

A general conclusion of our results from this section is that autoepistemic logic, at least as far as modal-free formulas are concerned, can be expressed in a very natural way by means of propositional logic. The theory we obtain, P_I, that carries the same information as I is, usually, very large. This seems to be in agreement with the expectation that nonmonotonic logics, as they were originally envisioned, are capable of expressing the same information as classical logic but by means of a smaller theory. The problem is that the price that is paid for this size efficiency is in the loss of efficiency in reasoning methods.

5 Applications

First, we will consider the question of computing the intersection of supported models of a finite, propositional logic program. Modifying slightly the results of [MS89], one can see that under the translation interpreting a clause

$$c \leftarrow a_1, \ldots, a_k, \neg b_1, \ldots, \neg b_m, \tag{6}$$

where a_i, b_i and c_i are propositional variables, by the epistemic program clause

$$La_1 \wedge \ldots \wedge La_k \wedge \neg Lb_1 \wedge \ldots \wedge \neg Lb_m \Rightarrow c, \tag{7}$$

there is a *one-to-one* correspondence between supported models of programs and expansions of translations. Thus, we get the following results as corollaries to the results of Section 3.

Corollary 5.1 *Let P be a finite propositional logic program and let I be a theory obtained by replacing each clause (6) of P with a formula (7).*
(i) *a belongs to all supported models of P if and only if $T_I \vdash a$.*
(ii) *a belongs to no supported model of P if and only if $T_I \vdash \neg a$.*
(iii) *a belongs to some supported model of P if and only if $T_I \cup \{a\}$ is consistent.*
(iv) *The problem of membership of an atom in all supported models of a logic program is co-NP complete.*
(v) *The problem of membership of an atom in some supported model of a logic program is NP complete.*

Let us note that in [MS89] an interpretation of logic programs is given that assigns to a program P expansions of (complete) *theories* determined by supported models of P. This requires formulas (7) and additional formulas, essentially similar to formulas of form Σ_s. Since here we are interested only in expansions of *sets of atoms* true in supported models, the formulas (7) are enough.

Using our general results of Section 4, we can reduce questions concerning membership in stable models to provability and satisfiability in propositional logic. This is achieved by means of the following result obtained independently by many authors ([Elk89, MT89]), but the credit for which has to be given to Michael Gelfond: under the interpretation of a clause (6) by a formula

$$\neg Lb_1 \wedge \ldots \wedge \neg Lb_m \Rightarrow (a_1 \wedge \ldots \wedge a_k \Rightarrow c), \tag{8}$$

there is a *one-to-one* correspondence between stable models of a logic program and sets of atoms of expansions of the corresponding 1-epistemic theory.

Thus, for example, to check if an atom a belongs to all stable models of P, we first build a corresponding 1-epistemic theory I interpreting each clause of P by the formula (8). Next we build the theory P_I and see whether $P_I \vdash a$. The details of this procedure, as well as precise statements of the relevant results will be given in the full version of the paper.

It was already mentioned that in general the theory P_I has much larger size than the theory I. The case of epistemic programs shows, however, that sometimes the size of P_I is of the same order as the size of I. There are several other case when the size of P_I is kept small (bounded by a polynomial in the size of I). For example, our method of Section 4 simplifies in the following two cases:

Case 1. Theory I consists of epistemic clauses (2), where α_i, β_i and γ are literals. This is a slightly more general case than that of epistemic programs. In this case, for each formula $\varphi \in U(I)$ (φ is a literal), there is at most one minimal set $\Gamma \subseteq H(I)$ such that $\Gamma \vdash \varphi$, namely, $\Gamma = \{\varphi\}$. Thus, the size of P_I is of the order of the size of I. Moreover, as in the case of epistemic programs in Section 3, introducing additional atoms allows us to convert P_I into a theory consisting of propositional clauses, without significant increase in size. The complete discussion of this case will be given in the full version of the paper.

Case 2. Theory I consists of epistemic clauses (2), where α_i, β_i and γ are disjunctions of atoms. In this case formulas of type (5) simplify. First of all notice that if Γ is a minimal subset of $H(I)$ such that $\Gamma \vdash \varphi$ then, since φ and all formulas in Γ are disjuntions of atoms, it follows that Γ has the size 1, and for the unique formula ψ in Γ, all the atoms in ψ appear in φ. Hence $S_\Gamma = \overline{\psi}$. Consequently, in this case, the formulas of the type (5) simplify to the following form:

$$\overline{\varphi} \Leftrightarrow \{\overline{\psi} \colon \psi \in H(I) \text{ and } \psi \text{ subsumes } \varphi\}$$

for each formula $\varphi \in U(I)$. This analysis allows us to estimate the size of P_I. The size of each simplified formula of type (5) is now $O(\mid I \mid)$. Consequently, the size of P_I is $O(size(I) + size(I) \cdot \mid I \mid)$. Thus the size of P_I is at most quadratic in the size of I.

6 Proof of Theorem 4.1

Theorem 4.1(a) *Let a valuation* V of atoms $\overline{\varphi}$ be a model of P_I. *Then*:

$$E(\{\varphi \in U(I) : V(\overline{\varphi}) = 1\})$$

is an expansion of I.

Proof: Consider a valuation V of atoms $\overline{\varphi}$ such that $V(P_I) = 1$. Let us denote $E = E(\{\varphi \in U(I) : V(\overline{\varphi}) = 1\})$. We need to show that:

(1) $I \subseteq E$, and

(2) for every $\varphi \in U(I)$ such that $V(\overline{\varphi}) = 1$, there is $\Gamma \subseteq H(I)$ such that $\Gamma \vdash \varphi$ and each $\gamma \in \Gamma$ has an I-support in E.

First, we prove a claim to be used in both parts of our argument.
Claim: *Let* $\alpha \in U(I)$. *Then,*

$$V(\overline{\alpha}) = 1 \quad \text{if and only if} \quad \{\varphi : V(\overline{\varphi}) = 1\} \vdash \alpha$$

Proof of the claim: Only the implication from right to left needs a proof. So assume $\{\varphi : V(\overline{\varphi}) = 1\} \vdash \alpha$. For each $\varphi \in U(I)$ and such that $V(\overline{\varphi}) = 1$,

$$V(\bigvee\{S_\Gamma : \Gamma \subseteq H(I) \text{ is minimal such that } \Gamma \vdash \varphi\}) = 1.$$

Hence, for each $\varphi \in U(I)$ and such that $V(\overline{\varphi}) = 1$ there is a $\Gamma_\varphi \subseteq H(I)$ such that $\Gamma_\varphi \vdash \varphi$ and $V(S_{\Gamma_\varphi}) = 1$. Consequently, for each $\gamma \in \Gamma_\varphi$ $V(S_\gamma) = 1$. Thus there is a clause $C \in I$ with γ in the head and such that $V(s_C) = 1$. Since $s_C \Rightarrow \overline{\gamma}$ belongs to P_I, it follows that $V(\overline{\gamma}) = 1$. Thus there exists a minimal subset $\Gamma \subseteq H(I) \cap \{\varphi : V(\overline{\varphi}) = 1\}$ such that $\Gamma \vdash \alpha$. In particular

$$V(\bigvee\{S_\Gamma : \Gamma \subseteq H(I) \text{ is minimal such that } \Gamma \vdash \alpha\}) = 1.$$

But all the instances of formula (5) are evaluated by valuation V as 1. Thus $V(\overline{\alpha} \Leftrightarrow \bigvee\{S_\Gamma : \Gamma \subseteq H(I) \text{ is minimal such that } \Gamma \vdash \alpha\}) = 1$. Therefore, $V(\overline{\alpha}) = 1$. $\square$ Claim

(1) Consider a 1-epistemic clause $C \in I$:

$$L\alpha_1 \wedge \ldots \wedge L\alpha_n \wedge \neg L\beta_1 \wedge \ldots \wedge \neg L\beta_m \Rightarrow \gamma$$

If $\alpha_i \notin E$ for some $1 \leq i \leq n$ or $\beta_j \in E$ for some $1 \leq j \leq m$ then by stability of E, $C \in E$. So assume that for all $1 \leq i \leq n$, $\alpha_i \in E$, and for all $1 \leq j \leq m$, $\beta_j \notin E$. By the claim, $V(\overline{\alpha_i}) = 1$, and $V(\overline{\beta_j}) = 0$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$. Thus $V(s_C) = 1$. Since $s_C \Rightarrow \overline{\gamma}$ belongs to P_I, $V(s_C \Rightarrow \overline{\gamma}) = 1$. Therefore $V(\overline{\gamma}) = 1$, and so $\gamma \in E$. But then $C \in E$.

(2) Let $\varphi \in U(I)$ and $V(\overline{\varphi}) = 1$. Since $V(\overline{\varphi}) = 1$, there is $\Gamma \subseteq H(I)$ such that $\Gamma \vdash \varphi$, and $V(S_\Gamma) = 1$. Let $\gamma \in \Gamma$. Then, $V(S_\gamma) = 1$. Consequently, there is a clause $C \in I$ of the form

$$L\alpha_1 \wedge \ldots \wedge L\alpha_n \wedge \neg L\beta_1 \wedge \ldots \wedge \neg L\beta_m \Rightarrow \gamma$$

such that $V(s_C) = 1$. Hence, $V(\overline{\alpha_i}) = 1$, $1 \leq i \leq n$ and $V(\overline{\beta_j}) = 0$, $1 \leq j \leq m$. By the claim, for each $1 \leq i \leq n$, $\alpha_i \in E$, and for each $1 \leq j \leq m$, $\beta_i \notin E$. Thus, γ has an I-support in E. $\Box$**4.1.(a)**

Theorem 4.1(b) *Let E be an expansion of I. Then the valuation V defined by*

$$V(\overline{\varphi}) = 1 \quad \textit{if and only if} \quad \varphi \in E$$

is a model of P_I.

Proof: Let E be an expansion of I. Then, by Theorem 2.3(a), $E = E(S)$, for some $S \subseteq H(I)$ and such that each $\gamma \in S$ has an I-support in E. Consequently, by the definition of the formulas S_γ and by the definition of the valuation V, for each $\gamma \in S$, $V(S_\gamma) = 1$.

Consider a formula $\varphi \in U(I)$ and the formula

$$\overline{\varphi} \Leftrightarrow \bigvee\{S_\Gamma : \Gamma \subseteq H(I) \text{ is minimal such that } \Gamma \vdash \varphi\}. \tag{9}$$

from P_I. Assume that $V(\overline{\varphi}) = 1$. Then, $\varphi \in E$. Consequently, $S \vdash \varphi$. Let $\Gamma \subseteq S$ be a minimal subset of S such that $\Gamma \vdash \varphi$. Since for each element γ of $\Gamma \subseteq S$, $V(S_\gamma) = 1$, we have $V(S_\Gamma) = 1$. Thus,

$$V(\bigvee\{S_\Gamma : \Gamma \subseteq H(I) \text{ is minimal such that } \Gamma \vdash \varphi\}) = 1.$$

Conversely, assume that for some minimal $\Gamma \subseteq H(I)$ such that $\Gamma \vdash \varphi$, $V(S_\Gamma) = 1$. As in part (a), it is easy to show now that each formula $\gamma \in \Gamma$ has an I-support in E. Thus, $\Gamma \subseteq E$. Consequently, $\varphi \in E$ and, by the definition of V, $V(\overline{\varphi}) = 1$. Summarizing, V satisfies each formula in P_I of the form (9).

Consider now a formula $s_C \Rightarrow \overline{\gamma}$ from P_I, where C is a 1-epistemic clause in I of the form

$$L\alpha_1 \wedge \ldots \wedge L\alpha_n \wedge \neg L\beta_1 \wedge \ldots \wedge \neg L\beta_m \Rightarrow \gamma.$$

Since E is an expansion of I, $C \in E$. Consequently, at least one of the following three possibilities holds:

(1) For some i, $1 \leq i \leq n$, $\alpha_i \notin E$. Then $V(\overline{\alpha_i}) = 0$ and $V(s_C) = 0$. Hence, $V(s_C \Rightarrow \overline{\gamma}) = 1$.

(2) For some i, $1 \le i \le m$, $\beta_i \in E$. Then, $V(\overline{\beta_i}) = 1$ and $V(s_C) = 0$. Hence, $V(s_C \Rightarrow \overline{\gamma}) = 1$.

(3) $\gamma \in E$. Then $V(\overline{\gamma}) = 1$ and $V(s_C \Rightarrow \overline{\gamma}) = 1$. $\Box$**4.1.(b)**

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