Chapter 2

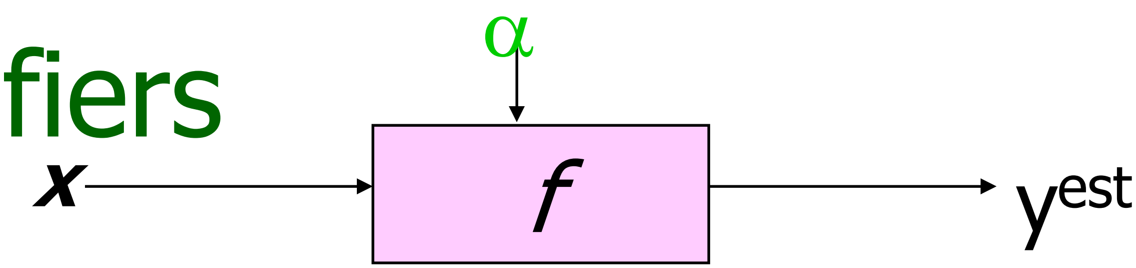
A Brief Introduction to Support Vector Machine (SVM)

January 25, 2011

Overview

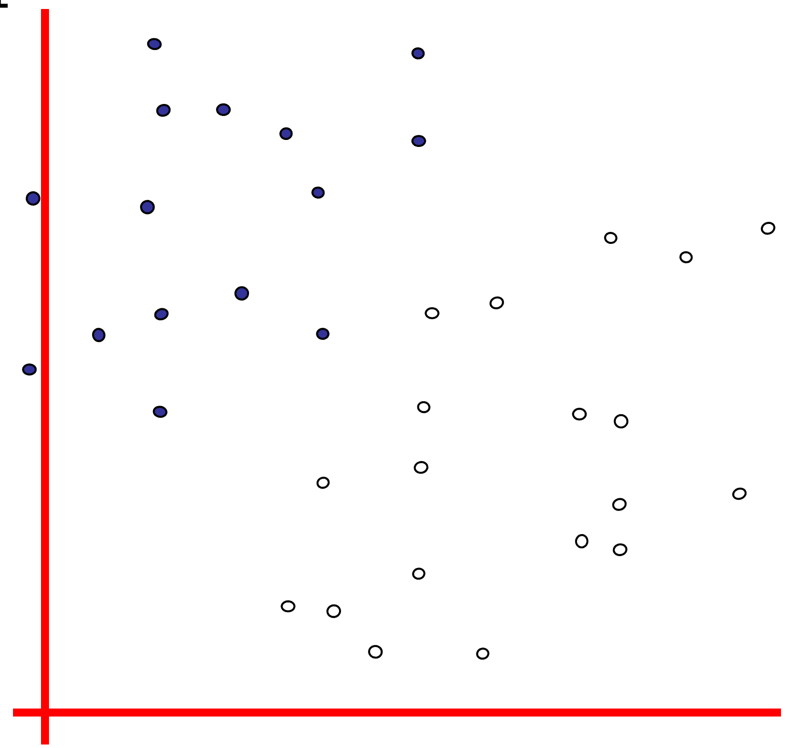
- A new, powerful method for 2-class classification
  - Original idea: Vapnik, 1965 for linear classifiers
  - SVM, Cortes and Vapnik, 1995
  - Became very hot since 2001
- Better generalization (less overfitting)
- Can do linearly unseparable classification with **global** optimal
- Key ideas
  - Use kernel function to transform low dimensional training samples to higher dim (for linear separability problem)
  - Use quadratic programming (QP) to find the best classifier boundary hyperplane (for global optima)

Linear Classifiers



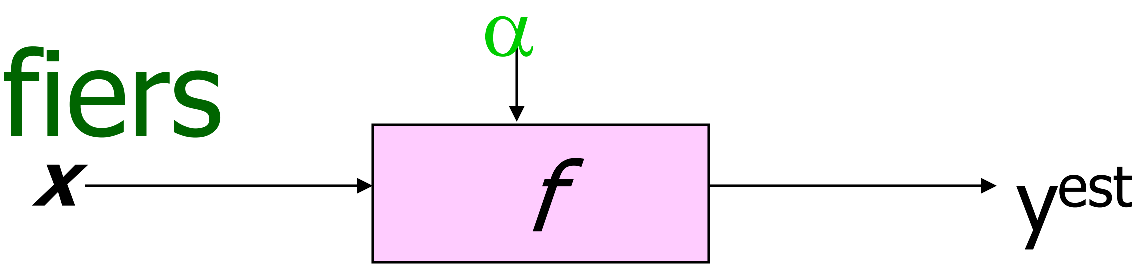
$f(\mathbf{x}, \mathbf{w}, b) = sign(\mathbf{w} \cdot \mathbf{x} - b)$

- • denotes +1
- ○ denotes -1



How would you classify this data?

Linear Classifiers



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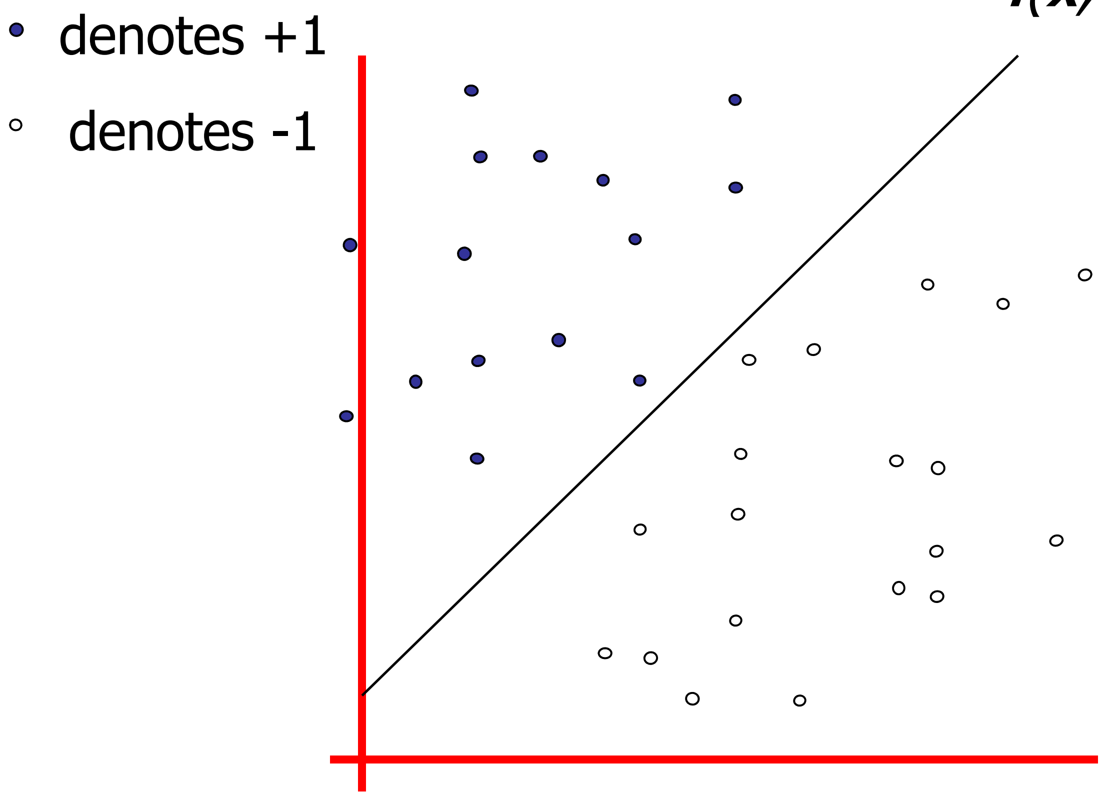

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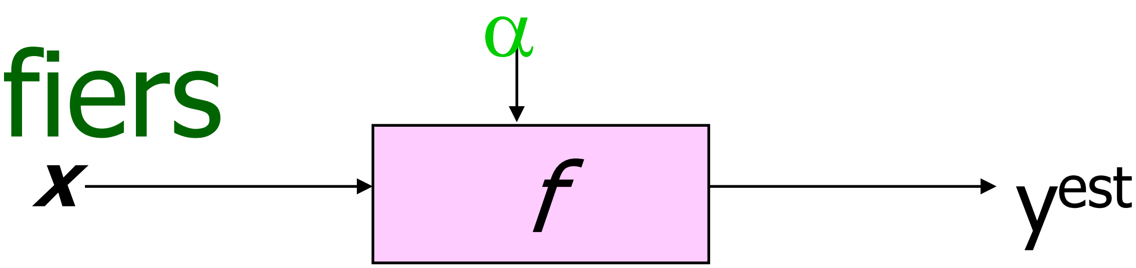

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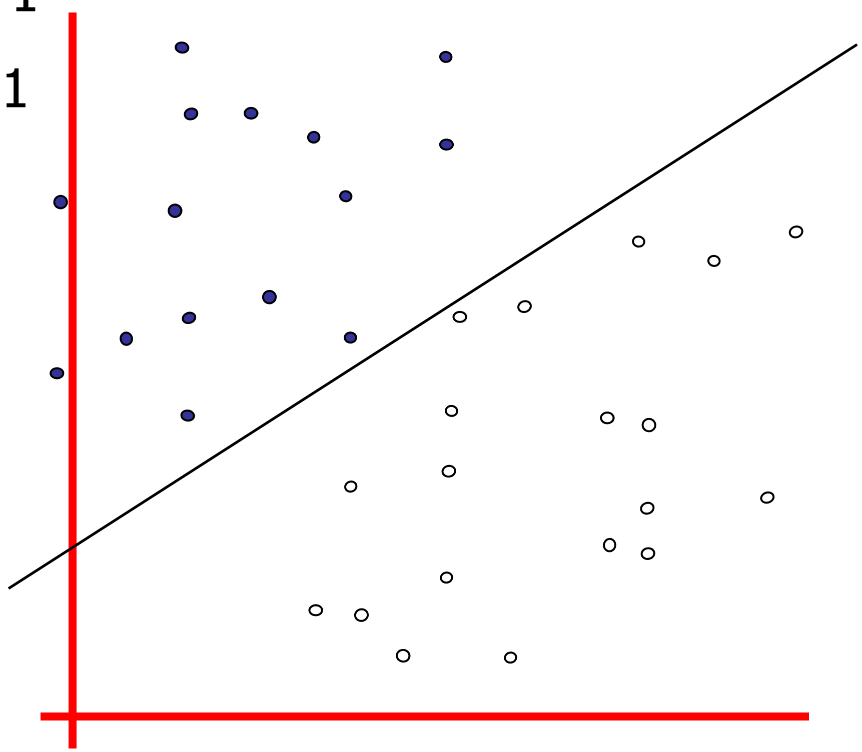

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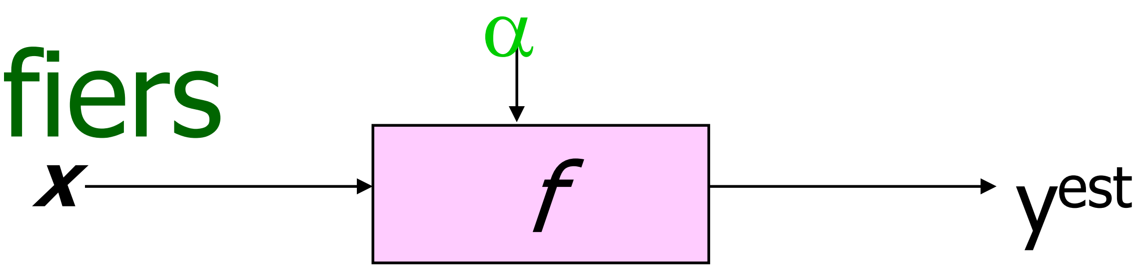

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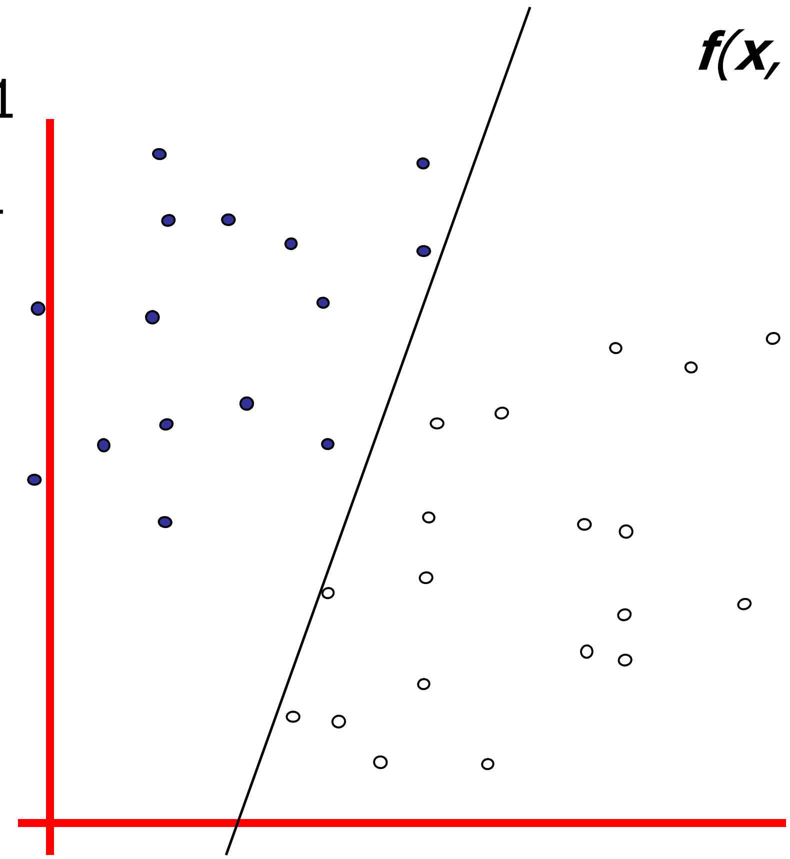

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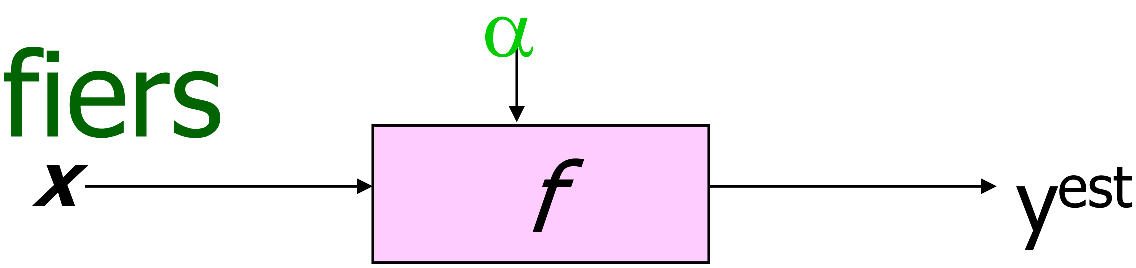

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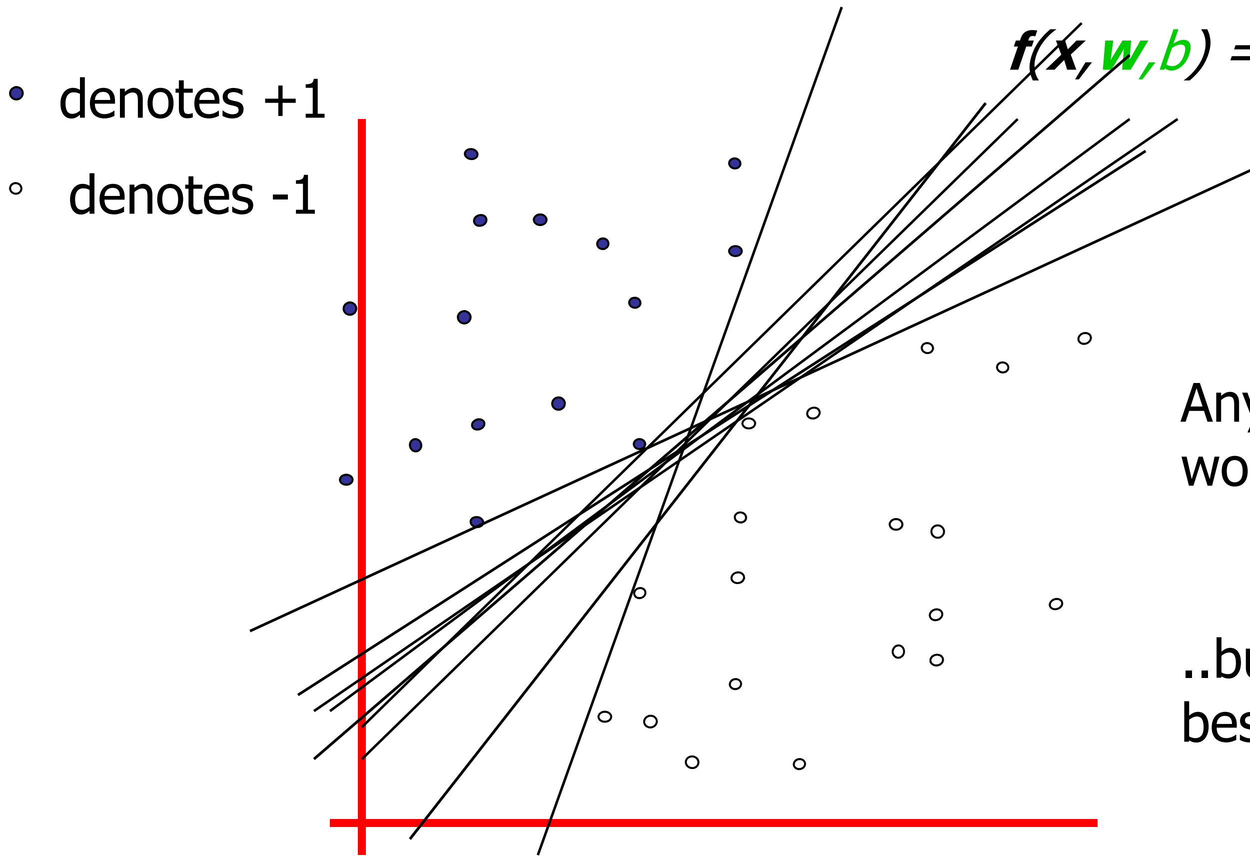
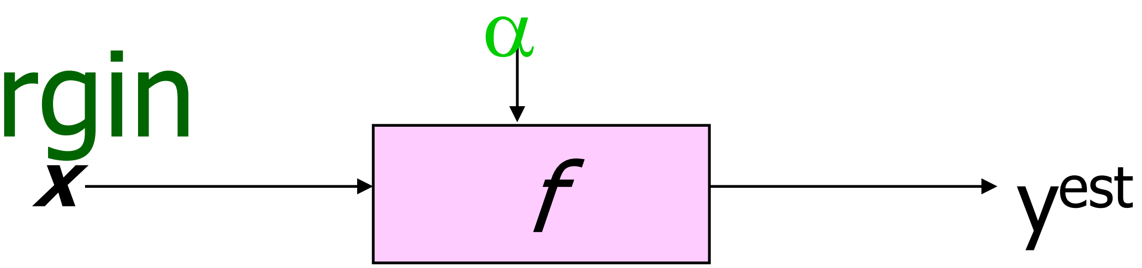


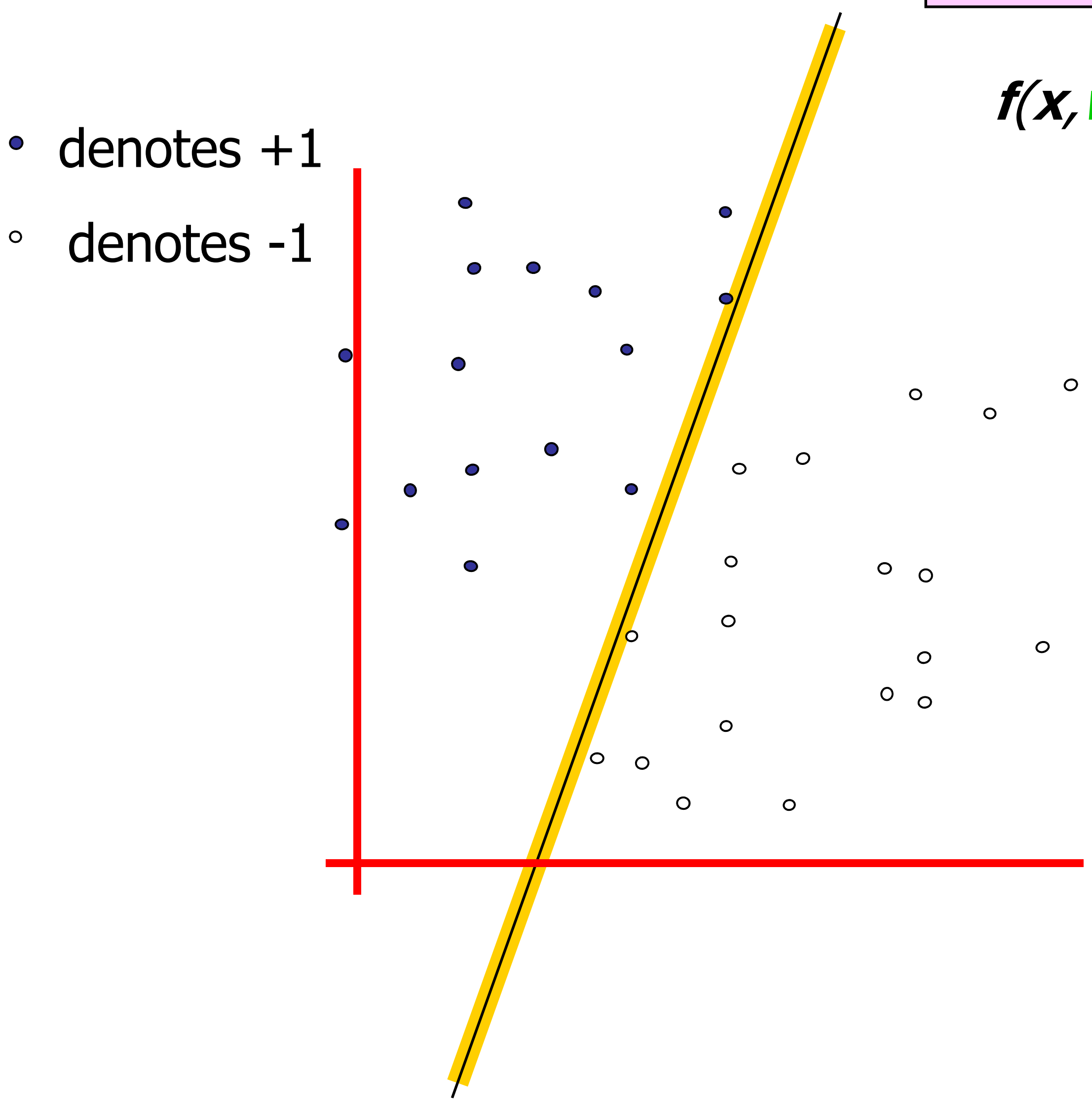
Any of these would be fine..

..but which is best?

Classifier Margin

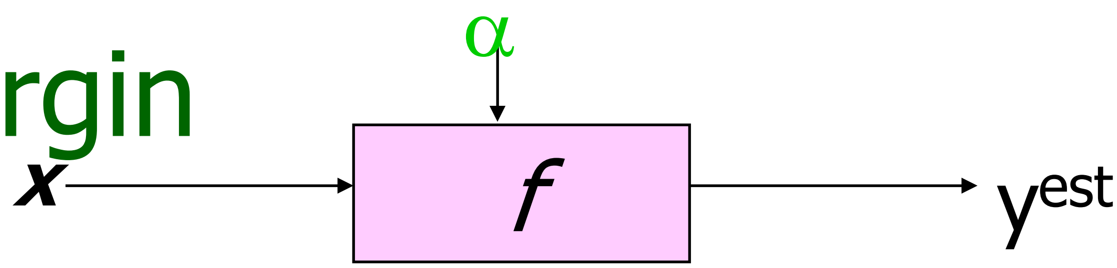


$f(x,w,b) = sign(w.\ x - b)$



Define the margin of a linear classifier as the width that the boundary could be increased by before hitting a datapoint.

Maximum Margin



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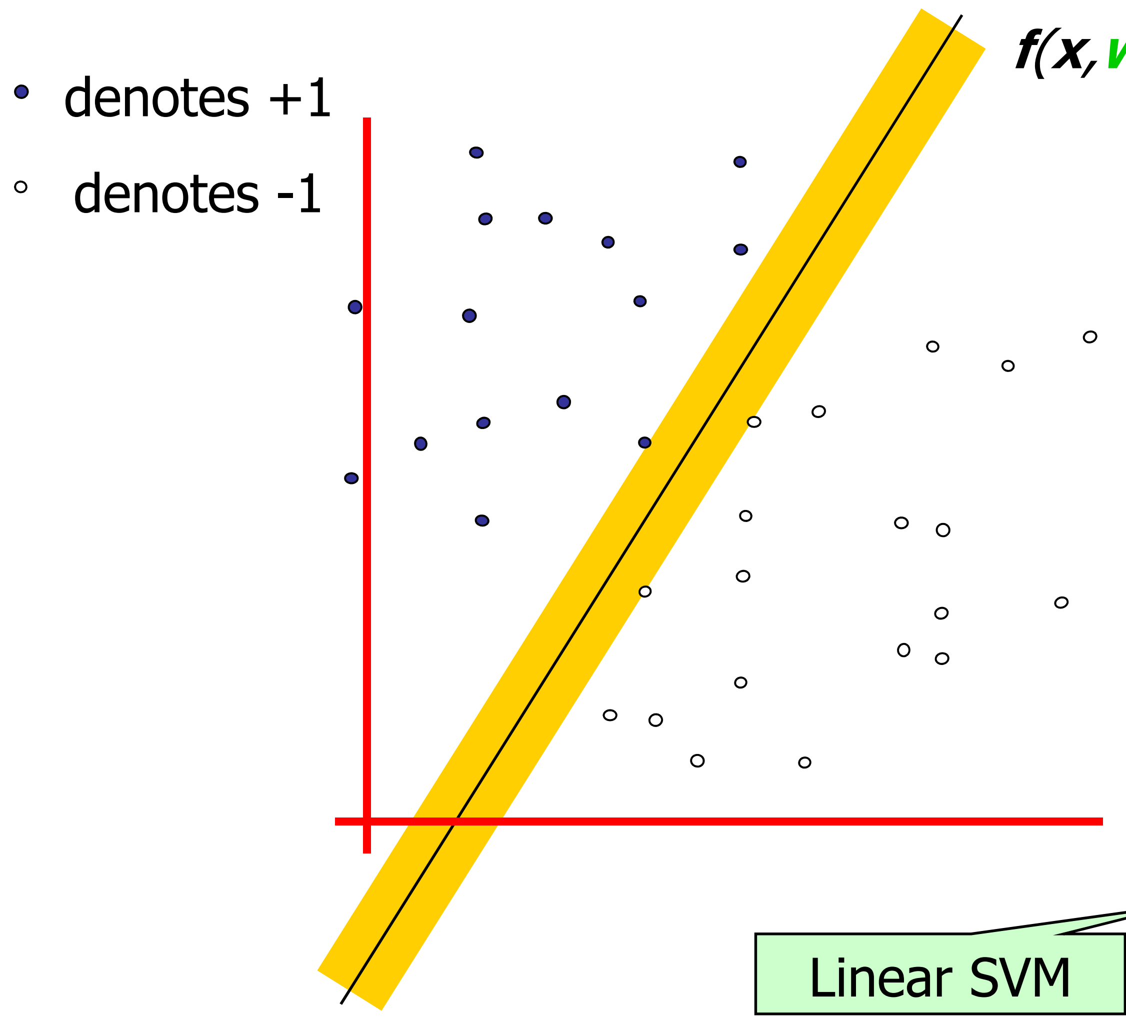
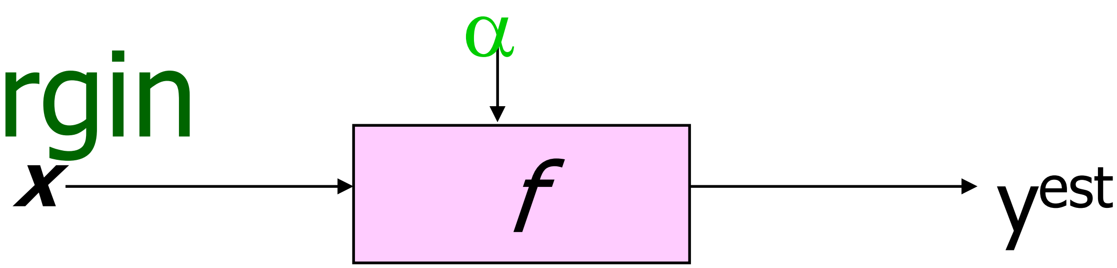


The maximum margin linear classifier is the linear classifier with maximum margin.

This is the simplest kind of SVM (Called an LSVM)

Maximum Margin



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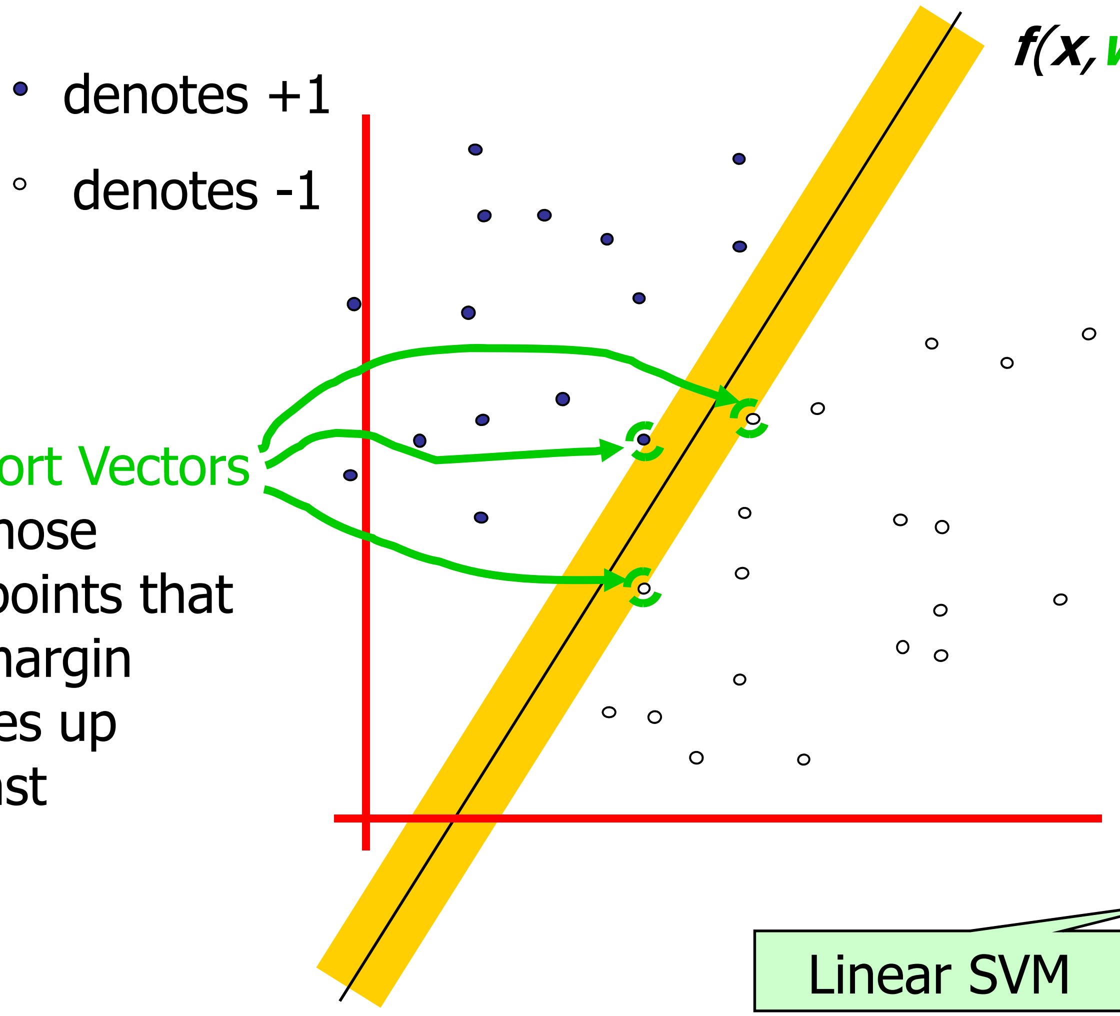

Support Vectors are those datapoints that the margin pushes up against

The maximum margin linear classifier is the linear classifier with maximum margin.

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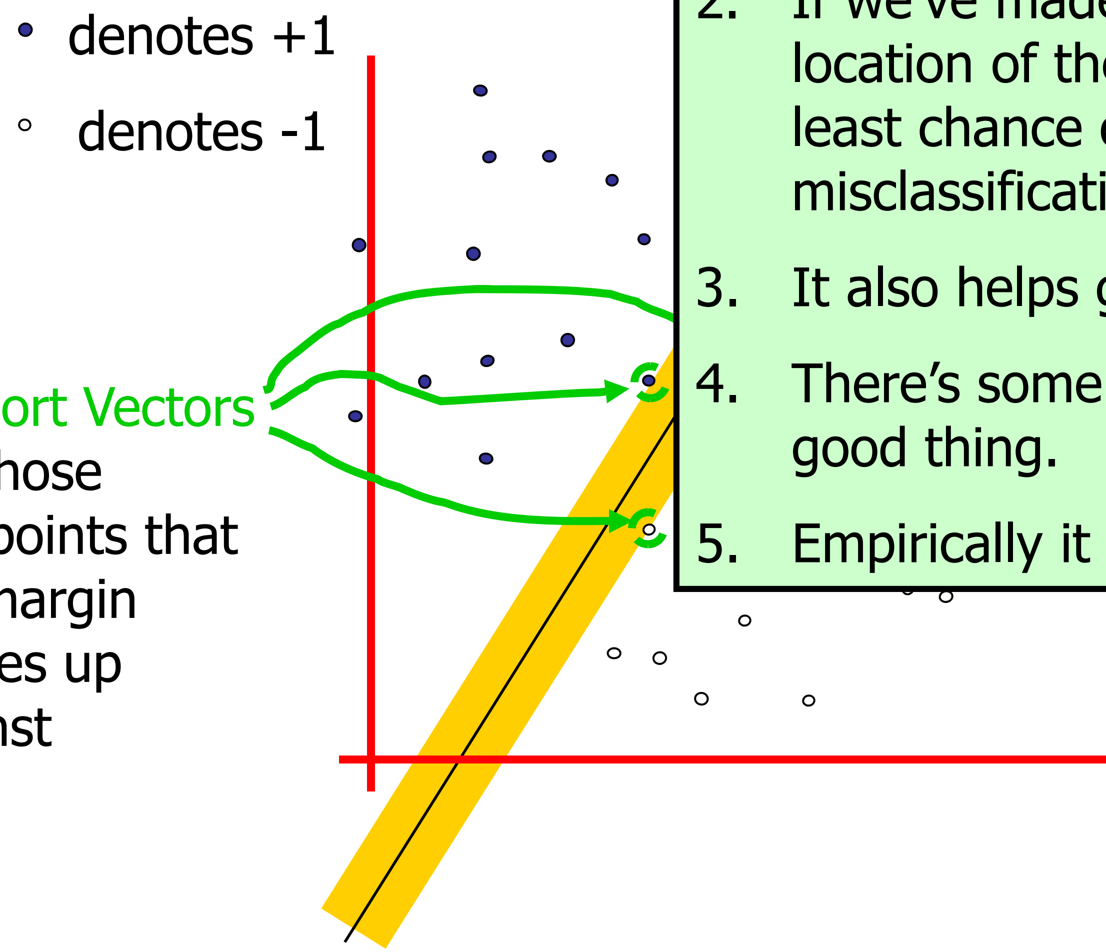
Why Maximum Margin?

1. Intuitively this feels safest.
2. If we've made a small error in the location of the boundary this gives us least chance of causing a misclassification.
3. It also helps generalization
4. There's some theory that this is a good thing.
5. Empirically it works very very well.



Support Vectors are those datapoints that the margin pushes up against

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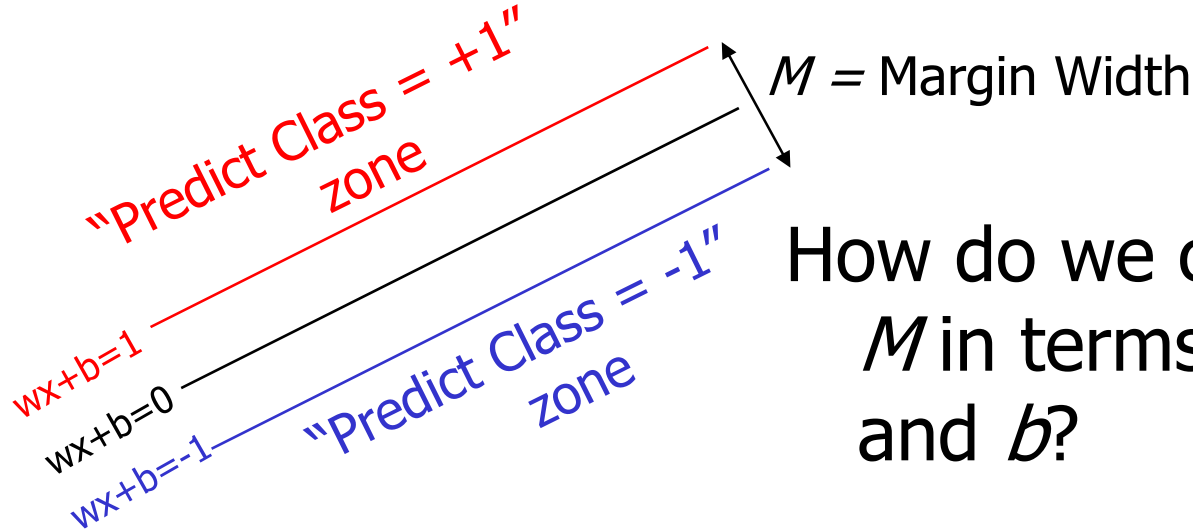
Computing the margin width



How do we compute *M* in terms of ***w*** and *b*?

- Plus-plane = $\{ \mathbf{x} : \mathbf{w} \cdot \mathbf{x} + b = +1 \}$
- Minus-plane = $\{ \mathbf{x} : \mathbf{w} \cdot \mathbf{x} + b = -1 \}$
- M= $\frac{2}{\sqrt{\mathbf{w}.\mathbf{w}}}$

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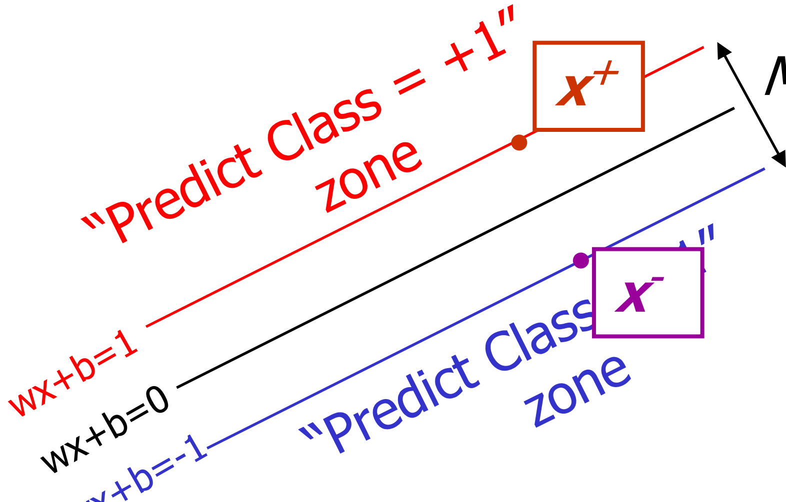
Learning the Maximum Margin Classifier



$$M = \text{Margin Width} = \frac{2}{\sqrt{\mathbf{w}.\mathbf{w}}}$$

Given a guess of ***w*** and *b* we can

- Compute whether all data points in the correct half-planes
- Compute the width of the margin

So now we just need to write a program to search the space of **w**'s and *b*'s to find the widest margin that matches all the datapoints. *How?*

Gradient descent? Simulated Annealing?

Learning via Quadratic Programming

- QP is a well-studied class of optimization algorithms to maximize a quadratic function of some real-valued variables subject to linear constraints.
- Minimize both $w \cdot w$ (to maximize M) and misclassification error

Quadratic Programming

Find $\underset{\mathbf{u}}{\arg\max} \quad c + \mathbf{d}^T\mathbf{u} + \dfrac{\mathbf{u}^T R\mathbf{u}}{2}$ ← Quadratic criterion

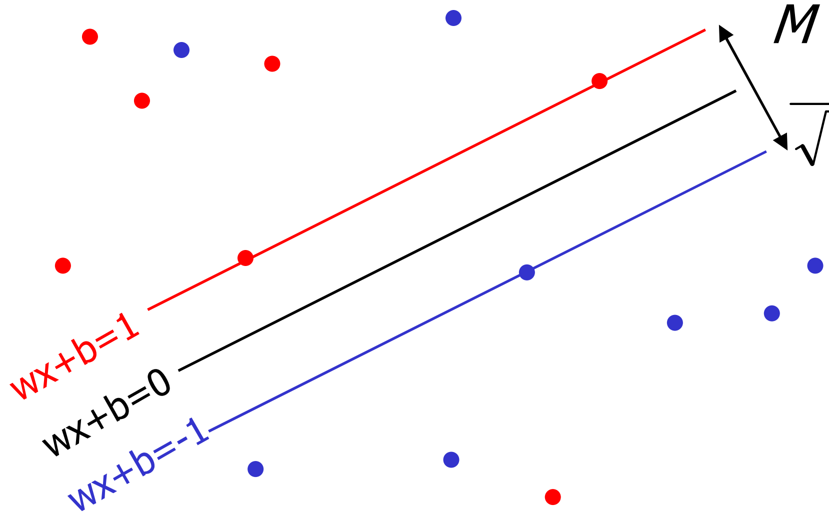
Subject to

$$\left.\begin{array}{c}
a_{11}u_1 + a_{12}u_2 + \ldots + a_{1m}u_m \le b_1 \\
a_{21}u_1 + a_{22}u_2 + \ldots + a_{2m}u_m \le b_2 \\
\vdots \\
a_{n1}u_1 + a_{n2}u_2 + \ldots + a_{nm}u_m \le b_n
\end{array}\right\} \text{ n additional linear inequality constraints}$$

And subject to

$$\left.\begin{array}{c}
a_{(n+1)1}u_1 + a_{(n+1)2}u_2 + \ldots + a_{(n+1)m}u_m = b_{(n+1)} \\
a_{(n+2)1}u_1 + a_{(n+2)2}u_2 + \ldots + a_{(n+2)m}u_m = b_{(n+2)} \\
\vdots \\
a_{(n+e)1}u_1 + a_{(n+e)2}u_2 + \ldots + a_{(n+e)m}u_m = b_{(n+e)}
\end{array}\right\} \text{ e additional linear equality constraints}$$

Learning Maximum Margin with Noise



$M = \frac{2}{\sqrt{\mathbf{w}.\mathbf{w}}}$

Given guess of $\boldsymbol{w}$, b we can

- Compute sum of distances of points to their correct zones
- Compute the margin width

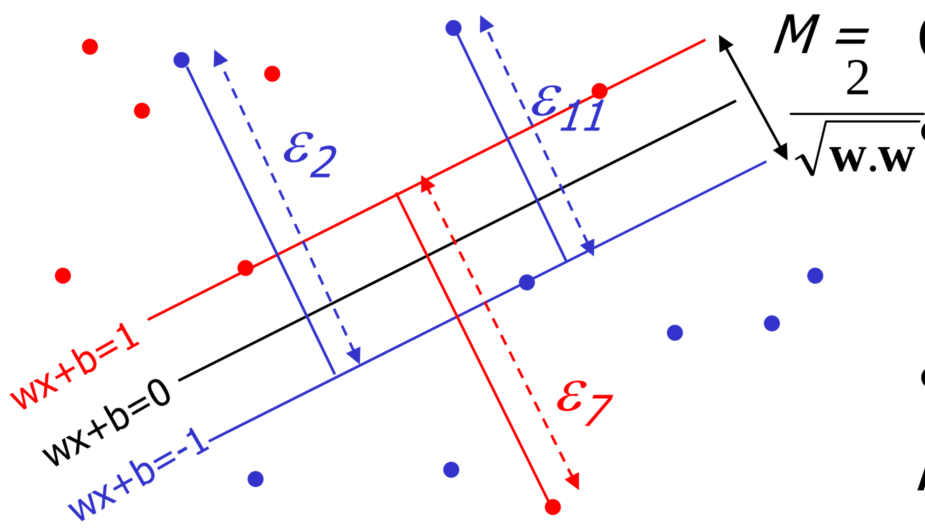
Assume R datapoints, each $(\boldsymbol{x}_k, y_k)$ where $y_k = +/- 1$

What should our quadratic optimization criterion be?

How many constraints will we have?

What should they be?

Learning Maximum Margin with Noise



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Assume R datapoints, each $(\mathbf{x}_k, y_k)$ where $y_k = +/- 1$

What should our quadratic optimization criterion be?

Minimize $$\frac{1}{2}\mathbf{w}.\mathbf{w} + C\sum_{k=1}^{R}\varepsilon_k$$

ε_k = distance of error points to their correct place

How many constraints will we have? $2R$

What should they be?

$\mathbf{w} \,.\, \mathbf{x}_k + b >= 1-\varepsilon_k$ if $y_k = 1$

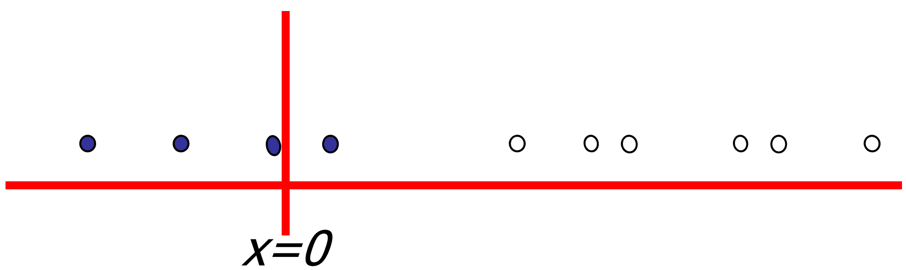
$\mathbf{w} \,.\, \mathbf{x}_k + b <= -1+\varepsilon_k$ if $y_k = -1$

$\varepsilon_k >= 0$ for all k

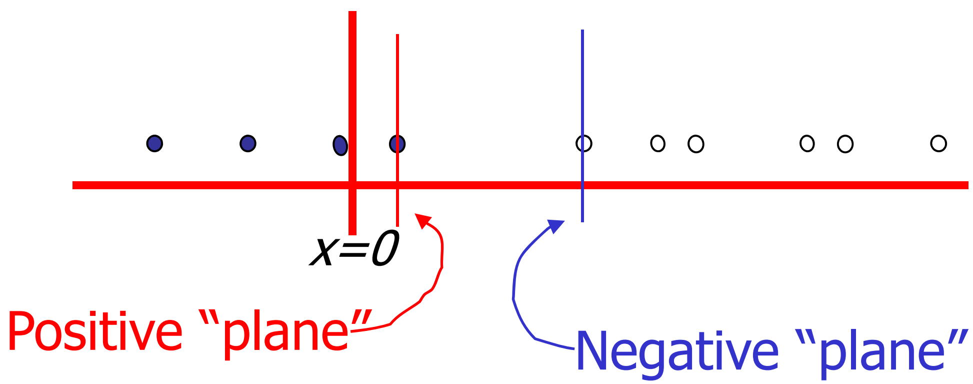
From LSVM to general SVM

Suppose we are in 1-dimension

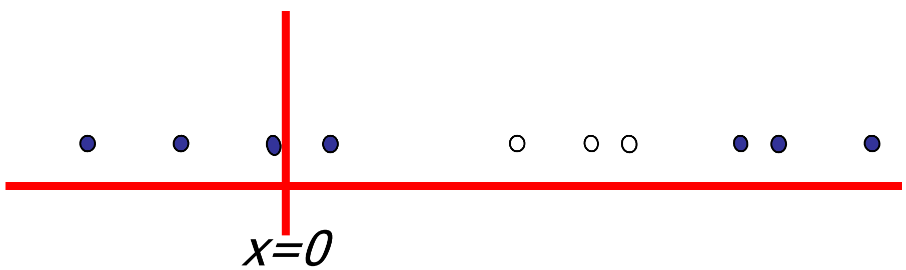
What would SVMs do with this data?



Not a big surprise



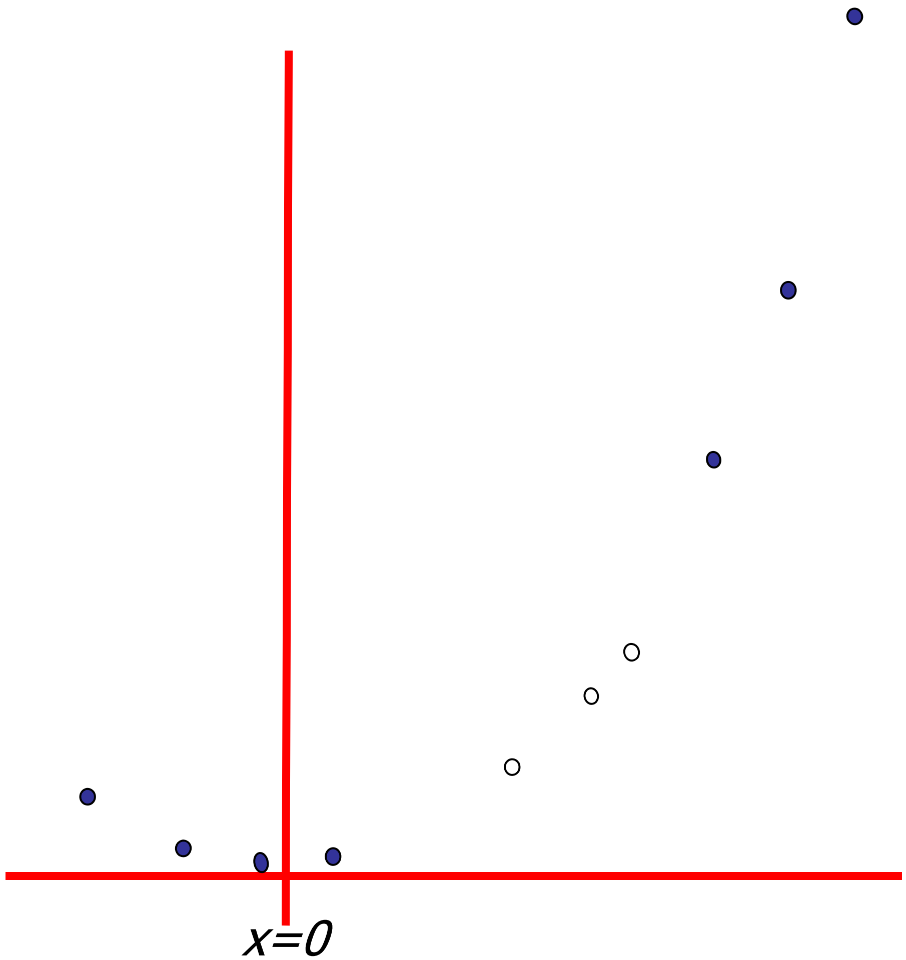
Harder 1-dimensional dataset



Points are not linearly separable.

What can we do now?

Harder 1-dimensional dataset

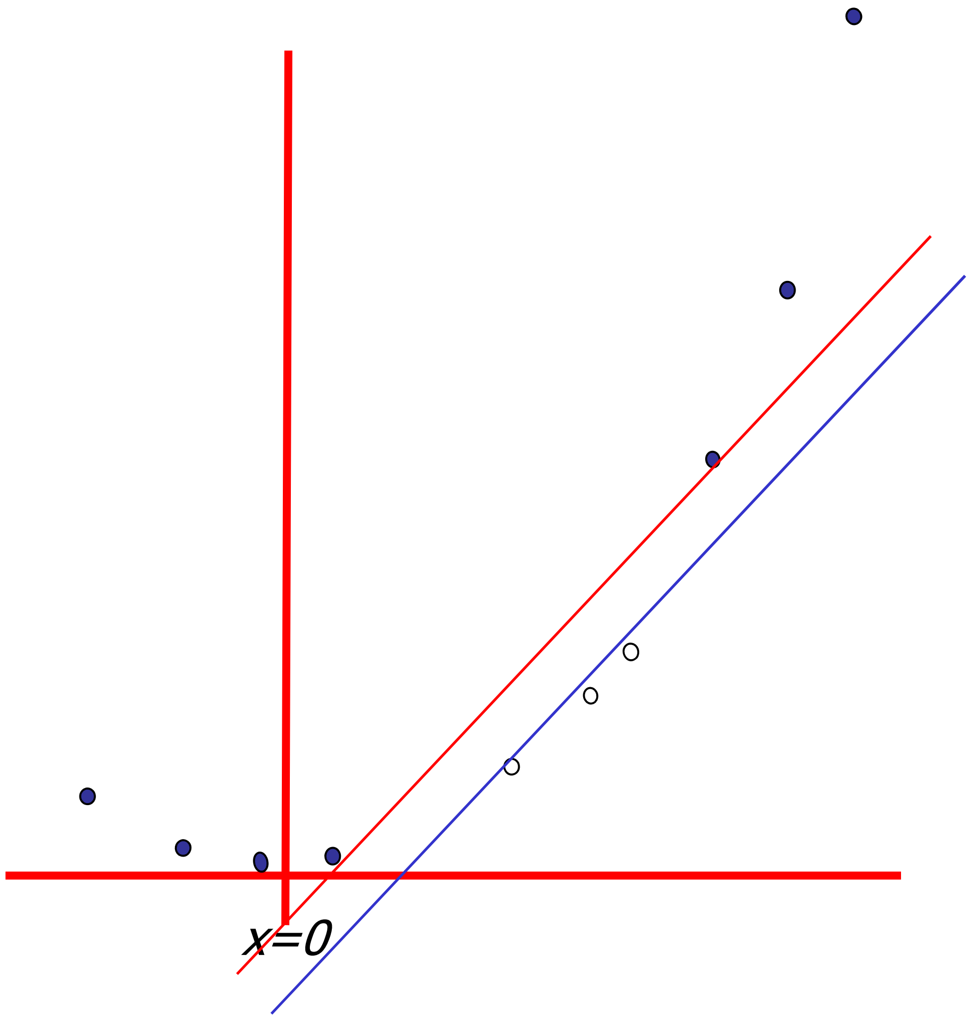


Transform the data points from 1-dim to 2-dim by some nonlinear basis function (called **Kernel functions**)

These points sometimes are called **feature vectors**.

$$\mathbf{z}_k = (x_k, x_k^2)$$

Harder 1-dimensional dataset



These points are linearly separable now!

Boundary can be found by QP

$$\mathbf{z}_k = (x_k, x_k^2)$$

Common SVM basis functions

$\boldsymbol{z}_k$ = (polynomial terms of $\boldsymbol{x}_k$ of degree 1 to q)

$\boldsymbol{z}_k$ = (radial basis functions of $\boldsymbol{x}_k$)

$$\mathbf{z}_k[j] = \varphi_j(\mathbf{x}_k) = \mathrm{KernelFn}\left(\frac{|\mathbf{x}_k - \mathbf{c}_j|}{\mathrm{KW}}\right)$$

$\boldsymbol{z}_k$ = (sigmoid functions of $\boldsymbol{x}_k$)

Doing multi-class classification

- SVMs can only handle two-class outputs (i.e., a categorical output variable with variety 2).
- What can be done?
- Answer: with N classes, learn N SVM's
  - SVM 1 learns "Output==1" vs "Output != 1"
  - SVM 2 learns "Output==2" vs "Output != 2"
  - :
  - SVM N learns "Output==N" vs "Output != N"
- Then to predict the output for a new input, just predict with each SVM and find out which one puts the prediction the furthest into the positive region.

Compare SVM with NN

- Similarity
  - SVM + sigmoid kernel ~ two-layer feedforward NN
  - SVM + Gaussian kernel ~ RBF network
  - For most problems, SVM and NN have similar performance
- Advantages
  - Based on sound mathematics theory
  - Learning result is more robust
  - Over-fitting is not common
  - Not trapped in local minima (because of QP)
  - Fewer parameters to consider (kernel, error cost C)
  - Works well with fewer training samples (number of support vectors do not matter much).
- Disadvantages
  - Problem need to be formulated as 2-class classification
  - Learning takes long time (QP optimization)

Advantages and Disadvantages of SVM

- Advantages
  - prediction accuracy is generally high
  - robust, works when training examples contain errors
  - fast evaluation of the learned target function
- Criticism
  - long training time
  - difficult to understand the learned function (weights)
  - not easy to incorporate domain knowledge

SVM Related Links

- Representative implementations
  - LIBSVM: an efficient implementation of SVM, multi-class classifications, nu-SVM, one-class SVM, including also various interfaces with java, python, etc.
  - SVM-light: simpler but performance is not better than LIBSVM, support only binary classification and only C language
  - SVM-torch: another recent implementation also written in C.