For each problem, explain your answer or show how it was derived.

1. For each of the following relations and each property { symmetric, anti-symmetric, reflexive, transitive}, state whether it has that property.
   (a) $R_1(x, y)$ holds iff x is y's sister, where x and y are drawn from all living women. **symmetric, not antisymmetric, not reflexive, transitive—if you mean full sister; not transitive if half-sisters are allowed**
   (b) $R_2(x, y)$ holds iff $x + y > 0$, where $x, y \in \mathbf{R}$ **symmetric, not antisymmetric, not reflexive (consider $x = y = -1$), not transitive (consider $x = z = -1$ and $y = 2$)**
   (c) $R_3(x, y)$ holds iff there is a road with no intersections connecting x and y, and x, y are intersections in Lexington. **not symmetric, not antisymmetric, not reflexive, not transitive**
2. If $f : A \to B$ is onto and $g : B \to C$ is one-one, what can we say about the relative values of $|A|$, $|B|$, and $|C|$? $|A| \geq |B|$ **and** $|B| \leq |C|$
3. Prove or disprove that $[(x \Rightarrow y) \Rightarrow z] \Rightarrow (x \Rightarrow z)$.
4. Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 3, 5\}$.
   (a) Give an element of $A \times B$ $(1, 5)$ **Note that there are many right answers, all of them ordered pairs.**
   (b) What is $|A \times B|$? $4 \cdot 3 = 12$
   (c) Give an element of $A \oplus B$ 5
   (d) What is $|A \oplus B|$? $|A \oplus B| = \{2, 4, 5\}$, so the answer is 3.
5. Suppose you are told that $|A| = 7$, $|B| = 12$, and $|A \cup B| = 15$. What is $|A \oplus B|$? (To insure partial credit if you are unsure of your answer, indicate how you derived the answer, showing a formula, a Venn diagram, or other.) $|A \cup B| = |A| + |B| - |A \cap B|$, so $|A \cap B| = 4$
6. Consider the following functions. For each function and each property from the set {one-one, onto, total}, list the properties that that function has.
   (a) $f : \mathbf{N} \to \mathbf{R}, f(x) = \log_2(x)$ **one-one, onto, not total (not defined at $x = 0$)**
   (b) $g : \mathbf{N} \times \mathbf{N} \to \mathbf{N}, g(\langle x, y\rangle) = 2^x 3^y$ **one-one, not onto (it does not map to 5, for instance), total**
   (c) $h : \mathbf{R}^+ \to (0, 1), h(x) = \frac{1}{x+1}$. **one-one, not onto (it does not map to $\frac{2}{3}$), total**

7. For each of the pairs f and g, state whether f is $\mathcal{O}(g)$, g is $\mathcal{O}(f)$, or f is $\Theta(g)$.

   (a) $f(n) = n$ and $g(n) = n \log n$
   $f \in \mathcal{O}(g)$

   (b) $f(n) = n^2$ and $g(n) = n \log n$
   $g \in \mathcal{O}(f)$

   (c) $f(n) = 2^n$ and $g(n) = n!$
   $f \in \mathcal{O}(g)$

8. Let
$$S(n) = \sum_{\{a_1,a_2,\ldots a_k\} \subseteq \{1,2,\ldots,n\}} \frac{1}{a_1 \cdot a_2 \cdots a_k}.$$
Here, the sum is over all nonempty subsets of $\{1, 2, \ldots n\}$.
Note: $S(3) = (\frac{1}{1} + \frac{1}{2} + \frac{1}{1\cdot 2}) + (\frac{1}{3}) + (\frac{1}{3\cdot 1} + \frac{1}{3\cdot 2} + \frac{1}{3\cdot 1\cdot 2})$.

   (a) What can you say about the third group of fractions, in terms of the first?
   If the groups of fractions are A, B and C, then $C = \frac{1}{3} \cdot A$.

   (b) How do you express $S(3)$ in terms of $S(2)$? $S(3) = S(2) + \frac{1}{3} \cdot S(2) + \frac{1}{3}$ because there are the subsets of $\{1, 2, 3\}$ that don't contain 3 (that gives us $S(2)$), and those with the addition of 3, and then the subset of just 3. This can be generalized, as in the proof below.

   (c) Prove $S(n) = n$ by induction.
   Base case: $n = 1$, where $S(1) = \frac{1}{1}$. Note that there is an implicit assumption that the subsets considered are nonempty.
   Inductive hypothesis: $S(k) = k$.
   Inductive step: We can derive that
$$\begin{aligned} S(k+1) &= S(k) + \frac{1}{k+1} + \frac{1}{k+1} \cdot S(k) \\ &= k + \frac{1}{k+1} + \frac{1}{k+1} \cdot k \\ &= k + \frac{1}{k+1} \cdot (1+k) \\ &= k + 1. \end{aligned}$$
   Thus, for all $n \in \mathbf{N}$, $S(n) = n$.