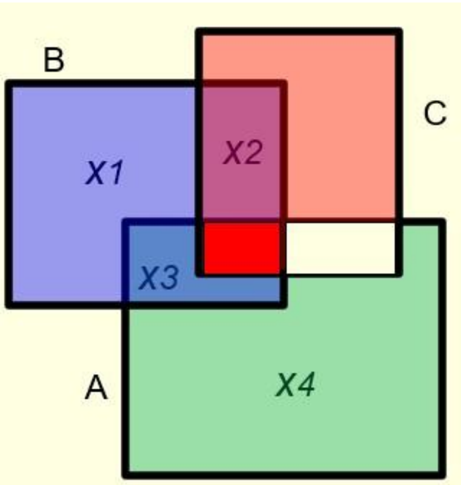
Solution to CS375 Homework Assignment 1 (40 points)

Due date: September 1, 2026

(The red boxes are text boxes. You can put your answers into the boxes directly)

1. In the following **Venn Diagram**, find an expression (in terms of set operations on the sets A, B and C; for instance, the green region can be expressed as A-(BUC)) for the region whose four components are marked with x1, x2, x3 and x4 and put the expression in the red box below. The expression is not unique. Try to make your expression as compact as possible. (4 points)



Sol:

Expression 1: Union of components X1 and X2 = B-A, union of components X3 and X4 = A - C. So, one answer is: $(B - A) \cup (A - C)$

Expression 2: We can subtract the intersection of A and C from the union of A and B to get the specified region. So a second answer is: $(A \cup B) - (A \cap C)$

2. In the above diagram, if |A| =53, |C| = 50, |A ∩ B| = 8, |A ∩ C| =10, |B ∩ C| = 11, |A U B U C| =145 and |A ∩ B ∩ C| = 6 then |B| = ? Show your derivation to get partial credit. (4 points)

Sol:

By the union rule and the difference rule, we can prove that

$$\begin{aligned}|A \cup B \cup C| &= |A \cup B| + |C| - |(A \cup B) \cap C| \\ &= |A| + |B| - |A \cap B| + |C| - |(A \cap C) \cup (B \cap C)| \\ &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |(A \cap C) \cap (B \cap C)| \\ &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|\end{aligned}$$

Hence, $|B|$ = 145- 53 -50 +8 + 10 + 11 - 6 = **65.**

3. R is a binary relation over N, the set of natural integers. (x, y) ϵ R iff $x = y \pmod 5$. R defined this way is an RST relation. List all equivalence classes of R in the following box. For each equivalence class, list all the elements in that class. (6 points)

Sol:

If you use the definition N={1, 2, 3, ….} then you have five equivalence classes [1], [2], [3], [4], [5] defined the following way:

[1] = {1, 6, 11, 16, …} = {5k-4 | k ϵ N }
[2] = {2, 7, 12, 17, …} = {5k-3 | k ϵ N }
[3] = {3, 8, 13, 18, …} = {5k-2 | k ϵ N }
[4] = {4, 9, 14, 19, …} = {5k-1 | k ϵ N }
[5] = {5, 10, 15, 20, …} = {5k | k ϵ N }

If your definition of N is N={0, 1, 2, 3, …. } then you have the following five Equivalence classes:

[0] = {0, 5, 10, 15, …} = {5k | k ϵ N }
[1] = {1, 6, 11, 16, …} = {5k+1 | k ϵ N }
[2] = {2, 7, 12, 17, …} = {5k+2 | k ϵ N }
[3] = {3, 8, 13, 18, …} = {5k+3 | k ϵ N }
[4] = {4, 9, 14, 19, …} = {5k+4 | k ϵ N }

4. Count the number of **strings** of length 5 over *A* = {*a, b, c, d*} that begins with *a*, ends with *b* and have exactly one *c*. (5 points).

Sol:

The second letter (from left to right) through the fourth letter of each string that satisfies the requirement fall into one of the following three groups:

- The second letter is *c*, and each of the two remaining letters is *a*, *b,* or *d*
- The third letter is *c*, and each of the two remaining letters is *a*, *b* or *d*
- The fourth letter is *c*, and each of the two remaining letters is *a, b* or *d*

The number of strings in each group is: $1 \times 1 \times 3^2 \times 1 = 9$
So the number of strings that satisfy the requirement is **9 x 3 = 27.**

5. Use the **pigeonhole principle** to determine how many people are needed in a group to say that at least ten were born on the same day of the week. (4 points)

Sol:

At least 64 people.
Number of pigeons = number of people
Number of holes = number of days (in a week)

63 people would not work because it could be that nine of them were born on each day of the week (so, 7x9, we get 63).
But if we have 64 people, and if nine or less people were born on each day, then we would get the conclusion that there are at most 63 people in that group, a contradiction.

6. For a set A with m elements and a set B with n elements,

(i) how many different functions *f : A → B* can be defined if m=4 and n=5? (3 points)

For each element of A, there are 5 choices for its image. There are 4 elements in A. So the number of different combinations is: $5*5*5*5=5^4$

(ii) if *m=3* and *n=5* then how many different *one-to-one* functions *f : A →B* can be defined? (3 points)

For the first element of A, there are five different choices for its image. Once an element of B has been chosen to be the image of the first element of A, then there are only four different choices left for the second element of A, and, similarly, three different choices for the third element of A. So, totally, the number of different one-to-one functions from A to B is:

$$5*4*3=\frac{5!}{(5-3)!}=60$$

(iii) if *m=4* and *n=3* then how many different *onto* functions *f : A → B* can be defined? (3 points)

In this case, the number of different **onto functions** that can be defined from A to B is: $3^4-F_1-F_2$
where F_1 is the number of onto functions from A to B whose image set has one element only and F_2 is the number of onto functions from A to B whose image set has two elements only.

$F_1=3$ and $F_2=(2^4-2)\binom{3}{2}$ (see slides 51 and 52 of the notes "Preliminaries" for the values of F_1 and F_2)

So, the answer is: $3^4-3-(2^4-2)\binom{3}{2}=36$

7. Let $f: N_7 \rightarrow N_7$ be defined by $f(x) = (5x + 1) \bmod 7$. Find f^{-1} if it exists. (4 points).

Sol:

We use the definition N={0, 1, 2, 3, …} here.
So $N_7 = \{0, 1, 2, 3, 4, 5, 6\}$

Since gcd(5, 7) = 1, **f** is bijective.
First, test values to find c such that **f**(c) = 0, e.g. **f**(4) = 0.
On the other hand, from gcd(5, 7) = 1, we can get that
1 = 5×(3)+ 7×(-2). Hence, we have k = 3.

Thus **f^{-1}(x) = (3x + 4) mod 7**

If the definition N= {1, 2, 3, 4, …} is used, then
$N_7 = \{1, 2, 3, 4, 5, 6, 7\}$

Again, since gcd(5, 7) = 1, **f** is bijective.
In this case, we find c such that f(c) = 7. c=4.
The rest is the same.

So, again, **f^{-1}(x) = (3x + 4) mod 7**

8. Find a recursive definition for $f : \mathrm{N} \rightarrow \mathrm{N}$ defined by $f(n) = 0+1+8+27+ \ldots +n^3$. (4 points)

Sol:

With the definition N = {0, 1, 2, …}, we have

$f(0) = 0$

$f(n+1) = f(n) + (n+1)^3$

With the definition N ={1,2, 3, …}, we have

$f(1) = 1$

$f(n+1) = f(n) + (n+1)^3$

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Note that either way $f(n) = \left(\frac{n(n+1)}{2}\right)^2$.

To prove this, simply show that

$$f(n+1) = \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 = \left(\frac{(n+1)(n+2)}{2}\right)^2.$$

But you don't have to prove it.