Homework 4: CS321-002, Fall 2014
Answer Sheet

1. (10 points) Let $h = 0.25$, and the composite Simpson’s rule is

$$
\int_1^2 x^{-2} dx \approx \frac{0.25}{3} \left[ 1 + 4(1.25)^{-2} + 2(1.5)^2 + 4(1.75)^2 + (2)^{-2} \right] \approx 0.5004
$$

The error is

$$
\frac{(b - a) h^4 f^{(4)}(\xi)}{180}
$$

Here $f(x) = x^{-2}$, $f'(x) = -2x^{-3}$, $f''(x) = 6x^{-4}$, $f'''(x) = -24x^{-5}$, and $f^{(4)}(x) = 120x^{-6}$. Hence we have $|f^{(4)}(x)| \leq 120$ and

$$
|\text{error}| \leq \frac{(2 - 1)}{180} (0.25)^4 (120) = 0.02604
$$

The exact integration is

$$
\int_1^2 x^{-2} dx = -x^{-1}\big|_1^2 = 0.5
$$

So the exact error is 0.0004 which is smaller than the bound 0.02604.

2. (10 points) We have

$$
|\text{error}| = \frac{(b - a) h^2 |f''(\xi)|}{12}
$$

Here $f(x) = \sin x^2$, $f'(x) = 2x \cos x^2$, $f''(x) = 2 \cos x^2 - 4x^2 \sin x^2$. So

$$
|f''(\xi)| \leq 2|\cos(\xi^2)| + 4\xi^4 |\sin(\xi^2)| \leq 2 + 4\xi^2 \leq 146
$$

Hence

$$
|\text{error}| \leq \frac{(6 - 0)}{12} \left( \frac{6}{100} \right) (146) \approx 0.2628
$$

3. (10 points) There are two ways to solve this problem. One is to use the method in the note by integrating the Lagrange cardinal functions. Another is to use undetermined coefficient method. Both should give the same answer. We use the second method here.

We use three basis polynomials of order 2 or less to do the integration exactly. They are $1, x,$ and $x^2$.

\[
\begin{align*}
2 &= \int_0^2 dx = A_0(1) + A_1(1) + A_2(1) \\
2 &= \int_0^2 x dx = A_0(0) + A_1(1) + A_2(2) \\
\frac{8}{3} &= \int_0^2 x^2 dx = A_0(0)^2 + A_1(1)^2 + A_2(2)^2
\end{align*}
\]
Solving the above 3 equations yields

\[ A_0 = A_2 = \frac{1}{3}, \quad A_1 = \frac{4}{3}, \]

which is the trapezoid rule.

4. (10 points)

\[
R(0,0) = \frac{1}{2} (b-a)[f(a) + f(b)] = \frac{1}{2}[1 + e^{-100}] \approx 0.5
\]

\[
R(1,0) = \frac{1}{2} R(0,0) + \frac{1}{2} f(h) = \frac{1}{2}(0.5) + \frac{1}{2}e^{-25} \approx 0.25
\]

\[
R(1,1) = \frac{4}{3} R(1,0) + \frac{1}{3} |R(1,0) - R(0,0)| \approx 0.1667
\]

5. (10 points) Since \( f(x) \) is a decreasing function, for each subinterval \( [x_i, x_{i+1}] \), \( m_i = f(x_i) \), and \( M_i = f(x_{i-1}) \). Let \( x_i = a + ih \) with \( h = \frac{b-a}{n} \). The lower sum is

\[
L = \sum_{i=1}^{n} f(x_i)h
\]

and the upper sum is

\[
U = \sum_{i=1}^{n} f(x_{i-1})h
\]

So we have

\[
U - L = f(x_0)h - f(x_n)h = h(f(a) - f(b)) = \frac{(b-a)}{n}[f(a) - f(b)]
\]

6. (10 points) From Problem 1, the exact value of this integral is 0.5. Note that the function is decreasing, so the lower sum is

\[
L = \frac{1}{4}[(1.25)^{-2} + (1.5)^{-2} + (1.75)^{-2} + (2)^{-2}] = 0.4152
\]

The composite trapezoid rule gives

\[
C = \frac{1}{8}[1 + 2(1.25)^{-2} + 2(1.5)^{-2} + 2(1.75)^{-2} + (2)^{-2}] = 0.5090
\]

So the composite trapezoid rule is more accurate.