1. (10 points) For this problem, we need to verify the definition of polynomial interpolation, i.e., we need to verify that

\[ p(x_i) = y_i, \quad q(x_i) = y_i, \quad i = 0, 1, 2, 3. \]

It turns that both \( p(x) \) and \( q(x) \) interpolate the table.

The Theorem says that there is a unique polynomial of degree less or equal to \( n \) interpolates a table with \( n + 1 \) data sets. For this problem, after simplifying \( p(x) \), we can see that \( p(x) = q(x) \), so they both should interpolate the table, since they are the same. The uniqueness theorem does hold here.

2. (10 points)

One can use step by step Newton’s method to construct the Newton’s interpolating polynomial, or use the divided difference table to compute the coefficients of the Newton’s interpolating polynomial.

The divided difference table is

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f[] )</th>
<th>( f[.] )</th>
<th>( f[,] )</th>
<th>( f[,] )</th>
<th>( f[,] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1</td>
<td>2/3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>-1/3</td>
<td>41/48</td>
<td>1/8</td>
<td>3/4</td>
<td>11/6</td>
</tr>
<tr>
<td>2.5</td>
<td>2/32</td>
<td>119/48</td>
<td>39/24</td>
<td>75/12</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>4/3</td>
<td>71/3</td>
<td>113/8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>25</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The interpolation polynomial is

\[
p_4(x) = -1 + \frac{2}{3}(x-1) + \frac{1}{8}(x-1)(x-2) + \frac{3}{4}(x-1)(x-2)(x-2.5) + \frac{11}{6}(x-1)(x-2)(x-2.5)(x-3)
\]
3. (10 points) The divided difference table is

<table>
<thead>
<tr>
<th>x</th>
<th>f[]</th>
<th>f[ , ]</th>
<th>f[ , , ]</th>
<th>f[ , , , ]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>−3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>−4</td>
<td>51/3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>82</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The interpolation polynomial is

\[ p_3(x) = 2 + (x) - 3(x)(x - 2) + 4(x)(x - 2)(x - 3) \]

4. (10 points) You need to make sure you understand the definition of polynomial interpolation.
Since \( g \) interpolates \( f \) at \( x_0, x_1, \ldots, x_{n-1} \), we have \( g(x_i) = f(x_i) \) for \( 0 \leq i \leq (n-1) \).
Similarly, \( h \) interpolating \( f \) at \( x_1, x_2, \ldots, x_n \) implies that \( h(x_i) = f(x_i) \) for \( 1 \leq i \leq n \).
Hence, \( g(x_i) = h(x_i) \) for \( 1 \leq i \leq (n-1) \).

Now let

\[ k(x) = g(x) + \frac{x_0 - x}{x_n - x_0} [g(x) - h(x)] \]

We want to show that \( k(x_i) = f(x_i) \) for \( 0 \leq i \leq n \).

First verify at \( x_0 \)

\[ k(x_0) = g(x_0) + \frac{x_0 - x_0}{x_n - x_0} [g(x_0) - h(x_0)] = g(x_0) = f(x_0). \]

For \( 1 \leq i \leq (n-1) \), we have

\[ k(x_i) = g(x_i) + \frac{x_0 - x_i}{x_n - x_0} [g(x_i) - h(x_i)] = f(x_i) + \frac{x_0 - x_i}{x_n - x_0} [f(x_i) - f(x_i)] = f(x_i) \]

For point \( x_n \), we have

\[ k(x_n) = g(x_n) + \frac{x_0 - x_n}{x_n - x_0} [g(x_n) - h(x_n)] = h(x_n) = f(x_n). \]

Since \( k(x_i) = f(x_i) \) for \( i = 0, 1, 2, \ldots, x_n \), we conclude that \( k(x) \) interpolates \( f(x) \) at \( x_0, x_1, x_2, \ldots, x_n \).

5. (10 points) Using Taylor series

\[ f(x + 3h) = f(x) + 3hf'(x) + \frac{1}{2!}(3h)^2f''(x) + \frac{1}{3!}(3h)^3f'''(x) + \cdots \] (1)

\[ f(x - h) = f(x) - hf'(x) + \frac{1}{2!}(-h)^2f''(x) + \frac{1}{3!}(-h)^3f'''(x) + \cdots \] (2)
What we need here is to keep the $f'(x)$ term, subtract the second equation from the first equation to get:

$$f(x + 3h) - f(x - h) = 4h f'(x) + 4h^2 f''(x) + \frac{14}{3} (h)^3 f'''(x) + \cdots.$$ 

Hence, the approximate formula is

$$f'(x) = \frac{1}{4h} [f(x + 3h) - f(x - h)] - hf''(x) - \frac{14}{12} h^2 f'''(x) - \cdots$$

and the leading truncation error is $hf''(x)$.

6. From

$$\phi(h) = L - c_1 h^{1/2} - c_2 h^{3/2} - c_3 h^{3/2} - \cdots \quad (3)$$

we have

$$\phi(h/2) = L - c_1 (h/2)^{1/2} - c_2 (h/2)^{3/2} - c_3 (h/2)^{3/2} - \cdots \quad (4)$$

$$= L - \frac{c_1}{\sqrt{2}} h^{1/2} - \frac{c_2}{2} h^{3/2} - \frac{c_3}{2\sqrt{2}} h^{3/2} + \cdots \quad (5)$$

Eq. (4) - $\sqrt{2} \times$ Eq. (3) to cancel the $h^{1/2}$ term, we have

$$\phi(h) - \sqrt{2} \phi(h/2) = (1 - \sqrt{2}) L - \frac{1}{2} c_2 h^{2/2} - \frac{2\sqrt{2} - 1}{2\sqrt{2}} c_3 h^{3/2} + \cdots$$

Finally,

$$\frac{\phi(h) - \sqrt{2} \phi(h/2)}{1 - \sqrt{2}} = L - \frac{1}{2(1 - \sqrt{2})} c_2 h^{2/2} - \frac{2\sqrt{2} - 1}{2(\sqrt{2} - 2)} c_3 h^{3/2} + \cdots$$