For each problem, explain your answer or show how it was derived.

1. Show that Partition is in NP. Guess a set $S' \subseteq S$, and check both sums.

2. Show that Hamiltonian Path is polynomial-time mapping reducible to Hamiltonian Cycle. Given a graph $G$, the reduction produces a graph $G'$ such that $G$ has a Hamiltonian path iff $G'$ has a Hamiltonian cycle. Let $G = \langle V, E \rangle$, and $G' = \langle V', E' \rangle$, where $V' = V \cup \{z\}$, where $z \notin V$, and $E' = E \cup \{\langle u, z \rangle : u \in V\}$. In other words, we add one vertex and attach it to all the original vertices. Any Hamiltonian cycle in $G'$ goes through $z$ once; if we excise (remove) $z$ and its edges, that leaves a Hamiltonian path in $G$. Any Hamiltonian path $v_1v_2, \ldots, v_n$ in $G$ can be extended to a Hamiltonian path by including edges $\langle z, v_1 \rangle$ and $\langle v_n, z \rangle$. Thus $G$ has a Hamiltonian path iff $G'$ has a Hamiltonian cycle.

3. (a) What do we know about the deterministic time and space complexity of co-NP? (Give a brief argument that your claims are true.) Like NP, co-NP is in deterministic polynomial space (PSPACE) and deterministic exponential time (EXP). The proofs of both of these use breadth-first search on the computation trees of the co-NP computations. Since the trees have depth polynomial in the size of the input, we can traverse the entire tree in time exponential in that polynomial $2^{O(p(n))}$ and in space $p(n)^2$.

(b) Is $\text{NP} = \text{EXP}$? We don't know.

4. For each of the following sets, determine whether it is

- Regular
• Context-free and not regular
• Not context-free
• In P, not regular or context-free
• In NP
• Decidable ( = computable)
• Turing recognizable (= Turing enumerable), not decidable
• Not Turing enumerable

For each language, prove your answer.

(a) The set of even natural numbers *Polynomial time; if coded in binary or decimal, regular.*

(b) \( \{a^n b^k : n \neq k \} \) Not regular; by Myhill-Nerode and the fact that for \( i \neq j \), we can distinguish \( a^i \) from \( a^j \) by \( b^j \). Context Free: \( S \rightarrow aSb|A|B \quad A \rightarrow aA|a \quad B \rightarrow bB|b. \)

(c) \( \{<e> : M_e(e) \text{ does not halt in } e^e \text{ steps} \} \) Decidable. Not in P, since \( e^e \) is not polynomial in \( |e| \), and in fact is not even exponential in \( |e| \). *Since NP} \subseteq \text{EXP, and this is super-exponential, not even in NP.}*

(d) The set of graphs that have paths of length at least 3. *In P. Use breadth-first from each node to look for long paths. There may be faster methods, but that’s } \mathcal{O}(n^2). \)

(e) Set Cover: Given a universe \( U = \{1, \ldots, n\} \) and a set \( S \subseteq \mathcal{P}(U) \) of subsets of \( U \), and \( k \in \mathbb{N} \), is there a subset \( S' \subseteq S \) of size \( k \) whose union is \( U \)? It’s in NP (guess a cover and check its union). One can show it’s NP-complete, which indicates that it might not be in P, but that’s too much to prove during the exam.

(f) \( \{0^n : n \text{ prime} \} \) In P, not CF. Suppose it were, and \( k \) was the pumping constant. Then consider some prime \( p > k \), and string \( 0^p = uvxyz, \)
where $|vxy| \leq k$ and $0 < |vy| = j < p$. Let $m = p - j$. If we pump the string $m$ times, we get $|uxz| = p - j$ and $(|vy|^m) = (p - j) \cdot j$ (note that order of alphabet symbols doesn’t matter, since they’re all 0s). Thus the pumped string has length $(p - j)(j + 1)$, which is not prime, contradicting the pumping lemma assumption that any pumped version of the string should be in the language.

(g) $\{<e>: M_e(x) \text{ halts for infinitely many } x\}$

By Rice’s Theorem, this is not decidable. One can show by reduction from $\overline{A_{TM}}$ that it is not Turing recognizable.