15.7 Bicubic Bezier Surface Patches

\[ S(u,v) = \sum_{i=0}^{3} \sum_{j=0}^{3} B_{i,3}(u) B_{j,3}(v) P_{i,j} \]

where

\[ B_{k,3}(t) = \binom{3}{k} t^k (1-t)^{3-k}, \quad 0 \leq u, v \leq 1 \]
15.7 Bicubic Bezier Surface Patches

Matrix form:

\[
S(u,v) = \begin{bmatrix} 1, u, u^2, u^3 \end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
-3 & 3 & 0 & 0 \\
3 & -6 & 3 & 0 \\
-1 & 3 & -3 & 1
\end{bmatrix}
\begin{bmatrix}
P_{0,0} & P_{0,1} & P_{0,2} & P_{0,3} \\
P_{1,0} & P_{1,1} & P_{1,2} & P_{1,3} \\
P_{2,0} & P_{2,1} & P_{2,2} & P_{2,3} \\
P_{3,0} & P_{3,1} & P_{3,2} & P_{3,3}
\end{bmatrix}
\begin{bmatrix}
1 & -3 & 3 & -1 \\
0 & 3 & -6 & 3 \\
0 & 0 & 3 & -3 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 \\
v \\
v^2 \\
v^3
\end{bmatrix}
= U \cdot M_b \cdot G \cdot M_b^t \cdot V^t
\]
15.7 Bicubic Bezier Surface Patches

- Satisfies **convex hull property**
- **Subdivision** process
  - Subdivide in \( u \) and then subdivide in \( v \)
- **Rendering** techniques
  - Wire frame: generate *iso-parametric curves in both directions*
  - Shaded images:
    - Ray tracing
    - Scan convert approximating polygons:
      - approximate the surface patch by fine polygons
      - (triangles or quadrilaterals), then shade the polygons
- Patches can be joined to form complicated shapes
15.8 Subdivision Techniques for Piecewise Surfaces

*Midpoint* subdivision

*(see previous section)*
Definition: Let \( \{t_i\} \) be an infinite sequence of points (called knots) on the real axis. The B-spline basis function \( N_{i,n}(t) \) of degree \( n \) with support \( [t_i, t_{i+m+1}] \) is defined by the following recursive procedure:

\[
N_{i,0} = \begin{cases} 
1, & t_i \leq t < t_{i+1} \\
0, & \text{otherwise}
\end{cases}
\]

and for \( m \geq 1 \)

\[
N_{i,m}(t) = \frac{t-t_i}{t_{i+m}-t_i} N_{i,m-1}(t) + \frac{t_{i+m+1}-t}{t_{i+m+1}-t_{i+1}} N_{i+1,m-1}(t)
\]
15.9 Non-Uniform B-Spline Curves

Intuitively, B-splines of degree \( n \) (order \( n + 1 \)) are piecewise polynomial curves that are zero at all subintervals but \( n + 1 \) of them and have continuous \((n - 1)\)st derivative. The following are examples of B-splines of degree 0, 1, 2, and 3.
15.9 Non-Uniform B-Spline Curves
15.9 Non-Uniform B-Spline Curves

Explicit forms of low degree B-splines:

1. Linear B-splines:

\[
N_{i,1}(t) = \begin{cases} 
\frac{t-t_i}{t_{i+1}-t_i}, & t_i \leq t < t_{i+1} \\
\frac{t_{i+2}-t}{t_{i+2}-t_{i+1}}, & t_{i+1} \leq t < t_{i+2} \\
0, & \text{elsewhere}
\end{cases}
\]
15.9 Non-Uniform B-Spline Curves

2. Quadratic B-splines:

\[
N_{i,2}(t) = \begin{cases} 
\frac{(t-t_i)^2}{(t_{i+1}-t_i)(t_{i+2}-t_i)}, & t_i \leq t < t_{i+1} \\
\frac{(t-t_i)(t_{i+2}-t)}{(t_{i+1}-t_i)(t_{i+2}-t_i+1)} + \frac{(t_{i+3}-t)(t-t_{i+1})}{(t_{i+3}-t_i+1)(t_{i+2}-t_i+1)}, & t_{i+1} \leq t < t_{i+2} \\
\frac{(t_{i+3}-t)^2}{(t_{i+3}-t_i+1)(t_{i+3}-t_i+2)}, & t_{i+2} \leq t < t_{i+3} \\
0, & \text{elsewhere}
\end{cases}
\]
15.9 Non-Uniform B-Spline Curves

3. Cubic B-splines:

\[ N_{i,3}(t) = \frac{(t-t_i)^3}{(t_{i+1}-t_i)(t_{i+2}-t_i)(t_{i+3}-t_i)}, \quad t_i \leq t < t_{i+1} \]

\[ = \frac{(t-t_i)^2(t_{i+2}-t)}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})(t_{i+3}-t_i)} + \frac{(t-t_i)(t_{i+3}-t)(t-t_{i+1})}{(t_{i+3}-t_i)(t_{i+3}-t_{i+1})(t_{i+3}-t_i)} \]

\[ + \frac{(t_{i+4}-t)(t-t_{i+1})}{(t_{i+4}-t_i)(t_{i+3}-t_{i+1})(t_{i+2}-t_{i+1})}, \quad t_{i+1} \leq t < t_{i+2} \]

\[ = \frac{(t-t_i)(t_{i+3}-t)^2}{(t_{i+3}-t_i)(t_{i+3}-t_{i+1})(t_{i+3}-t_{i+2})} + \frac{(t_{4}-t)(t_{i+3}-t)(t-t_{i+1})}{(t_{i+4}-t_i)(t_{i+3}-t_{i+1})(t_{i+3}-t_{i+2})} \]

\[ + \frac{(t_{i+4}-t)^2(t-t_{i+2})}{(t_{i+4}-t_i)(t_{i+3}-t_{i+1})(t_{i+2}-t_{i+2})}, \quad t_{i+2} \leq t < t_{i+3} \]

\[ = \frac{(t_{i+4}-t)^3}{(t_{i+4}-t_i+1)(t_{i+4}-t_{i+2})(t_{i+4}-t_{i+3})}, \quad t_{i+3} \leq t < t_{i+4} \]

\[ = 0. \quad \text{elsewhere} \]
15.9 Non-Uniform B-Spline Curves

What happens if the knots are uniformly distributed, for instance, \( t_i = i \) for all \( i \)?

In this case, we have

1. Uniform Linear B-splines:

\[
N_{i,1}(t) = \begin{cases} 
  t-i, & i \leq t < i+1 \\
  i+2-t, & i+1 \leq t < i+2 \\
  0, & \text{elsewhere}
\end{cases}
\]
15.9 Non-Uniform B-Spline Curves

2. Uniform Quadratic B-splines:

\[ N_{i,2}(t) = \begin{cases} 
\frac{(t-i)^2}{2} & i \leq t < i+1 \\
\frac{(t-i)(i+2-t)}{2} + \frac{(i+3-t)(t-i-1)}{2} & i+1 \leq t < i+2 \\
\frac{(i+3-t)^2}{2} & i+2 \leq t < i+3 \\
0, & \text{elsewhere}
\end{cases} \]
15.9 Non-Uniform B-Spline Curves

3. Uniform cubic B-splines:

\[ N_{i,3}(t) = \begin{cases} 
\frac{(t-i)^3}{6}, & i \leq t < i+1 \\
\frac{(t-i)^2(i+2-t)}{6} + \frac{(t-i)(i+3-t)(t-i-1)}{6}, & i+1 \leq t < i+2 \\
\frac{(i+4-t)(t-i-1)^2}{6}, & i+1 \leq t < i+2 \\
\frac{(t-i)(i+3-t)^2}{6} + \frac{(i+4-t)(i+3-t)(t-i-1)}{6}, & i+2 \leq t < i+3 \\
\frac{(i+4-t)^2(t-i-2)}{6}, & i+2 \leq t < i+3 \\
\frac{(i+4-t)^3}{6}, & i+3 \leq t < i+4 \\
0, & \text{elsewhere} 
\end{cases} \]
What are the relationship between the uniform cubic B-splines defined here and the cubic B-spline blending functions defined in Section 15.3?
15.9 Non-Uniform B-Spline Curves

**Definition:** A B-spline curve of degree $k$ is defined as follows

$$C(t) = \sum_{i=0}^{n} N_{i,k}(t) P_i$$

where $N_{i,k}(t)$ are B-spline basis functions of degree $k$ defined by the knot vector $\{t_i \mid 0 \leq i \leq n+k+1\}$ and $P_i$, $0 \leq i \leq n$, are 2D or 3D control points. The parameter space of this curve is the interval between $t_k$ and $t_{n+1}$. 
15.9 Non-Uniform B-Spline Curves

Each interval $[t_i, t_{i+1}]$, of the parameter space $[t_k, t_{n+1}]$ is called a span. The portion of the curve defined by a span is called a segment. So, $C(t)$ is a curve with $n-k+1$ segments defined by $n+1$ control points.
15.9 Non-Uniform B-Spline Curves

Example of a cubic B-spline curve:

If knot $t_i = i$ for all $i$, then we get a uniform cubic B-spline curve. In that case, would the curve defined here be the same as the one given in Section 3.1.3?
Questions:

1. Would a non-uniform cubic B-spline curve satisfy convex hull property?

2. What would happen if \( t_0 = t_1 = t_2 = t_3 \) and \( t_{n+1} = t_{n+2} = t_{n+3} = t_{n+4} \)？

3. What is the relationship between a composite cubic Bezier curve and a cubic B-spline curve?
Theorem: Let \( \{ t_i \} \) be an infinite sequence of knots on the real axis and \( N_{i,n}(t) \) be the corresponding B-spline basis functions of degree \( n \). Then the summation of \( N_{i,n}(t) \) for any \( t \) of the real axis is always equal to 1, i.e.,

\[
\sum_i N_{i,n}(t) = 1, \quad t \in R
\]

It is okay that the bounds of the index are not given explicitly because the sum has only \( n+1 \) non-zero terms for each value of \( t \).
15.9 Non-Uniform B-Spline Curves

**Proof.** If \( t \in [t_k, t_{k+1}) \), then

\[
\sum N_{i,n}(t) = \sum_{i=k-n}^{k} N_{i,n}(t).
\]

The definition of the B-spline basis functions shows that

\[
N_{i-1,n}(t) = \frac{t-t_{i-1}}{t_{i+n-1}-t_{i-1}} N_{i-1,n-1}(t) + \frac{t_{i+n}-t}{t_{i+n}-t_i} N_{i,n-1}(t)
\]

and

\[
N_{i-1,n}(t) + N_{i,n}(t) = \frac{t-t_{i-1}}{t_{i+n-1}-t_{i-1}} N_{i-1,n-1}(t)
\]

\[
+ N_{i,n-1}(t) + \frac{t_{i+n+1}-t}{t_{i+n+1}-t_{i+1}} N_{i+1,n-1}(t)
\]
Hence,

$$\sum_{i} N_{i,n}(t) = \frac{t-t_{k-n}}{t_{k}-t_{k-n}} N_{k-n,n-1}(t) + \sum_{i=k-n+1}^{k} N_{i,n-1}(t)$$

$$+ \frac{t_{k+n+1}-t}{t_{k+n+1}-t_{k+1}} N_{k+1,n-1}(t)$$

$$= \sum_{i=k-n+1}^{k} N_{i,n-1}(t)$$

since $N_{k-n,n-1}$ and $N_{k+1,n-1}$ are both zero on $[t_{k},t_{k+1})$. Iteratively repeat this process, we get
15.9 Non-Uniform B-Spline Curves

\[
\sum_{i=k-n}^{k} N_{i,n}(t) = \sum_{i=k-n+1}^{k} N_{i,n-1}(t) \\
= \sum_{i=k-n+2}^{k} N_{i,n-2}(t) \\
\ldots\ldots\ldots
\]

\[
= \sum_{i=k}^{k} N_{i,0}(t) \\
= N_{k,0}(t) \\
= 1.
\]
15.9 Non-Uniform B-Spline Curves

Answer to Question 1:
The above theorem shows that a cubic B-spline curve satisfies a stronger convex hull property: each segment of a (non-uniform) cubic B-spline curve is contained in the convex hull of the four control points that determine the segment.

Answer to Question 2:
The resulting cubic B-spline curve interpolates the first and last control points. Why?
15.9 Non-Uniform B-Spline Curves

When \( t_0 = t_1 = t_2 = t_3 \), the first three cubic B-spline basis functions are of the following forms:

\[
N_{0,3}(t) = \begin{cases} 
(t_4-t)^3, & t_3 \leq t < t_4 \\
(t_4-t_3)^3, & t_0 = t_1 = t_2 = t_3 \\
0, & \text{elsewhere} 
\end{cases}
\]
15.9 Non-Uniform B-Spline Curves

\[ N_{1,3}(t) = \begin{cases} 
\frac{(t_4-t)^2}{(t_4-t_3)^3} + \frac{(t_5-t)(t_4-t)}{(t_5-t_3)(t_4-t_3)^2} + \frac{(t_5-t)^2}{(t_5-t_3)^2(t_4-t_3)} & \text{if } t_3 \leq t < t_4 \\
\frac{(t_5-t)^3}{(t_5-t_3)^2(t_5-t_4)} & \text{if } t_4 \leq t < t_5 \\
0, & \text{otherwise}
\end{cases} \]
15.9 Non-Uniform B-Spline Curves

\[ N_{2,3}(t) = \begin{cases} \frac{(t_4-t)(t-t_3)^2}{(t_5-t_3)(t_4-t_3)^2} + \frac{(t_5-t)(t-t_3)^2}{(t_5-t_3)^2(t_4-t_3)} \\ + \frac{(t_6-t)(t-t_3)^2}{(t_6-t_3)(t_5-t_3)(t_4-t_3)}, & t_3 \leq t < t_4 \\
\end{cases} \]

\[ = \frac{(t_5-t)^2(t-t_3)}{(t_5-t_3)^2(t_5-t_4)} + \frac{(t_6-t)(t-t_3)(t_5-t)}{(t_6-t_3)(t_5-t_3)(t_5-t_4)} \\ + \frac{(t_6-t)^2(t-t_4)}{(t_6-t_3)(t_6-t_4)(t_5-t_4)}, & t_4 \leq t < t_5 \]

\[ = \frac{(t_6-t)^3}{(t_6-t_3)(t_6-t_4)(t_6-t_5)}, & t_5 \leq t < t_6 \]

= 0, \quad \text{elsewhere} \]
15.9 Non-Uniform B-Spline Curves

Hence, when \( t = t_3 \), we have \( C(t_3) = N_{0,3}(t_3)P_0 = P_0 \).

Similarly, \( C(t_{n+1}) = N_{n,3}(t_{n+1})P_n = P_n \).

How are \( N_{0,3} \), \( N_{1,3} \), and \( N_{2,3} \) computed?

The Cox-de Boor recurrence formula shows that \( N_{i,3} \) may be computed using the following chart:

\[
\begin{align*}
N_{i,0} & \rightarrow N_{i,1} \\
N_{i+1,0} & \rightarrow N_{i+1,1} \\
N_{i+2,0} & \rightarrow N_{i+2,1} \\
N_{i+3,0} & \rightarrow N_{i+3,1} \\
N_{i+4,0} & \rightarrow N_{i+4,1} \\
\vdots & \\
N_{i,2} & \rightarrow N_{i,3} \\
N_{i+1,2} & \rightarrow N_{i+1,3} \\
N_{i+2,2} & \rightarrow N_{i+2,3} \\
N_{i+3,2} & \rightarrow N_{i+3,3} \\
\vdots & \\
\end{align*}
\]
15.9 Non-Uniform B-Spline Curves

The above chart can be simplified if one observes that on any given interval, \([t_i, t_{i+1})\), there are only \(n+1\) B-splines of degree \(n\) that are non-zero. On that interval, \(N_{i,n}\) depends on \(N_{i,n-1}\) only, while \(N_{i-l,n}\), \(0 < l \leq n\), depends on both \(N_{i-l+1,n-1}\) and \(N_{i-l,n-1}\). Therefore, for \(t \in [t_i, t_{i+1})\), we have the following chart:

This chart will be used in the solution for question 3.
15.9 Non-Uniform B-Spline Curves

The above chart can be simplified if one observes that on any given interval, \([t_i, t_{i+1})\), there are only \(n+1\) B-splines of degree \(n\) that are non-zero. On that interval, \(N_{i,n}\) depends on \(N_{i,n-1}\) only, while \(N_{i-l,n}, \ 0 < l \leq n\), depends on both \(N_{i-l+1,n-1}\) and \(N_{i-l,n-1}\). Therefore, for \(t \in [t_i, t_{i+1})\), we have the following chart:

This chart will be used in the solution for question 3.
15.9 Non-Uniform B-Spline Curves

To answer Question 3, we need to study the following problem:

If a cubic B-spline curve has only one segment, and its knots satisfy the condition: 
\[ t_0 = t_1 = t_2 = t_3 \quad \text{and} \quad t_4 = t_5 = t_6 = t_7, \]
then what would happen?

For simplicity, we shall consider the simple case 
\[ t_0 = t_1 = t_2 = t_3 = 0 \quad \text{and} \quad t_4 = t_5 = t_6 = t_7 = 1 \] first. The corresponding cubic B-spline basis functions will be denoted \( N_{i,k}, 0 \leq k \leq 3. \)
15.9 Non-Uniform B-Spline Curves

This special cubic B-spline curve segment, denoted \( \overline{C}(t) \), is defined as follows:

\[
\overline{C}(t) = \sum_{i=0}^{3} \overline{N}_{i,3}(t) \overline{P}_i, \quad \text{for} \quad t \in [t_3, t_4] = [0, 1]
\]

where \( t_0 = t_1 = t_2 = t_3 = 0 \) and \( t_4 = t_5 = t_6 = t_7 = 1 \), and \( \overline{P}_i \) are the control points of the curve segment.

The property of a parametric curve depends on the definition of its blending functions. For \( \overline{C}(t), \overline{N}_{i,3}(t) \) can be computed using the chart on page 37. When \( t \in [t_3, t_4) = [0, 1) \), we have...
15.9 Non-Uniform B-Spline Curves
Hence, when \( t_0 = t_1 = t_2 = t_3 = 0 \) and \( t_4 = t_5 = t_6 = t_7 = 1 \), the corresponding cubic B-spline curve segment defined on page 39 is a cubic Bezier curve segment with control points \( \bar{P}_i \):

\[
\bar{C}(t) = \sum_{i=0}^{3} N_{i,3}(t) \bar{P}_i = \sum_{i=0}^{3} B_{i,3}(t) \bar{P}_i
\]
15.9 Non-Uniform B-Spline Curves

On the other hand, let $C(t)$ be a cubic B-spline curve with one segment too

$$C(t) = \sum_{i=0}^{3} N_{i,3}(t) P_i, \quad t \in [t_3, t_4)$$

but the knots are all distinct and $t_i = i - 3$, for $i = 0, 1, \ldots, 7$. (Hence, $[t_3, t_4] = [0, 1]$)
15.9 Non-Uniform B-Spline Curves

If $\mathbf{C}(t)$ and $\mathbf{c}(t)$ represent the same curve, then what is the relationship between control points of $\mathbf{C}(t)$, $\mathbf{P}_i$, and control points of $\mathbf{C}(t)$, $\mathbf{P}_i$?

Solution:

$$\mathbf{P}_1 = \mathbf{P}_1 + \frac{1}{3}(\mathbf{P}_2 - \mathbf{P}_1), \quad \mathbf{P}_2 = \mathbf{P}_1 + \frac{2}{3}(\mathbf{P}_2 - \mathbf{P}_1).$$

$$\mathbf{P}_0 = \frac{[\mathbf{P}_0 + \frac{2}{3}(\mathbf{P}_1 - \mathbf{P}_0)] + \mathbf{P}_1}{2} = \frac{1}{6}\mathbf{P}_0 + \frac{4}{6}\mathbf{P}_1 + \frac{1}{6}\mathbf{P}_2.$$

$$\mathbf{P}_3 = \frac{[\mathbf{P}_2 + \frac{1}{3}(\mathbf{P}_3 - \mathbf{P}_2)]}{2} = \frac{1}{6}\mathbf{P}_1 + \frac{4}{6}\mathbf{P}_2 + \frac{1}{6}\mathbf{P}_3.$$
This means a uniform cubic B-spline curve segment can be converted to a cubic Bezier curve segment, and vice versa.

The relationship between there control points is as follows:

\[ P_i : \text{B-spline control points} \]
\[ \bar{P}_i : \text{Bezier control points} \]
If the cubic Bezier curve $\vec{C}(t)$ defined on page 39 is the same as the uniform cubic B-spline curve segment defined on page 42,

$$\vec{C}(t) = \sum_{i=0}^{3} B_{i,3}(t) \vec{P}_i = C(t) = \sum_{i=0}^{3} N_{i,3}(t) P_i,$$

then we must have

$$\vec{C}(0) = C(0), \quad \vec{C}(1) = C(1)$$

$$\vec{C}'(0) = C'(0), \quad \vec{C}'(1) = C'(1)$$
15.9 Non-Uniform B-Spline Curves

\[ \bar{P}_0 = \frac{1}{6} P_0 + \frac{4}{6} P_1 + \frac{1}{6} P_2, \quad \bar{P}_3 = \frac{1}{6} P_1 + \frac{4}{6} P_2 + \frac{1}{6} P_3 \]

\[ 3(\bar{P}_1 - \bar{P}_0) = -\frac{1}{2} P_0 + \frac{1}{2} P_2, \quad 3(\bar{P}_3 - \bar{P}_2) = -\frac{1}{2} P_1 + \frac{1}{2} P_3 \]

equivalent to the conditions on page 43.