15 Curves and Surfaces

- Can be used for font, carton character, car body, ..., design/representation
15 Curves and Surfaces

Types of Curves:

- **Explicit**
  \[ y = mx + b \]
  \[ r = A_r x + B_r y + C_r \]

- **Implicit**
  \[ Ax + By + C = 0 \]
  \[ (x - x_0)^2 + (y - y_0)^2 - r^2 = 0 \]

- **Parametric**
  \[ x = x_0 + (x_1 - x_0)t \]
  \[ x = x_0 + r\cos\theta \]
  \[ y = y_0 + (y_1 - y_0)t \]
  \[ y = y_0 + r\sin\theta \]

- **Generative**: generated with a procedure, e.g., subdivision schemes and fractals
15 Curves and Surfaces

Why parametric?

• Flexible
• Not required to be functions
  - curves can be multi-valued with respect to any coordinate system
• Parameter count gives the object's dimension
  \((x(u,v), y(u,v), z(u,v))\)
• Coord functions independent
Specifying Curves

- **Control Points:**
  - a set of points that influence the curve’s shape

- **Knots:**
  - control points that lie on the curve

- **Interpolating spline:**
  - curve passes through the control points

- **Approximating spline:**
  - control points merely influence shape
Piecewise Parametric Curves

We can represent an arbitrary length curve as a series of curve segments pieced together. WHY?

But we will want to control how these curve segments fit together.
Piecewise Cubic Curves

• In order to assure C2 continuity our functions must be of at least degree 3. This is also the lowest degree to describe a non-planar curve.

• Cubic has 4 degrees of freedom and can control 4 things.

• Use polynomials: $x(t)$ of degree $n$ is a function of $t$. $y(t)$ and $z(t)$ are similar and each is handled independently.

• That is:

$$x(t) = \sum_{i=0}^{n} a_i x^i$$
15.1 Bezier Curve Segments of Degree 3

\[ C(t) = (1-t)^3 P_0 + 3t(1-t)^2 P_1 + 3t^2(1-t) P_2 + t^3 P_3 \]

\[ 0 \leq t \leq 1 \]

Matrix form:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
-3 & 3 & 0 & 0 \\
3 & -6 & 3 & 0 \\
-1 & 3 & -3 & 1
\end{bmatrix}
\begin{bmatrix}
P_0 \\
P_1 \\
P_2 \\
P_3
\end{bmatrix}
\]
15.1 Bezier Curve Segments of Degree 3

- $P_i = (x_i, y_i)$ are called control points
- The polygon $P_0P_1P_2P_3$ is called the control polygon
- The weights $(1-t)^3$, $3t(1-t)^2$, $3t^2(1-t)$, and $t^3$ are called blending functions

Notes:
- Blending functions are always non-negative
- Blending functions always sum to 1
15.1 Bezier Curve Segments of Degree 3

- A Bezier curve always starts at $P_0$ and ends at $P_3$
- A Bezier curve is tangent to the control polygon at the endpoints
- Bezier curve segments satisfy convex hull property
- Bezier curves have intuitive appeal for interactive users
15.2 General Bezier Curve Segments

\[ C(t) = \sum_{i=0}^{n} B_{i,n}(t) P_i = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} P_i, \]

where \( 0 \leq t \leq 1 \) and \( \binom{n}{i} = \frac{n!}{i!(n-i)!} \). \( B_{i,n}(t) \) are again called blending functions and \( P_i \) control points.
15.2 General Bezier Curve Segments

- All the properties mentioned on page 5 hold for general Bezier curves

\[
C(t) = (1 - t) \left\{ \sum_{i=0}^{n-1} B_{i,n-1}(t) P_i \right\} + t \left\{ \sum_{i=0}^{n-1} B_{i,n-1}(t) P_{i+1} \right\}
\]

\[
= (1 - t) \left\{ \sum_{i=0}^{n-1} \binom{n-1}{i} t^i (1 - t)^{n-1-i} P_i \right\}
\]

\[
- t \left\{ \sum_{i=0}^{n-1} \binom{n-1}{i} t^i (1 - t)^{n-1-i} P_{i+1} \right\}
\]
15.2 General Bezier Curve Segments

- Curve computation

If degree = 3 then

\[
C\left(\frac{1}{3}\right) = \frac{2}{3} \left[ \frac{2}{3} P_0 + \frac{1}{3} P_1 \right] + \frac{1}{3} \left[ \frac{2}{3} P_1 + \frac{1}{3} P_2 \right]
\]

\[
+ \frac{1}{3} \left[ \frac{2}{3} P_2 + \frac{1}{3} P_3 \right]
\]
15.2 General Bezier Curve Segments

- **Midpoint Curve Subdivision**

\[ P_0, M, N, O \text{ are control points of } C(t), \ 0 \leq t \leq 1/2, \]

\[ \text{and } O, P, Q, P_3 \text{ are control points of } C(t), \]

\[ 1/2 \leq t \leq 1. \]
Proof:

Define \( C_1(t) = C(t/2) \). We have

\[
C_1(t) = \left[ 1, t/2, t^2/4, t^3/8 \right] M P = T S M P
\]

where

\[
T = \left[ 1, t, t^2, t^3 \right]^T,
\]

\[
M = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
-3 & 3 & 0 & 0 & 0 \\
3 & -6 & 3 & 0 & 0 \\
-1 & 3 & -3 & 1 & 0
\end{bmatrix},
\]

\[
S = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1/2 & 0 & 0 & 0 \\
0 & 0 & 1/4 & 0 & 0 \\
0 & 0 & 0 & 1/8 & 0
\end{bmatrix},
\]

\[
P = \begin{bmatrix}
P_0, P_1, P_2, P_3
\end{bmatrix}^T.
\]
Proof:

On the other hand, as a Bezier curve of degree 3, $C_1(t)$ can also be expressed as

$$C_1(t) = TMQ$$ (**)

where $Q = [Q_0, Q_1, Q_2, Q_3]$ is the control polygon of $C_1(t)$.

From (*) and (**), we have

$$MQ = SMP$$

or

$$Q = M^{-1}SMP$$
Proof:

It is easy to see that

\[ M^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1/3 & 0 & 0 \\ 1 & 2/3 & 1/3 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \]

Hence, we have

\[ Q = \begin{bmatrix} P_0 \\ \frac{P_0 + P_1}{2} \\ \frac{P_0}{4} + \frac{P_1}{2} + \frac{P_2}{4} \\ \frac{P_0}{8} + \frac{3P_1}{8} + \frac{3P_2}{8} + \frac{P_3}{8} \end{bmatrix} = \begin{bmatrix} P_0 \\ M \\ N \\ O \end{bmatrix} \]
15.2 General Bezier Curve Segments

- **Recursively subdivide** the control polygons at the midpoints, we can divide the curve into many small segments, each with its own control points.

- These control points, when **connected**, form a good linear approximation of the curve $C(t)$. (This linear approximation is usually used to find the intersection points of two Bezier curves)
15.3 Cubic Uniform B-spline Curves

(a curve representation with local property)

A Cubic Uniform B-Spline Curve segment

For four given control points $P_0$, $P_1$, $P_2$ and $P_3$, a cubic uniform B-spline curve segment is defined as follows:

$$C_{bs}(t) = \frac{(1-t)^3}{6} P_0 + \frac{(4-6t^2+3t^3)}{6} P_1 + \frac{(1+3t+3t^2-3t^3)}{6} P_2 + \frac{t^3}{6} P_3$$

$$0 \leq t \leq 1$$
15.3 Cubic Uniform B-spline Curves

Matrix form:

\[
C_{bs}(t) = \left[1, t, t^2, t^3\right] \frac{1}{6} \begin{bmatrix}
1 & 4 & 1 & 0 \\
-3 & 0 & 3 & 0 \\
3 & -6 & 3 & 0 \\
-1 & 3 & -3 & 1
\end{bmatrix} \begin{bmatrix}
P_0 \\
P_1 \\
P_2 \\
P_3
\end{bmatrix}
\]

Blending functions:
15.3 Uniform Cubic B-spline Curves

Properties of B-spline blending functions

- Non-negative
- Sum = 1
- Hence, again, a B-spline curve segment is always contained in the convex hull of its control points.
- However, the curve does not interpolate the first and the last control points. Actually

\[
C_{bs}(0) = \frac{1}{6} P_0 + \frac{2}{3} P_1 + \frac{1}{6} P_2
\]

\[
C_{bs}(1) = \frac{1}{6} P_1 + \frac{2}{3} P_2 + \frac{1}{6} P_3
\]
A **Cubic Uniform B-Spline Curve**:

- Given a set of *n* control points, one can define a cubic (uniform) B-spline curve with *(n - 3)* segments.
- The first segment, $C_1(t)$, is defined by the first four control points: $P_0, P_1, P_2, P_3$.
- The second segment, $C_2(t)$, is defined by the second four control points: $P_1, P_2, P_3, P_4$.
- The last one, $C_{n-3}(t)$, by $P_{n-3}, P_{n-2}, P_{n-1}, P_n$. 
15.3 Cubic Uniform B-spline Curves

Properties/Advantages of a B-spline curve:

- **Local property** (changing one control point will affect at most four segments)
- **C2 continuity** at the joints
- **Compact form** for multiple segments
- **Can use multiple control points** to achieve exact point interpolation

\[ P_0 = P_1 = P_2 \]

\[ P_6 = P_7 = P_8 \]
15.4 Composite Bezier Curves

- Bezier curve segments can be joined together to form complicated shapes

\[ P_0, P_1, P_2, \text{ and } P_3 \text{ are control points of the 1}\text{st} \text{ segment} \]
\[ P_3, P_4, P_5, \text{ and } P_6 \text{ are control points of the 2}\text{nd} \text{ segment} \]
\[ P_2, P_3, \text{ and } P_4 \text{ are } \text{collinear} \text{ (to guarantee smooth joint)} \]
15.4 Composite Bezier Curves

**Smoothness (continuity) at Join Points:**
- $C^0$: the endpoints coincide
- $G^1$: tangents have the same slope
- $C^1$: first derivatives on both segments match at join point
- $C^2$: 2nd derivatives on both segments match at join point
15.4 Composite Bezier Curves

- **G1-continuity:** $P_2$, $P_3$, and $P_4$ are collinear

- **C1-continuity:** $P_2$, $P_3$, and $P_4$ are collinear and $P_3$ is the midpoint of $P_2P_4$
15.4 Composite Bezier Curves

- **C2-continuity:**
  * $P_2$, $P_3$, and $P_4$ are collinear
  * $P_3$ is the midpoint of $P_2P_4$
  * $P_5 = P_1 + 4(P_3 - P_2)$
15.5 Curve Interpolation using Composite Bezier Curves

- Give a set of data points \( D_0, D_1, ..., D_n \) (\( n \geq 2 \)), how can a composite cubic Bezier curve that interpolates these points be constructed?

- The composite cubic Bezier curve has \( n \) segments \( C_1(t) \), \( C_2(t) \), ..., \( C_n(t) \) with \( D_{i-1} \) and \( D_i \) being the start and end points of \( C_i(t) \).

- The composite cubic Bezier curve is \( C^2 \)-continuous.
An analysis of the problem:

- To construct the curve, how many control points are needed?
- How many of them are known to us now?

So, how many of them remain to be computed? $2n$

And how should they be computed?
(How should the $C1$- and $C2$-continuity conditions be used?)
15.5 Curve Interpolation using Composite Bezier Curves

Let \( P_{i,0} \), \( P_{i,1} \), \( P_{i,2} \), \( P_{i,3} \) be the control points of the \( C_i(t) \).

Then for each two adjacent Bezier segments \( C_i(t) \) and \( C_{i+1}(t) \), we have

\[
\begin{align*}
    P_{i,3} &= D_i = P_{i+1,0} \\
    P_{i+1,1} - D_i &= D_i - P_{i,2} \\
    P_{i+1,2} - P_{i,1} &= 2(P_{i+1,1} - P_{i,2})
\end{align*}
\]
15.5 Curve Interpolation using Composite Bezier Curves

Hence, we have a system of $2(n-1)$ equations in $2n$ unknowns $\{P_{i,1}, P_{i,2}\}_{i=1}^{n}$

\[
\begin{align*}
P_{i,2} + P_{i+1,1} &= 2D_i \\
2P_{i,1} - P_{i,2} + 2P_{i+1,1} - P_{i+1,2} &= 0
\end{align*}
\]

$i = 1, 2, \ldots, n-1$ (3.1)

Two extra conditions can be given as follows:

1. $P_{1,1}$ and $P_{n,2}$ are specified by the user, or
2. requiring the composite Bezier curve to have zero 2nd derivative at $D_0$ and $D_n$. 
If we use the 2nd approach for the extra conditions, (3.2), together with (3.1), we get a system of 2n equations in 2n unknowns, as follows:

\[
\begin{align*}
\dot{C}_1''(0) &= 6(P_{1,2} - 2P_{1,1} + P_{1,0}) = 0 \\
\dot{C}_n''(1) &= 6(P_{n,3} - 2P_{n,2} + P_{n,1}) = 0
\end{align*}
\]
This system of equations can be solved using **Gaussian elimination without pivoting**.
End of 15.5