

CS 115 Lecture 20

Recursion

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1 December 2015

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$$\text{Fib}(0) = 1, \text{Fib}(1) = 1, \text{Fib}(2) = 2, \\ 3, 5, 8, 13, 21, 34, \dots$$

What's the pattern?

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