Modal nonmonotonic logics: ranges, characterization, computation

Wiktor Marek
Department of Computer Science
University of Kentucky
Lexington, KY 40506-0027

Grigori F. Shvarts
Program Systems Institute
USSR Academy of Sciences
152140 Pereslavl-Zalessky, USSR

Mirosław Truszczyński
Department of Computer Science
University of Kentucky
Lexington, KY 40506-0027

Abstract

In the paper, we investigate the way in which nonmonotonic modal logics depend on their underlying **monotonic** modal logics. Most notably, we study when different monotonic modal logics define the same nonmonotonic system. In particular, we show that for an important class of the so called *stratified* theories all nonmonotonic logics considered in the paper, with the exception of **S5**, coincide.

It turns out that in some cases, nonstandard (that is, non-normal) logics have interesting nonmonotonic counterparts. Two such systems are investigated in the paper in detail.

For the case of finite theories, all nonmonotonic logics considered are shown to be decidable and an appropriate algorithm is presented.

1 INTRODUCTION

Many types of commonsense reasonings can be faithfully represented within modal nonmonotonic logic. These reasonings include: default logic of Reiter [Rei80], nonmonotonic logic of belief of Moore [Moo85], truth maintenance of Doyle [Doy79], and some important aspects of logic programming [GL88]. Therefore, in this paper we will study three fundamental issues of modal nonmonotonic logics:
(a) dependence of such logics on the underlying monotonic logic,
(b) characterization of expansions,
(c) computation of expansions.

Most known modal nonmonotonic logics belong to the family proposed by McDermott [McD82]. In this paper we focus on these logics only. The reader is referred to [Kam90], [Tru91], [Sho87] and [LS90] for the discussion of other modal nonmonotonic systems.

There is a rich variety of different monotonic modal logics [Che80], [HC84]. An important question is: how much of this variety carries over to the nonmonotonic case. The most important and somewhat unexpected result of our paper is that the structure of the family of nonmonotonic modal logics is much simpler. Speaking more formally, **it is often the case that different monotonic modal logics collapse to the same nonmonotonic system**.

The results of this paper focus on modal logics whose nonmonotonic counterparts are applicable in knowledge representation. We identified three such logics: **K45**, **N** and **W5**. It turns out that for each of these logics there is a whole family of monotonic modal logics that generate the same system in the nonmonotonic case. For nonmonotonic logics **K45**, **N** and **W5**, as well as many others, we characterize expansions and provide algorithms for computing them.

Throughout the paper, by Cn we denote the operator of propositional consequence, and by $Cn_{\mathcal{S}}$, where $\mathcal{S}$ stands for a modal logic, the operator of provability in logic $\mathcal{S}$. Our further discussion assumes familiarity with basic notions of modal logics as given in [HC84] and [Che80]. The mechanism of nonmonotonic modal consequence operation based on the (monotonic) modal logic $\mathcal{S}$ has been introduced in [MD80] and [McD82]. This method can be shortly presented as follows: Given a modal logic $\mathcal{S}$ and a theory I in the language with one modal operator L, denoted $\mathcal{L}_L$, a theory $T \subseteq \mathcal{L}_L$ is called an $\mathcal{S}$-*expansion* of I if it satisfies the equation:

$$T = Cn_{\mathcal{S}}(I \cup \{\neg L\varphi : \varphi \notin T\}), \qquad (1)$$

With the interpretation of modal operator L as "is known" or "is believed", T is an expansion of I if T is precisely the collection of these formulas which can be derived in $\mathcal{S}$ from I and statements about "ignorance" or "negative introspection" *with respect to* T. This circular aspect and self-reference of the concept of expansion results in the fact that the equation (1) may have single or multiple solutions, and sometimes even no solution. Once all such "points of view" T are identified, we compute the nonmonotonic consequence of I in $\mathcal{S}$ as the intersection of all $\mathcal{S}$-expansions of I.

Once the nonmonotonic counterparts for modal logics are defined, the following question becomes of fundamental importance: **Which modal nonmonotonic logics are applicable in knowledge representation?** Early attempts to identify such logics have been unsuccessful. The case of $\mathcal{S}$ equal to propositional calculus (in the language $\mathcal{L}_L$) leads to counterintuitive expansions, for example, containing both p and $\neg Lp$. The other extreme, $\mathcal{S} = \mathbf{S5}$, collapses to monotonic **S5** that is, the resulting nonmonotonic consequence operator coincides with the monotonic **S5**, [McD82], [MT90].

In a reaction to the above mentioned failures [Moo85] introduced a seemingly different scheme:

$$T = Cn(I \cup \{L\varphi : \varphi \in T\} \cup \{\neg L\varphi : \varphi \notin T\}) \qquad (2)$$

and argued that the solutions to the equation (2) better capture the intuitions associated with the states of belief of a fully introspective agent than the general scheme (1). This scheme is very specialized — notice the absence of modal parameter $\mathcal{S}$ in the equation (2). Yet, it turns out that for consistent theories T, Moore's expansions, called *autoepistemic* or *stable* expansions, coincide with **K45**-expansions [Shv90]. In other words, the logic of Moore belongs to the family of nonmonotonic formalisms definable by the scheme (1).

One of the most widely studied and used nonmonotonic formalisms is the default logic of Reiter [Rei80]. Intensive studies were undertaken to find a modal counterpart of the default logic [Kon88], [MT89a], [MT90], [LS90]. It turned out that under the translation that assigns to a default rule $\frac{\alpha:\beta_1\ldots\beta_n}{\gamma}$ a modal formula: $L\alpha \wedge \neg LL\neg\beta_1 \wedge \ldots \wedge \neg LL\neg\beta_n \Rightarrow \gamma$, extensions of default theories can be faithfully described as **N**-expansions, where **N** is the modal logic of necessitation, that is the modal logic without any scheme for handling modalities [MT90]. In addition, the same modal logic can be used to represent stable semantics for logic programs, as well as ordinary truth maintenance systems ([MT89b]).

Third nonmonotonic modal logic with natural applications in knowledge representation is the modal nonmonotonic logic associated with the modal logic **W5**, which contains one axiom schema:

W5: $\quad \neg L\neg L\varphi \Rightarrow (\varphi \Rightarrow L\varphi)$,

a weaker variant of **5**. It is shown in [MT90] that the nonmonotonic logic **W5** allows to provide a natural semantics for two important modes of nonmonotonic reasoning: logic programming with classical negation (cf. [GL90]), and truth maintenance systems in which we admit rules containing literals (not only atoms).

The nonmonotonic logic **W5** possesses a semantic characterization similar to Moore's characterization of stable (that is, **K45**-) expansions. The following fixpoint equation characterizes **W5**-expansions [MT90]:

$$T = Cn(I \cup \{\varphi \Rightarrow L\varphi : \varphi \in T\} \cup \{\neg L\varphi : \varphi \notin T\}). \quad (3)$$

This fixpoint equation relaxes Moore's definition of *autoepistemic valuation with index* T. Recall that V is an *autoepistemic* valuation with index T if $V(L\varphi) = 1$ precisely when $\varphi \in T$. Here, the class of valuations V contains, as before, valuations that satisfy condition $V(L\varphi) = 0$ if $\varphi \notin T$. But, on positive side, for $\varphi \in T$, we relax the condition as follows: we require that $V(\varphi \Rightarrow L\varphi) = 1$. All autoepistemic valuations satisfy this condition, but there are other valuations which need to be considered as well. The intuition here is that we want V to evaluate $L\varphi$ as 1 *providing* that V evaluates φ as 1. Since we extended the class of valuations under consideration, a stronger condition is imposed on a fixpoint. Any solution of equation (3) is called a *strict* expansion of I. The important point here is that such strict expansions are definable semantically, and in a natural fashion.

Summarizing our discussion, three logics **K45**, **N** and **W5** seem to be of particular interest in knowledge representaion. These three logics have several puzzling things in common. First of all, they are clearly off the main research track of classical modal logic. To our knowledge, even **K45** was very little studied, even though it has an elegant and well understood Kripke semantics. Secondly, two of the logics — **N** and **W5** are subnormal that is, do not satisfy axiom schema K. The third one, **K45** is normal but, as proved in [MT90], it is equivalent to the subnormal modal logic **5** satisfying only axiom schema 5, in the sense that both logics have the same "nonmonotonic variant".

The meaning of this last result (collapse of nonmonotonic modal logics **5** and **K45**) was deeply intriguing; it indicated that *different* monotonic logics may generate the same notion of (consistent) expansion, and consequently the same nonmonotonic consequence operation! In other words, **the realm of nonmonotonic modal logics is much less diversified than**

that of monotonic modal logics. This, in turn, means that in the nonmonotonic case the axioms to manipulate modality play a different role than in the monotonic case.

On a closer inspection these observations seem to be less puzzling. After all, nonmonotonic $\mathcal{S}$-consequences of a theory I are often strictly larger than monotonic $\mathcal{S}$-consequences of I due to the powerful principle of "negation as failure to prove" which allows us to use in the reasonings formulas expressing negative introspection ($\{\neg L\varphi\colon \varphi \notin T\}$). Precisely speaking, let $\mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \mathbf{S5}$ be two modal logics. If for every stable theory $T \subseteq \mathcal{L}_L$, and for every instance ψ of any axiom schema of $\mathcal{S}_2$, $\{\neg L\varphi\colon \varphi \notin T\} \vdash_{\mathcal{S}_1} \psi$, then nonmonotonic logics $\mathcal{S}_1$ and $\mathcal{S}_2$ are identical.

A number of basic questions arise on the dependence of the nonmonotonic $\mathcal{S}$-consequences on the underlying logic $\mathcal{S}$. The first question is this: Are there nonmonotonic logics different from the autoepistemic logic of Moore that can be equivalently defined through McDermott's scheme (1) for different monotonic logics $\mathcal{S}$? If so, what are the properties of these logics and their mutual relationships. Although we do not have complete answers to these questions, it turns out that a number of nontrivial facts can be proved about "ranges" of modal logics collapsing to the same nonmonotonic logic. We shall give a number of such results in this paper.

Formally, a *range* is a collection of monotonic modal logics generating the same concept of a consistent expansion. A number of ranges will be exhibited below namely, for the logics **N**, **W5** and **K45**. Important open questions are: Are there any trivial (one-element) ranges? Are there any other nontrivial ranges than those exhibited in this paper? One needs to point that the fact that nontrivial ranges exist is, potentially, quite beneficial. For instance a range may contain very different logics. Some may have a nice automated theorem proving mechanism, whereas others may have an elegant semantics. In such case, for the *nonmonotonic* logic associated with the range, we might have completely different mechanisms for semantical and for syntactical manipulations. In fact we may have completely different semantics, all "nonmonotonically" equivalent. This is the case for the range associated with Moore's logic. The freedom of selecting different theorem provers may turn out useful in practical implementations.

In addition, it turns out that syntactic restrictions on I make some ranges coagulate. Two important types of syntactic restrictions we study are stratification and restriction to formulas with negative introspection only. In each case, the ranges of equivalent (nonmonotonically) modal logics are exhibited.

Our results on the ranges can be summarized as follows:

- General theories: Logics **N, W5, K45** possess nontrivial, ranges.
- Theories consisting of formulas with negative introspection only: A wide range from **N** to **KD45**.
- Stratified theories: All known ranges collapse to one.

Next, we consider the problem of characterization of $\mathcal{S}$-expansions. Several general results were obtained in [Shv90]. In particular, they imply characterizations of $\mathcal{S}$-expansions for several normal modal logics including **K, T, S4, K45**. We strengthened the results of [Shv90] and derived characterizations of **N**-expansions and **W5**-expansions. All characterizations involve modal atoms appearing as subformulas in formulas of a theory whose expansions we study.

Most importantly, the characterizations we obtained form the basis for the third component of our investigation, namely computation of $\mathcal{S}$-expansions. We present algorithms for computing $\mathcal{S}$-expansions for a large class of logics (including **N, W5** and **K45**). We illustrate the algorithms with an example.

The paper is organized as follows. The next section contains our results on the ranges both for the case of general theories, as well as those subject to some syntactic restrictions. The results are gathered at the end of the section in appropriate diagrams. Secton 3 discusses characterizations of expansions and algorithms for their computation. Section 4 contains conclusions.

2 $\mathcal{S}$-EXPANSIONS — RANGES

Because of space limitations, we will assume some familiarity with the results of [Moo85], [Kon89] and [MT89a]. In addition to standard modal logics such as **K, T, S4, S5, K45** etc., we shall consider logic **N** of necessitation (with no axiom scheme at all) and the following two schemes: W5 (introduced above) and:

WK: $L(\varphi \Rightarrow \psi) \wedge L\varphi \Rightarrow \neg L\neg L\psi$ (a variant of K that, given $L\varphi$ and $L(\varphi \Rightarrow \psi)$, instead asserting $L\psi$ as K does, asserts $\neg L\neg L\psi$).

We start with investigating the general properties of $\mathcal{S}$-expansions. First, let us mention several facts concerning *stable* sets [Sta80, McD82, Moo85, Mar89]. For each theory $S \subseteq \mathcal{L}$ there is a unique stable set T such that $T \cap \mathcal{L}$ is exactly the set of logical consequences of S. This unique stable set will be denoted by $E(S)$. A constructive definition of $E(S)$, for a finite S, is given in [Mar89]. In fact, it can be shown, that

$$E(S) = Cn_{\mathbf{S5}}(S \cup \{\neg L\varphi : \varphi \in \mathcal{L} \setminus Cn(S)\}).$$

Clearly, if a modal logic $\mathcal{S}$ contains the necessitation rule then each $\mathcal{S}$-expansion is stable.

Let us now pass on to investigations of mutual relationships between classes of $\mathcal{S}$-expansions for various logics $\mathcal{S}$. We have the following simple result.

Proposition 2.1 ([McD82]) *Let $\mathcal{S}$ and $\mathcal{T}$ be two modal logics, $\mathcal{S} \subseteq \mathcal{T} \subseteq$ **S5**. Then each $\mathcal{S}$-expansion of I is a $\mathcal{T}$-expansion of I.*

The following three theorems show that the notion of range is nontrivial. First result, obtained in [MT90] shows that the autoepistemic logic of Moore can be defined not only by means of the logic **K45** or **KD45**, which was known already to Konolige [Kon88], but by any logic in a much wider class. In fact, our result shows that **the crucial role in the autoepistemic logic is played by the axiom schema 5, which allows to derive positive introspection from the negative introspection.**

Theorem 2.2 *Let $\mathcal{S}$ be a modal logic, $\mathbf{5} \subseteq \mathcal{S} \subseteq$ **KD45**. Let $I, T \subseteq \mathcal{L}_L$. The theory T is a stable expansion of I if and only if T is an $\mathcal{S}$-expansion of I.*

Thus, for the whole range of logics between **5** and **KD45** the same notion of expansion is obtained. A similar result holds for **N**-expansions.

Theorem 2.3 *Let $\mathcal{S}$ be a modal logic, $\mathbf{N} \subseteq \mathcal{S} \subseteq$ **WK**. Let $I, T \subseteq \mathcal{L}_L$. The theory T is an **N**-expansion of I if and only if T is an $\mathcal{S}$-expansion of I.*

Our final result of that type concerns the case of strict expansions. Firstly, it shows that strict expansions can also be characterized by means of scheme (1). Secondly, it exhibits the whole range of logics that can be used for that purpose.

Theorem 2.4 *Let $\mathcal{S}$ be a modal logic, $\mathbf{W5} \subseteq \mathcal{S} \subseteq$ **D4W5**. Let $I, T \subseteq \mathcal{L}_L$. Then, the theory T is a strict expansion of I if and only if T is an $\mathcal{S}$-expansion of I.*

Theorems 2.2, 2.3 and 2.4 give rise to an interesting theoretical problem. Let $\mathcal{S}$ be a monotonic modal logic. By $R(\mathcal{S})$ we denote the range of all monotonic modal logics $\mathcal{T} \subseteq$ **S5** "nonmonotonically equivalent" to $\mathcal{S}$. A general question that arises is: What is the structure of the set $R(\mathcal{S})$? It is easy to see that if $\mathcal{T}_1$, $\mathcal{T}_2 \in R(\mathcal{S})$, then for every modal logic $\mathcal{T}$ such that $\mathcal{T}_1 \subseteq \mathcal{T} \subseteq \mathcal{T}_2$ we have $\mathcal{T} \in R(\mathcal{S})$. Thus, each $R(\mathcal{S})$ is an interval in the partial ordering of modal logics by inclusion relation. The following question is still open: Does each such interval have the least element, the greatest element? Another general question is: Does each interval $R(\mathcal{S})$ always contain logics other than $\mathcal{S}$?

We have already seen that for the whole ranges of logics $\mathcal{S}$, the notions of expansions defined by these logics are equivalent (Theorems 2.2, 2.3 and 2.4). Below we show that if we restrict the class of theories I, even stronger results hold. Namely, we have the following two theorems. First of them was proved in [Shv90], the second one in [MT90].

Theorem 2.5 *Let $I \subseteq \mathcal{L}$ be consistent. For each logic $\mathcal{S}$ such that*
(a) $\mathbf{N} \subseteq \mathcal{S}$, *and*
(b) $\mathcal{S} \subseteq$ **KD45** *or* $\mathcal{S} \subseteq$ **S4**,
the theory $E(I)$ is the only $\mathcal{S}$-expansion of I.

Theorem 2.5 says that there is a big range of modal logics in the case of objective theories.

Theorem 2.6 *Let $\mathcal{S}$ be a modal logic contained in* **KD45**. *If I consists only of modal clauses with negative introspection, then T is an $\mathcal{S}$-expansion of I if and only if T is an **N**-expansion of I.*

Theorem 2.5 can be generalized to a wider class of theories. A theory I is *strongly stratified* if

STRAT1 Each formula in I is of the form $a(\varphi) \Rightarrow c(\varphi)$, $c(\varphi) \in \mathcal{L}$, for $\varphi \in I$, and $\{c(\varphi) : \varphi \in I\}$ is consistent.

STRAT2 I has a partition (called *stratification*) $I = I_1 \cup \cdots \cup I_n$ into disjoint and nonempty sets such that for each $\varphi \in I_j$, $1 \leq j \leq n$, propositional letters occurring in $c(\varphi)$ do not occur in formulas of $I_1 \cup \cdots \cup I_{j-1}$ and do not occur in formulas of I_j under the scope of the operator L.

This notion of stratification is closely related to stratification of logic programs, as introduced in [ABW87] and extends the concept of stratification as introduced in [Gel87]. The difference is that where Gelfond requires that the consequents $c(\varphi)$ of formulas in I are disjunctions of atoms, we impose a weaker condition, namely that these consequents form a consistent theory. We will see that as long as we restrict ourselves to strongly stratified theories all ranges collapse into one large range. This is the subject of the next theorem.

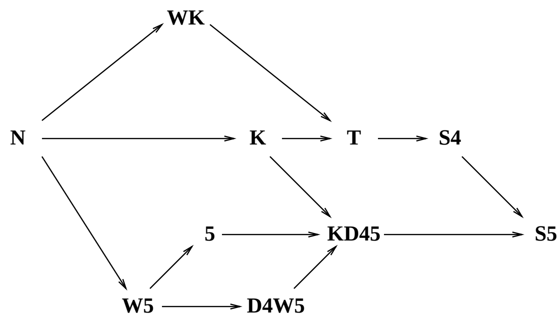

Figure 1: The hierarchy of modal logics

Theorem 2.7 *Let $I \subseteq \mathcal{L}$ be strongly stratified. For each logic $\mathcal{S}$ such that*
(a) $\mathbf{N} \subseteq \mathcal{S}$, *and*
(b) $\mathcal{S} \subseteq \mathbf{KD45}$ *or* $\mathcal{S} \subseteq \mathbf{S4}$,
the theory I has exactly one $\mathcal{S}$-expansion and for all these logics $\mathcal{S}$, $\mathcal{S}$-expansions of I coincide.

We collect now the results on the relationships among the classes of expansions discussed in the paper. We considered the following logics: **N, WK, K, T, S4, S5, W5, D4W5, 5, KD45**. Inclusion relation diagram for these logics is shown in Figure 1. An arrow in the diagram directed from a logic $\mathcal{S}_1$ to a logic $\mathcal{S}_2$ indicates that $\mathcal{S}_1 \subseteq \mathcal{S}_2$.

All of these inclusions are straightforward. To see that $\mathbf{WK} \subseteq \mathbf{T}$ simply check that the axiom schema **WK** holds in every **T**-Kripke model (that is, in every Kripke model with a reflexive admissibility relation).

According to Proposition 2.1 (b), the same inclusion relations hold for classes of $\mathcal{S}$-expansions. In addition, for each theory I, a **WK**-expansion of I is also an **N**-expansion of I (Theorem 2.3) and, therefore, a **K**-expansion of I. Theorems 2.2, 2.3 and 2.4 state that logics **N** and **WK**, **5** and **KD45**, and **W5** and **D4W5**, respectively, define the same notion of expansion. All these properties are summarized in the diagram in Figure 2. The arrow between the symbols for two logics $\mathcal{S}$ and $\mathcal{T}$ indicates that for each theory I, every $\mathcal{S}$-expansion of I is a $\mathcal{T}$-expansion of I. The ranges of logics that define the same notion of expansion are indicated by ovals. In all other cases, there are theories, for which different logics define different classes of expansions. This is illustrated by examples given in Section 3.

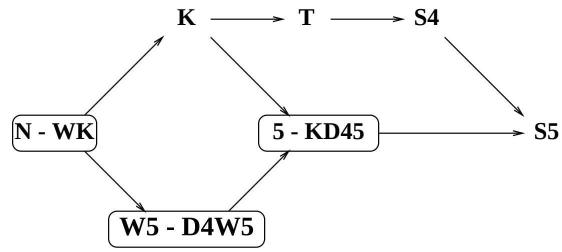

Figure 2: The hierarchy of modal nonmonotonic logics

Thus, the only generally true inclusion relationships between the classes of expansions shown in Figure 2 are those indicated by arrows.

3 CHARACTERIZATIONS OF $\mathcal{S}$-EXPANSIONS AND ALGORITHMS

Continuing our program, we move to characterize $\mathcal{S}$-expansions. In the case of many modal logics including logics **K**, **T**, **S4**, **K45**, the problem was solved by Shvarts [Shv90]. Below we strengthen his results and extend them to logics **N** and **W5**.

By I^L we denote the set of all subformulas of the formulas from I of the form $L\psi$. Let T be a consistent stable theory containing I. Let $\Psi = I^L \cap T$, and $\Phi = I^L \setminus \Psi$. We denote by $\neg\Phi$ the set $\{\neg\varphi : \varphi \in \Phi\}$. Obviously, if $L\psi \in T$ then $\psi \in T$, and if $L\varphi \in \Phi$ then $\varphi \notin T$ and $\neg L\varphi \in T$. Hence $I \cup \neg\Phi \cup \Psi \cup \{\psi : L\psi \in \Psi\}$ is consistent and is contained in T. Furthermore, for every $L\varphi \in \Phi$, $\varphi \notin Cn(I \cup \neg\Phi \cup \Psi \cup \{\psi : L\psi \in \Psi\})$. These observations motivate the following definition. Let $\Phi \subseteq I^L, \Psi = I^L \setminus \Phi$. Φ is said to be *admissible for* I iff $I \cup \neg\Phi \cup \Psi \cup \{\psi : L\psi \in \Psi\}$ is propositionally consistent and for each $L\varphi \in \Phi$, $\varphi \notin Cn(I \cup \neg\Phi \cup \Psi \cup \{\psi : L\psi \in \Psi\})$. Given a modal logic $\mathcal{S}$, we call a set $\Phi \subseteq I^L$ $\mathcal{S}$-*admissible* if Φ is admissible for I and for each $L\psi \in \Psi$, $I \cup \neg\Phi \vdash_{\mathcal{S}} \psi$.

In what follows, the stable set generated by $Cn(I \cup \neg\Phi \cup \Psi \cup \{\psi : L\psi \in \Psi\}) \cap \mathcal{L}$ plays a special role. We will denote it by

$$T_{I,\Phi} = E(Cn(I \cup \neg\Phi \cup \Psi \cup \{\psi : L\psi \in \Psi\}) \cap \mathcal{L}).$$

The role of the stable set $T_{I,\Phi}$ is explained by the following result of Shvarts [Shv90].

Theorem 3.1 *Let $\mathcal{S}$ be any modal logic contained in* **S5**, *and let Φ be $\mathcal{S}$-admissible for I. Then $T_{I,\Phi}$ is an $\mathcal{S}$-expansion of I.*

We will first extend general results of [Shv90] to a wider class of logics. A class of frames C is *closed* if for all $(M_1, R_1),\ (M_2, R_2) \in C$ such that $M_1 \cap M_2 = \emptyset$, the frame $(M_1 \cup M_2, R_1 \cup (M_1 \times M_2) \cup R_2)$ belongs to C. In the next theorem, we need yet another concept. If F is a complete frame, that is $F = (M, M \times M)$, then by *one-element extension* (1-extension) of F we mean a frame of the form $(M \cup \{a\}, (\{a\} \times M) \cup M \times M)$ or $(M \cup \{a\}, (\{< a, a >\} \cup \{a\} \times M) \cup M \times M)$, where $a \notin M$. Thus, 1-extension of the complete frame F adds to F one new world a from which all the worlds of F are accessible. In addition a may be accessible from itself or not. By the (1-ext) property of a logic $\mathcal{S}$ we mean the following property:

(1-ext) For any complete frame F, at least one 1-extension G of F is a frame for $\mathcal{S}$.

Theorem 3.2 *Let $\mathcal{S} \subseteq$ **S5** be a modal logic.*
(a) *If $\mathcal{S}$ has the property (1-ext) then for each set Φ that is $\mathcal{S}$-admissible for a theory I, $T_{I,\Phi}$ is the only consistent $\mathcal{S}$-expansion of I containing $I \cup \neg\Phi$.*
(b) *If $\mathcal{S}$ has the property (1-ext) and, in addition, $\mathcal{S}$ is characterized by a closed class of frames C, then T is an $\mathcal{S}$-expansion of I if and only if $T = T_{I,\Phi}$, for some set Φ which is $\mathcal{S}$-admissible for I.*

This theorem is a slight extension of Theorems 3.2 and 3.3 from [Shv90]. A minor modification of the original arguments from [Shv90] can be used to prove it. We omit the details here.

Theorem 3.2 (a) applies to the logics **N**, **W5**, **K**, **T**, **KD45** and **S4**, and Theorem 3.2 (b) applies to logics **K**, **T** and **S4**. In this paper, using different methods, we show that its statement holds also for the logic **N**. All these results are gathered in the following proposition.

Theorem 3.3 **(a)** *Let $\mathcal{S}$ be any of* **N**, **K**, **T**, **S4**, **K45**, *and let Φ be $\mathcal{S}$-admissible for I. Then $T_{I,\Phi}$ is the unique $\mathcal{S}$-expansion of I containing $I \cup \neg\Phi$.*
(b) *Let $\mathcal{S}$ be any of* **N**, **K**, **T**, **S4**. *Then, T is an $\mathcal{S}$-expansion of I if and only if $T = T_{I,\Phi}$, for some $\mathcal{S}$-admissible Φ.*

Theorem 3.3(b) does not hold for logics **K45** and **W5**. For example, theory $I = \{Lp \Rightarrow p\}$ has a stable expansion $T_{I,\emptyset}$, but $\emptyset$ is not **K45**-admissible for I. To deal with the case of logic **K45** we need the following notion. A set Φ, admissible for I, is said to be *propositionally admissible for I* if for each $L\psi \in \Psi$, $I \cup \neg\Phi \cup \Psi \vdash \psi$. Shvarts [Shv88, Shv90] proved the following characterization of **K45**-expansions (stable expansions).

Theorem 3.4 *T is a stable expansion (that is, a* **K45***-expansion) of I if and only if T is $T_{I,\Phi}$ for some Φ that is propositionally admissible for I.*

We apply here a similar technique to deal with the case of strict expansions that is, **W5**-expansions. Let $\Phi \subseteq I^L$, $\Psi = I^L \setminus \Phi$. The set Φ is said to be *strictly admissible for I*, if Φ is admissible for I and for each $L\psi \in \Psi$, $I \cup \neg\Phi \cup \{\varphi \Rightarrow L\varphi : L\varphi \in \Psi\} \vdash \psi$.

Theorem 3.5 *If T is consistent, then T is a strict expansion (that is, a* **W5***-expansion) of I if and only if $T = T_{I,\Phi}$ for some Φ being strictly admissible for I.*

Now we are in a position to show that the inclusions between classes of $\mathcal{S}$-expansions shown in Figure 2 are the only ones that hold in general.
Example:

1. Let $I_1 = \emptyset$. Each stable theory is an **S5** - expansion of I_1. Not every stable theory is an **S4**-expansion of I_1, or a stable expansion (**KD45**-expansion) of I_1. Thus, in general, the notions of **S5**-expansion and **S4**-expansion, and of **S5**-expansion and **KD45**-expansion are different.

2. Let $I_2 = \{L(Lp \Rightarrow LLp) \Rightarrow p\}$. It is easy to see that $E(p)$ is an **S4**-expansion of I_2 (in fact, the only **S4**-expansion of I_2). By Theorem 3.3 (b), $E(p) = T_{I_2,\Phi}$, where $\Phi = I_2^L \setminus (I_2^L \cap E(p))$. Clearly, in our case, $\Phi = \emptyset$. Consider the **T**-model $\mathcal{M} = (M, R, W)$, where $M = \{a, b, c\}$, $R = \{(a,a), (b,b), (c,c), (a,b), (b,c), (c,a)\}$, and $W(a) = W(b) = \{p\}$ and $W(c) = \emptyset$. It is easy to see that $\mathcal{M} \models I_2$ but $\mathcal{M} \not\models p$. Thus, $I_2 \not\vdash_{\mathbf{T}} p$. Consequently, $\Phi = \emptyset$ is not **T**-admissible for I_2 and $E(p)$ is not a **T**-expansion of I_2.

3. Let $I_3 = \{L(LLp \wedge L(Lp \Rightarrow p) \Rightarrow Lp) \Rightarrow p\}\}$. Using similar methods as before, we show that **K**-expansions are not, in general, **W5**-expansions.

4. Theory $I_4 = \{L(Lp \Rightarrow p) \Rightarrow p\}$ shows that **T**-expansions are not **K**-expansions.

5. Theory $I_5 = \{L(\neg L\neg Lp \Rightarrow (p \Rightarrow Lp)) \Rightarrow p\}$ shows that **W5**-expansions are not , in general, **S4**-expansions, and that **KD45**-expansions are not, in general, **S4**-expansions.

6. Let $I_6 = \{Lp \Rightarrow p\}$. Put $T = E(p)$. It is easy to see that $\Phi = \{Lp\}$ is propositionally admissible and that $T_{I_6,\Phi} = E(p)$. Thus, T is a **K45**-expansion of I_6. Consider a valuation v of $\mathcal{L}_L$ such that $v(p) = 0$, and $v(L\varphi) = 1$ if and only if $\varphi \neq p$ and $\varphi \in T$. Then, $v(I_6 \cup \{\varphi \Rightarrow L\varphi : \varphi \in T\} \cup \neg L\overline{T}) = 1$ and $v(p) = 0$. But then, T is not a **W5**-expansion of I_6.

7. Let $I_7 = \{Lp\}$. Using similar techniques as in 2 and in 6, one easily shows that $E(p)$ is a **T**-expansion of I_7 but not a **5**-expansion of I_7.

Theories given in the example indicate that the only generally true inclusion relationships between the classes of expansions shown in Figure 2 are those indicated by arrows.

Theorems 3.2, 3.4 and 3.5 imply algorithms for computing $\mathcal{S}$-expansions of finite theories for a wide class of logics $\mathcal{S}$. The algorithm we give below can be used for each logic $\mathcal{S}$ to which Theorem 3.2(b) applies and for which there exists a decision procedure for the membership problem: given a finite $I \subseteq \mathcal{L}_L$ and $\varphi \in \mathcal{L}$, does $I \vdash_{\mathcal{S}} \varphi$ hold? In particular, the algorithm applies to the logics **N, K, T** and **S4**.

Algorithm:

compute I^L;
for each $\Phi \subseteq I^L$ **do**
  $\Psi := I^L \setminus \Phi$;
  if Φ is $\mathcal{S}$-admissible **then**
    compute a finite set A such that
    $Cn(A) = Cn(I \cup \neg\Phi \cup \Psi \cup \{\psi \colon L\psi \in \Psi) \cap \mathcal{L}$;
    output $E(A)$ as an $\mathcal{S}$-expansion of I
  fi
rof

To check $\mathcal{S}$-admissibility of Φ we need a decision procedure for propositional calculus to check admissibility of Φ (many such procedures are available), and a decision procedure for logic $\mathcal{S}$, to check whether $I \cup \neg\Phi \vdash_{\mathcal{S}} \psi$, for each ψ such that $L\psi \in \Psi$. Since all the theories involved (Φ, Ψ, $I \cup \neg\Phi \cup \Psi \cup \{\psi \colon L\psi \in \Psi\}$ and $I \cup \neg\Phi$) are finite, verifying $\mathcal{S}$-admissibility can be done in a finite number of steps. For the same reason, computing the set A can be executed in finite time.

Only small changes are needed in the algorithm above to produce a method for computing all **K45**- or **W5**-expansions. One simply has to replace checking $\mathcal{S}$-admissibility of Φ by checking propositional admissibility of Φ (for **K45**-expansions) and strict admissibility of Φ for **W5**-expansions.

Another approach to computing expansions is possible. It is not directly based on the notion of admissibility. This other approach results in simpler and more efficient algorithms and will be presented in the full version of the paper.

We conclude this section with an example illustrating how our algorithm works. Suppose we want to compute all **S4**-expansions of the theory $I = \{\neg Lp \Rightarrow q, \neg Lq \Rightarrow p\}$. First, the theory I^L is computed. Clearly, $I^L = \{Lp, Lq\}$. There are four sets Φ that have to be verified for **S4**-admissibility: $\Phi_1 = \emptyset$, $\Phi_2 = \{Lp\}$, $\Phi_3 = \{Lq\}$ and $\Phi_4 = \{Lp, Lq\}$. For the set Φ_1, the condition $I \cup \neg\Phi_1 \vdash_{\mathbf{S4}} \psi$, for all $L\psi \in I^L \setminus \Phi_1$ is not met. For instance, $I \cup \neg\Phi_1 \not\vdash_{\mathbf{S4}} p$. Sets Φ_2 and Φ_3 are **S4**-admissible and generate **S4**-expansions $E(p)$ and $E(q)$. For Φ_4, it is the case that $p \in Cn(I \cup \neg\Phi_4 \cup \Psi \cup \{\psi \colon L\psi \in \Psi\})$. Therefore Φ_4 is not even admissible.

4 CONCLUSIONS

In this paper we found that the structure of the family of modal nonmonotonic logics is much simpler than that of the family of underlying modal logics. This phenomenon is explained by the fact, first observed by McDermott [McD82], that the additional tool employed by modal nonmonotonic logics, namely "negation as failure to prove", permits to prove nonmonotonically various (monotone) axiom schemata that are not provable monotonically. This phenomenon, although expected, has not been known until now.

An additional result is the demonstration of applicability of subnormal logics in the domain of knowledge representation. This seems to indicate certain incompatibility of the research in classical modal logic (where most effort has been devoted to normal logics) and the needs of knowledge representation.

Finally, expansions for a variety of modal nonmonotonic logics have been characterized and procedures to compute them described.

5 PROOFS

In this section, we give proofs of the new results of the paper.

Theorem 2.2 *Let $\mathcal{S}$ be a modal logic, $\mathbf{5} \subseteq \mathcal{S} \subseteq \mathbf{KD45}$. Let $I, T \subseteq \mathcal{L}_L$. The theory T is a stable expansion of I if and only if T is an $\mathcal{S}$-expansion of I.*

Proof: It is easy to see that if $T \subseteq \mathcal{L}_L$ is stable and consistent, then $Cn(LT \cup \neg L\overline{T})$ contains all instances of axiom schemata K, D, 4 and 5. Consider for example a formula $\psi = \neg L\varphi \vee LL\varphi$, equivalent to an instance of 4. If $\varphi \notin T$, then $\neg L\varphi \in \neg L\overline{T}$ and so $\psi \in Cn(\neg L\overline{T})$. If $\varphi \in T$, then $L\varphi \in T$ and $\psi \in Cn(LT)$. The remaining axiom schemata can be dealt with similarly.

In addition, $\neg L\overline{T} \vdash_{\mathbf{5}} LT$. Indeed, let $\varphi \in T$. Then, since T is stable and consistent, $\neg L\varphi \notin T$, $\neg L\neg L\varphi \in T$, and $\neg L\neg L\varphi \vdash_{\mathbf{5}} L\varphi$.

These observations imply that if T is stable and consistent, then

$$Cn(I \cup LT \cup \neg L\overline{T}) = Cn_{\mathcal{S}}(I \cup \neg L\overline{T}),$$

which immediately implies the assertion. □

Theorem 2.3 *Let $\mathcal{S}$ be a modal logic, $\mathbf{N} \subseteq \mathcal{S} \subseteq \mathbf{WK}$. Let $I, T \subseteq \mathcal{L}_L$. The theory T is an $\mathbf{N}$-expansion of I if and only if T is an $\mathcal{S}$-expansion of I.*

Proof: Let T be stable and consistent. Consider $\varphi, \psi \in \mathcal{L}_L$. If $\varphi \notin T$ or $(\varphi \Rightarrow \psi) \notin T$, then $L\varphi \wedge L(\varphi \Rightarrow \psi) \Rightarrow \neg L \neg L\psi \in Cn(\neg L\overline{T})$. Otherwise, $\psi \in T$ and, since T is stable and consistent, $L\psi \in T$ and $\neg L\psi \notin T$. Thus, again $L\varphi \wedge L(\varphi \Rightarrow \psi) \Rightarrow \neg L \neg L\psi \in Cn(\neg L\overline{T})$. Consequently, $Cn_{\mathbf{WK}}(I \cup \neg L\overline{T}) \subseteq Cn_{\mathbf{N}}(I \cup \neg L\overline{T})$. The converse inclusion is evident. Hence $Cn_{\mathbf{WK}}(I \cup \neg L\overline{T}) = Cn_{\mathbf{N}}(I \cup \neg L\overline{T})$, and the result follows. □

Theorem 2.4 *Let $\mathcal{S}$ be a modal logic, $\mathbf{W5} \subseteq \mathcal{S} \subseteq \mathbf{D4W5}$. Let $I, T \subseteq \mathcal{L}_L$. Then, the theory T is a strict expansion of I if and only if T is an $\mathcal{S}$-expansion of I.*

Proof: The proof is almost identical to that of Proposition 2.2. We show that if $T \subseteq \mathcal{L}_L$ is stable and consistent, then $Cn(\{\varphi \Rightarrow L\varphi : \varphi \in T\} \cup \neg L\overline{T})$ contains all instances of axiom schemata D, 4 and W5, and that $\neg L\overline{T} \vdash_{\mathbf{W5}} \{\varphi \Rightarrow L\varphi : \varphi \in T\}$. These observations imply that if T is stable and consistent, then

$$Cn(I \cup \{\varphi \Rightarrow L\varphi : \varphi \in T\} \cup \neg L\overline{T}) = Cn_{\mathcal{S}}(I \cup \neg L\overline{T}),$$

which immediately implies the assertion. □

Now we will prove our results on stratification. To this end, we need two auxiliary lemmas. Let I be strongly stratified and let $I_1 \cup \cdots \cup I_n$ be a stratification of I. For any propositional variable p, let $r(p) = 0$, if p does not occur in $c(\varphi)$ for any $\varphi \in I$. If p occurs in $c(\varphi)$ for $\varphi \in T_i$, then put $r(p) = i$. By the definition of strong stratifability, such i is unique. By $r(\varphi)$ we denote $\max\{r(p) : p \text{ occurs in } \varphi\}$. By $m(\varphi)$ we denote the maximal depth of nesting of L in φ.

Lemma 5.1 *Let I be strongly stratified, Φ propositionally admissible for I and let $\Psi = I^L \setminus \Phi$. Consider $L\psi \in \Psi$ such that $r(\psi) = r$ and $m(L\psi) = m$. Then*

$$I \cup \neg\Phi \cup$$
$$\{L\eta \in \Psi : r(\eta) < r \textit{ or } (r(\eta) = r,\ m(L\eta) < m)\} \vdash \psi.$$

Proof: Let V be any propositional valuation such that $V(I) = 1$, $V(\Phi) = 0$ and $V(L\eta) = 1$ for each $L\eta \in \Psi$ such that $r(\eta) < r$ or ($r(\eta) = r$ and $m(L\eta) < m$).

Since $\{c(\varphi) : \varphi \in I\}$ is consistent, there is a valuation W such that $W(c(\varphi)) = 1$ for each $\varphi \in I$. Define a valuation U as follows. For a propositional variable p, if $r(p) \leq r$ then put $U(p) = V(p)$. Otherwise, put $U(p) = W(p)$. For a modal atom $L\varphi$, put $U(L\varphi) = 0$, if $L\varphi \in \Phi$, and $U(L\varphi) = 1$, otherwise.

Observe that if $\varphi \in I_j$, $j > r$, then $U(c(\varphi)) = W(c(\varphi)) = 1$. Thus, $U(\varphi) = 1$. Consider $\varphi \in I_j$ with $j \leq r$. For each propositional variable p occurring in φ, $r(p) \leq r$. Hence, $U(p) = V(p)$. Let $L\alpha$ be a modal atom occurring in φ. If $L\alpha \in \Phi$, then $U(L\alpha) = 0 = V(L\alpha)$. If $L\alpha \in \Psi$ then, since $L\alpha$ occurs in the formula $\varphi \in I_j$, $r(\alpha) < j \leq r$. Thus, $U(L\alpha) = 1 = V(L\alpha)$. Consequently, $U(\varphi) = V(\varphi) = 1$. Thus, $U(I) = 1$. By the definition of U, $U(\neg\Phi \cup \Psi) = 1$. Since Φ is propositionally admissible for I, $U(\psi) = 1$.

Now, let p be a propositional atom occurring in ψ. Then, $r(p) \leq r$ and $U(p) = V(p)$. Consider a modal atom $L\alpha$ occurring in ψ. If $L\alpha \in \Psi$, then $r(\alpha) \leq r$ and $m(L\alpha) < m$. Thus, $U(L\alpha) = V(L\alpha)$. Since U and V agree on all modal atoms $L\alpha \in \Phi$, it follows that U and V agree on all atoms occurring in ψ. Consequently, $V(\psi) = 1$. This completes the proof of the lemma. □

Lemma 5.2 *Let I be strongly stratified and let $I_1 \cup \cdots \cup I_n$ be a stratification of I. Let Φ be $\mathbf{S4}$-admissible for I and define $\Psi = I^L \setminus \Phi$. Let $L\psi \in \Psi$ and $r(L\psi) = r$. Then*

$$I \cup \{\neg L\varphi : L\varphi \in \Phi, r(\varphi) < r\} \vdash_{\mathbf{S4}} \psi.$$

Proof: Assume that

$$I \cup \{\neg L\varphi : L\varphi \in \Phi, r(\varphi) < r\} \not\vdash_{\mathbf{S4}} \psi.$$

Then for some $\mathbf{S4}$-Kripke model $\mathcal{N} =< N, Q, W >$, $\mathcal{N} \models I \cup \{\neg L\varphi : L\varphi \in \Phi, r(\varphi) < r\}$, but for some $a \in N$, $\mathcal{N}, a \not\models \psi$. (Let us recall that in a Kripke model $< N, Q, W >$, N stands for a nonempty set (of *worlds*), $Q \subseteq N \times N$ is an *accessibility relation* and for each $b \in N$, $W(b)$ is the set of all propositional variables true in the world b.)

We will construct another $\mathbf{S4}$-Kripke model $\mathcal{K}$ such that $\mathcal{K} \models I \cup \neg\Phi$ and $\mathcal{K}, a \not\models \psi$. This will contradict the assumption that Φ is $\mathbf{S4}$-admissible for I, and will prove the assertion of the lemma.

First, observe that since Φ is $\mathbf{S4}$-admissible for I, $T = T_{I,\Phi}$ is a consistent, stable theory containing $I \cup \neg\Phi \cup \Psi$. Thus, there exists a $\mathbf{S4}$-Kripke model $\mathcal{M} = (M, R, V)$ such that $\mathcal{M} \models T$.

Next, since I is strongly stratified, there is a valuation U such that for every $\varphi \in I$, $U(c(\varphi)) = 1$. We use U to modify the valuation W of N as follows: for each $b \in N$ put

$$W'(b) = \{p : \quad (r(p) < r \text{ and } p \in W(b)) \text{ or } (r(p) \geq r \text{ and } U(p) = 1)\}.$$

Now we define $\mathcal{K}$ to be the concatenation of (N, Q, W') and (M, R, V), that is

$$\mathcal{K} = (N \cup M, Q \cup (N \times M) \cup R, W' \cup V).$$

Clearly, $\mathcal{K}$ is an **S4**-Kripke model.

By an *I-formula* we mean a formula constructed from elements of I^L and propositional variables by means of propositional connectives. We will show now by the induction on the complexity of a formula that for each I-formula φ such that $r(\varphi) < r$, and for each $b \in N$,

$$\mathcal{N}, b \models \varphi \quad \text{if and only if} \quad \mathcal{K}, b \models \varphi. \tag{4}$$

In the case when φ is a propositional atom p with $r(p) < r$, (4) follows easily from the fact that for each world $b \in N$, $p \in W(b)$ if and only if $p \in W'(b)$. The cases when a formula φ is of the form $\neg\varphi_1$, $\varphi_1 \vee \varphi_2$ or $\varphi_1 \wedge \varphi_2$ are easy and their discussion is omitted. Let us consider now the last case, when φ is of the form $L\gamma$. Consider $b \in N$ and assume that $\mathcal{N}, b \models L\gamma$. Then, $\mathcal{N}, c \models \gamma$ for each c such that $(b, c) \in Q$. By the induction hypothesis, $\mathcal{K}, c \models \gamma$ for each $c \in N$ such that $(b, c) \in Q$.

Since $r(L\gamma) < r$ and $\mathcal{N}, b \models L\gamma$, it follows that $L\gamma \notin \Phi$. Thus, since $\varphi = L\gamma$ is an I-formula, $L\gamma \in \Psi$. Consequently, since T is consistent, $\gamma \in T$. Thus, for each $c \in M$, $\mathcal{M}, \alpha \models \gamma$. Consequently, $\mathcal{K}, b \models L\gamma$, as required. The converse implication in (4) is evident.

The property (4) implies that $\mathcal{K}, a \not\models \psi$. Let $\varphi \in I$. Consider $b \in \mathcal{M}$. Then, $\mathcal{K}, b \models \varphi$ because $\mathcal{M}, b \models \varphi$. Assume now that $b \in N$. If $\varphi \in I_j$, $j < r$, then $\mathcal{K}, b \models \varphi$, by (4). If $\varphi \in I_j$, where $j \geq r$, then for every propositional atom p of $c(\varphi)$, $r(p) \geq r$ and, consequently, $p \in W'(b)$ if and only if $U(p) = 1$. Hence, $\mathcal{K}, b \models c(\varphi)$ which, in turn, implies that $\mathcal{K}, b \models \varphi$. Consequently, $\mathcal{K} \models I$.

Consider now $L\gamma \in \Phi$. Then $\mathcal{M} \models \neg L\gamma$. This means that for each $b \in M$, $\mathcal{K}, b \models \neg L\gamma$. In addition, it follows that for some $b_0 \in M$, $\mathcal{K}, b_0 \not\models \gamma$. Thus, by the definition of the accessibility relation of $\mathcal{K}$, $\mathcal{K}, b \models \neg L\gamma$, for each $b \in N$. Consequently, $\mathcal{K} \models \neg\Phi$. □

Now, we are ready to prove Theorem 2.7.

Theorem 2.7 *Let $I \subseteq \mathcal{L}$ be strongly stratified. For each logic $\mathcal{S}$ such that*
(a) $\mathbf{N} \subseteq \mathcal{S}$*, and*
(b) $\mathcal{S} \subseteq \mathbf{KD45}$ *or* $\mathcal{S} \subseteq \mathbf{S4}$*,*
the theory I has exactly one $\mathcal{S}$-expansion and for all these logics $\mathcal{S}$, $\mathcal{S}$-expansions of I coincide.

Proof: Assume that I is strongly stratified and Φ is propositionally admissible for I. Define $\Psi = I^L \setminus \Phi$ and let $L\psi \in \Psi$. Then, for each $L\psi \in \Psi$, where $\Psi = I^L \setminus \Phi$, we have $I \cup \neg\Phi \vdash_{\mathbf{N}} \psi$. Indeed, this claim follows easily by induction on (r, m), where $r = r(\psi)$ and $m = m(L\psi)$ — both the basis of the induction and the induction step follow from Lemma 5.1. In other words, for a strongly stratified theory I, if Φ is propositionally admissible for I then Φ is **N**-admissible for I. Consequently, each **KD45**-expansion of I is an **N**-expansion of I. The converse implication always holds thus, the classes of **N**-expansions and **KD45**-expansions coincide for strongly stratified theories. Since, by the result of [MT91], a strongly stratified theory I has a unique **KD45**-expansion, it is also the unique **N**-expansion of I.

Next we show that if T_1 and T_2 are **S4**-expansions of a strongly stratified theory I, then $T_1 = T_2$. To this end we proceed as follows. Let T_1 and T_2 be determined by sets Φ_1 and Φ_2 that are **S4**-admissible for I. Put $\Psi_i = I^L \setminus \Phi_i$, $i = 1, 2$. Let r be the smallest integer such that for some $L\psi$ with $r(L\psi) = r$, $L\psi \in (\Phi_1 \setminus \Phi_2) \cup (\Phi_2 \setminus \Phi_1)$. Without loss of generality, suppose that $L\psi \in \Phi_1 \setminus \Phi_2$. Then, $L\psi \in \Psi_2$ and, by Lemma 5.2,

$$I \cup \{\neg L\varphi : L\varphi \in \Phi_2,\ r(\varphi) < r\} \vdash_{\mathbf{S4}} \psi.$$

By the choice of r,

$$\{\neg L\varphi : \quad L\varphi \in \Phi_2,\ r(\varphi) < r\} = \{\neg L\varphi : L\varphi \in \Phi_1,\ r(\varphi) < r\}.$$

Thus, we obtain that $I \cup \neg\Phi_1 \vdash_{\mathbf{S4}} L\psi$. This is a contradiction with the consistency of $I \cup \neg\Phi_1$. Thus, $\Phi_1 = \Phi_2$ and $T_1 = T_2$.

In other words, a strongly stratified theory has at most one **S4**-expansion. Since each **N**-expansion is an **S4**-expansion, that statement of the theorem follows from our previous remarks and from Proposition 2.1. □

Theorem 3.3 (a) *Let $\mathcal{S}$ be any of* **N**, **K**, **T**, **S4**, **K45** *, and let Φ be $\mathcal{S}$-admissible for I. Then $T_{I,\Phi}$ is the unique $\mathcal{S}$-expansion of I containing $I \cup \neg\Phi$.*
(b) *Let $\mathcal{S}$ be any of* **N**, **K**, **T**, **S4***. Then, T is an $\mathcal{S}$-expansion of I if and only if $T = T_{I,\Phi}$, for some $\mathcal{S}$-admissible Φ.*

Proof: Part (a) follows from Theorem 3.2(a). The proof of part (b), for for all logics except for **N**, is given in Shvarts [Shv90].

To prove part (b) for logic **N**, one has to show that if T is an **N**-expansion of I, then $T = T_{I,\Phi}$, for some **N**-admissible set Φ. To this end, observe that T is also a **K**-expansion of T. Define $\Psi = T \cap I^L$ and $\Phi = I^L \setminus \Psi$. By (b) for the logic **K**, it follows that Φ is **K**-admissible for I. To complete the proof, it remains to show that Φ is **N**-admissible for I. We will prove a slightly stronger fact that if $I \cup \neg L\overline{T} \vdash_{\mathbf{N}} \gamma$, where $\gamma = \psi$ or $\gamma = L\psi$, for $L\psi \in \Psi$, then $I \cup \neg\Phi \vdash_{\mathbf{N}} \gamma$.

We will proceed by the induction on the length of the proof of γ in the logic **N** and from $I \cup \neg L\overline{T}$. If the length of the proof is 1, then γ is a tautology or $\gamma \in I \cup \neg L\overline{T}$. The former case is obvious. In the latter, since $\{\gamma\}^L \subseteq I^L$, it follows that $\gamma \in I \cup \neg\Phi$.

Consider now a proof P of γ (in $\mathbf{N}$ and from $I \cup \neg L\overline{T}$) of length $n > 1$. If γ is derived by means of the necessitation rule, then $\gamma = L\alpha$ and α has a shorter proof than γ. Moreover, by the definition of γ, $L\alpha \in \Psi$ or $LL\alpha \in \Psi$. In this latter case, since T is consistent and stable, it follows that $L\alpha \in \Psi$. Thus, in both cases the induction hypothesis applies to α and $I \cup \neg\Phi \vdash_{\mathbf{N}} \alpha$. Consequently, $I \cup \neg\Phi \vdash_{\mathbf{N}} L\alpha$.

Otherwise, if γ is not derived by an application of necessitation, it follows that $I \cup \neg L\overline{T} \cup X \vdash \gamma$, where X is the set of all modal atoms $L\alpha$ that appear in the proof P and were derived by necessitation. Since $\{\gamma\}^L \subseteq I^L$, it follows that $I \cup \neg\Phi \cup (X \cap I^L) \vdash \gamma$. Clearly, for each atom $L\alpha \in X$, $L\alpha \in T$. Thus, $X \cap I^L \subseteq \Psi$. Since each α such that $L\alpha \in X$ has a shorter proof than γ, by the induction hypothesis, $I \cup \neg\Phi \vdash_{\mathbf{N}} \alpha$, for $L\alpha \in X$. Consequently, $I \cup \neg\Phi \vdash_{\mathbf{N}} \gamma$. □

Theorem 3.5 *If T is consistent, then T is a* **W5**-*expansion of I if and only if $T = T_{I,\Phi}$ for some Φ being strictly admissible for I.*

Proof: Consider a set Φ strictly admissible for I. Define $I' = I \cup \{\varphi \Rightarrow L\varphi :\ L\varphi \in \Psi\}$. Then, for each $L\psi \in \Psi$,

$$I' \cup \neg\Phi \vdash \psi.$$

This implies that $I' \cup \neg\Phi \vdash_{\mathbf{W5}} \psi$. Consequently, Φ is **W5**-admissible for I'. Thus, by Theorem 3.1, $T = T_{I',\Phi}$ is a **W5**-expansion of I'. By Proposition 2.4

$$\begin{aligned} T = Cn(I \quad &\cup \quad \{\varphi \Rightarrow L\varphi :\ L\varphi \in \Psi\} \\ &\cup \quad \{\varphi \Rightarrow L\varphi :\ \varphi \in T\} \cup \neg L\overline{T}). \end{aligned}$$

But $\Psi \subseteq T_{I',\Phi} = T$. Thus, since T is consistent,

$$\{\varphi \Rightarrow L\varphi :\ L\varphi \in \Psi\} \subseteq \{\varphi \Rightarrow L\varphi :\ \varphi \in T\}.$$

Consequently,

$$T = Cn(I \cup \{\varphi \Rightarrow L\varphi :\ \varphi \in T\} \cup \neg L\overline{T}),$$

that is, T is a **W5**-expansion of I. It remains to show that $T_{I',\Phi} = T_{I,\Phi}$, but this follows immediately from the definition of $T_{I,\Phi}$ and the equality:

$$\begin{gathered} Cn(I \cup \neg\Phi \cup \Psi \cup \{\varphi :\ L\varphi \in \Psi\}) = \\ Cn(I \cup \{\varphi \Rightarrow L\varphi :\ L\varphi \in \Psi\} \cup \neg\Phi \cup \Psi \cup \{\varphi :\ L\varphi \in \Psi\}). \end{gathered}$$

To prove the "only if" part, we proceed as follows. Let T be a **W5**-expansion of I. Then T is a **K45**-expansion of I. Define $\Psi = T \cap I^L$ and $\Phi = I^L \setminus \Psi$. Then Φ is propositionally admissible for I and $T = T_{I,\Phi}$. It remains to prove that Φ is strictly admissible for I. Let $L\psi \in \Psi$. Then, $\psi \in T$ and, since T is a strict expansion, we have

$$I \cup \{\neg L\varphi : \varphi \notin T\} \cup \{\varphi \Rightarrow L\varphi : \varphi \in T\} \vdash \psi. \tag{5}$$

n order to show that Φ is strictly admissible, it is sufficient to prove that

$$I \cup \{\neg L\varphi : L\varphi \in I^L \setminus T\} \cup \{\varphi \Rightarrow L\varphi : L\varphi \in \Psi\} \vdash \psi. \tag{6}$$

Consider any propositional valuation V evaluating the premises of (6) as 1. Define a valuation W by

1. $W(L\varphi) = 0$, for $L\varphi \notin T \cup I^L$,
2. $W(L\varphi) = 1$, for $L\varphi \in T \setminus I^L$,
3. $W(\varphi) = V(\varphi)$, for all other atoms.

Since T is stable, $\varphi \in T$ if and only if $L\varphi \in T$. Thus, all the premises in (5) are evaluated as 1 by W. Hence $W(\psi) = 1$. Since $\{\psi\}^L \subseteq I^L$, $W(\psi) = V(\psi)$. Thus, for each valuation V satisfying the premises of (6), $V(\psi) = 1$. Consequently, (6) holds. Hence, Φ is strictly admissible for I. □

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