# A theory of nonmonotonic rule systems II

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## Contents



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# 1 Introduction

This is the continuation of [Marek, Nerode and Remmel, 1990]. We often refer to it as "Part I", when we quote theorems or definitions.

We continue development of the theory of nonmonotonic rule systems, as introduced in the Part I. There are three directions pursued.

First, we study extensions of "highly recursive" nonmonotonic rule systems. These are systems $\langle U, N\rangle$ where $U = \omega$, is the set of natural numbers, $N$ is a recursive collection of nonmonotonic rules, and $\langle U, N\rangle$ satisfies an additional "boundedness condition" on the collection of proof schemas. These systems are closely connected with recursively bounded $\Pi^0_1$ classes and also with the "marriage problem" for highly recursive societies. As a corollary we get a large number of facts concerning the stable semantics of logic programs.

Second, we investigate a semantics for nonmonotonic rule systems. Here the goal is to get a semantical characterizations of classes of structures associated with nonmonotonic rule systems as models of theories in $\mathcal{L}_{\infty,\omega}$. We get a semantical characterization of extensions, weak extensions, deductively closed sets and minimal deductively closed sets. When $U = \omega$, these characterizations provide sharp estimates of the arithmetical class of sets of extensions, weak extensions, sets closed under rules etc.

Third, we to investigate computation extensions, weak extensions and minimal closed sets. We apply the tableaux method to compute membership in the least fixed point of a monotonic operator.

## 2 Extensions of Highly Recursive Rule Systems

In this section we define the notions of recursive and highly recursive nonmonotonic rule systems. We show that the problem of finding an extension in a highly recursive nonmonotonic rule system is effectively equivalent to finding an infinite path through a recursive binary tree. That is, we prove that, given any highly recursive nonmonotonic rule system $\mathcal{S} =< U, N >$, there is a recursive binary tree $T_{\mathcal{S}}$ and an effective one-to-one degree-preserving correspondence between the set of extensions of $\mathcal{S}$ and the set of infinite paths through $T_{\mathcal{S}}$. Conversely, we show that given any recursive binary tree $T$, there is a highly recursive nonmonotonic rule system $\mathcal{S}_T =< U_T, N_T >$ such that there is an effective one-to-one degree of unsolvability preserving correspondence between the set of infinite paths through $T$ and the set of extensions of $\mathcal{S}_T$.

It follows from the result of [Jockusch and Soare, 1972a] that any recursively bounded $\Pi^0_1$-class can be coded as the set of infinite paths through a recursive binary tree. There have been a number of papers in the literature on the study of the set of the possible degrees of recursively bounded $\Pi^0_1$-classes. The basic equivalence between the problem of finding extensions of highly recursive nonmonotonic systems and the problem of finding infinite paths through recursive binary trees described above allows us to transfer all the results about degrees of elements of recursively bounded $\Pi^0_1$-classes to results about degrees of extensions in highly recursive nonmonotonic rule systems.

## 2.1 Paths through the Binary Trees and Extensions

To make the program outlined above precise, we first need some notation. Let $\omega = \{0, 1, 2, \ldots\}$ denote the set of natural numbers and let $<,>: \omega \times \omega \to \omega$ be some fixed one-to-one and onto recursive pairing function such that the projection functions $\pi_1$ and $\pi_2$ defined by $\pi_1(< x, y >) = x$ and $\pi_2(< x, y >) = y$ are also recursive. We extend our pairing function to code $n$-tuples for $n > 2$ by the usual inductive definition, that is $< x_1, \ldots, x_n >=< x_1, < x_2, \ldots, x_n >>$ for $n \geq 3$. We let $\omega^{<\omega}$ denote the set of all finite sequences from $\omega$ and $2^{<\omega}$ denote the set of all finite sequences of 0's and 1's. Given $\alpha =< \alpha_1, \ldots, \alpha_n >$ and $\beta =< \beta_1, \ldots, \beta_k >$ in $\omega^{<\omega}$, we write $\alpha \sqsubseteq \beta$ if $\alpha$ is initial segment of $\beta$, that is if $n \leq k$ and $\alpha_i = \beta_i$ for $i \leq n$. For the rest of this paper, we identify a finite sequence $\alpha =< \alpha_1, \ldots, \alpha_n >$ with its code $c(\alpha) =< n, < \alpha_1, \ldots, \alpha_n >>$ in $\omega$. We let 0 be the code of the empty sequence $\emptyset$. Thus, when we say a set $S \subseteq \omega^{<\omega}$ is recursive, recursively enumerable, etc., we mean the set $\{c(\alpha) : \alpha \in S\}$ is recursive, recursively enumerable, etc. A *tree* $T$ is a nonempty subset of $\omega^{<\omega}$ such that $T$ is closed under initial segments. A function $f: \omega \to \omega$ is an infinite *path* through $T$ if for all $n$, $< f(0), \ldots, f(n) >\in T$. We let $\mathcal{P}(T)$ denote the set of all infinite paths through $T$. A set $A$ of functions is a $\Pi^0_1$-class if there is a recursive predicate $R$ such that $A = \{f: \omega \to \omega : \forall_n(R((f(0), \ldots, f(n)))\}$. A $\Pi^0_1$-class $A$ is *recursively bounded* if there is a recursive function $g: \omega \to \omega$ such that $\forall_{f \in A} \forall_n (f(n) \leq g(n))$. It is not difficult to see that if $A$ is a $\Pi^0_1$-class, then $A = \mathcal{P}(T)$ for some recursive tree $T \subseteq \omega^{<\omega}$. We say that a tree $T \subseteq \omega^{<\omega}$ is *highly recursive* if $T$ is a recursive, finitely branching tree such that there is a recursive procedure which, given $\alpha =< \alpha_1, \ldots, \alpha_n >$ in $T$ produces a canonical index of the set of immediate successors of $\alpha$ in $T$, that is, produces a canonical index of $\{\beta =< \alpha_1, \ldots, \alpha_n, k >: \beta \in T\}$. Here we say the canonical index, $can(X)$, of the finite set $X = \{x_1 < \ldots < x_n\} \subseteq \omega$ is $2^{x_1} + \ldots + 2^{x_n}$ and the canonical index of $\emptyset$ is 0. We let $D_k$ denote the finite set whose canonical index is $k$, that is $can(D_k) = k$. It is then the case that if $A$ is a recursively bounded $\Pi^0_1$-class, then $A = \mathcal{P}(T)$ for some highly recursive tree $T \subseteq \omega^{<\omega}$, see [Jockusch and Soare, 1972a]. We note that if $T$ is a tree contained in $2^{<\omega}$, then $\mathcal{P}(T)$ is a collection of $\{0, 1\}$-valued functions and by identifying each $f \in \mathcal{P}(T)$ with the set $A_f$, $A_f = \{x: f(x) = 1\}$ of which $f$ is the characteristic function, we can think of $\mathcal{P}(T)$ as a $\Pi^0_1$ class of sets.

Next we need to define the notions of recursive and highly recursive nonmonotonic rule systems $\mathcal{S} =< U, N >$. For the rest of this section we shall assume that $U \subseteq \omega$ and we shall identify a rule $r = \frac{\alpha_1, \ldots, \alpha_n : \beta_1, \ldots, \beta_m}{\varphi}$ in $N$ with its code $c(r) =< k, l, \varphi >$ where $D_k = \{\alpha_1, \ldots, \alpha_n\}$ and $D_l = \{\beta_1, \ldots, \beta_m\}$. In this way, we can think of $N$ as a subset of $\omega$. We say that $\mathcal{S} =< U, N >$ is *recursive* if $U$ and $N$ are recursive subsets of $\omega$. To define the notion of a highly recursive nonmonotonic rule system $\mathcal{S} =< U, N >$, we must first introduce the concept of a *proof scheme* for $\varphi$ in $< U, N >$. An (annotated) proof scheme for $\varphi$ is a finite sequence

$$p =<< \varphi_0, r_0, can(G_0) >, \ldots, < \varphi_m, r_m, can(G_m) >> \quad (1)$$

such that $\varphi_m = \varphi$ and

(1) If $m = 0$ then:

(a) $\varphi_0$ is an axiom (that is there exists a rule $r \in N, r = \frac{:}{\varphi_0}$), $r_0 = r$, and $G_0 = \emptyset$

or

(b) $\varphi$ is a conclusion of a rule $r = \frac{:\beta_1,\ldots,\beta_r}{\varphi}$, $r_0 = r$, and $G_0 = \{\beta_1, \ldots, \beta_r\}$,

(2) $m > 0$, $<< \varphi_i, r_i, can(G_i) >>_{i=0}^{m-1}$ is a proof scheme of length $m$ and $\varphi_m$ is a conclusion of $r = \frac{\varphi_{i_0},\ldots\varphi_{i_s}:\beta_1,\ldots,\beta_r}{\varphi_m}$ where $i_0, \ldots, i_s < m$, $r_m = r$, and $G_m = G_{m-1} \cup \{\beta_1, \ldots, \beta_r\}$

The formula $\varphi_m$ is called the *conclusion* of $p$ and denoted by $cln(p)$, the set $G_m$ is called the *support* of $p$ and denoted by $supp(p)$.

The idea behind this concept is this: given an $S$-derivation in the system $< U, N >$, say, $p$, it uses some negative information about $S$ to ensure that the restraints of rules that were used are outside of $S$. But this negative information is finite, that is, it involves a finite subset of complement of $S$. Thus, there exists a finite subset $G$ of complement of $S$, such that as long as $G \cap S_1 = \emptyset$, $p$ is an $S_1$ derivation as well. In the notion of proof scheme we capture this finitary character of $S$-derivation.

A proof scheme with the conclusion $\varphi$ may include a number of rules irrelevant to the enterprise of deriving $\varphi$. There is a natural preordering $\prec$ on proof schemes namely we say that $p \prec p_1$ if every rule appearing in $p$ appears in $p_1$ as well. The relation $\prec$ is not a partial ordering, and it is not a partial ordering if we restrict ourselves to proof schemes with a fixed conclusion $\varphi$. Yet it is a well-founded relation, namely, for every proof scheme $p$ there exists a proof scheme $p_1 \prec p$ such for every $p_2$, if $p_2 \prec p_1$ then $p_1 \prec p_2$. Moreover we can, if desired, require the conclusion of $p_1$ to be the same as that of $p$.

We also set $p \sim p_1 \equiv (p \prec p_1 \wedge p_1 \prec p)$ and see that $\sim$ is an equivalence relation and that its cosets are finite.

We say that the system $< U, N >$ is locally finite if for every $\varphi \in U$ there are finitely many $\prec$-minimal proof schemes with conclusion $\varphi$. This concept is motivated by the fact that, for locally finite systems, for every $\varphi$ there is a finite set of derivations $Dr_\varphi$, such that all the derivations of $\varphi$ are inessential extensions of derivations in $Dr_\varphi$. That is, if $p$ is a derivation of $\varphi$, then there is a derivation $p_1 \in Dr_\varphi$ such that $p_1 \prec p$. Finally, we say that $\mathcal{S}$ is *highly recursive* if $\mathcal{S}$ is recursive, locally finite, and the map $\varphi \mapsto can(Dr_\varphi)$ is partial recursive, that is, there exists an effective procedure which, given any $\varphi \in U$, produces a canonical index of the set of all $\prec$-minimal proof schemes with conclusion $\varphi$. We let $\mathcal{E}(\mathcal{S})$ denote the set of *extensions* of $\mathcal{S}$.

Formally, when we say that there is an effective, one-to-one degree preserving correspondence between the set of extensions $\mathcal{E}(\mathcal{S})$ of a highly recursive nonmonotonic rule system $\mathcal{S} =< U, N >$ and the set of infinite paths $\mathcal{P}(T)$ through a highly recursive tree $T$, we mean that there are indices $e_1$ and $e_2$ of oracle Turing machines such that

(i) $\forall_{f \in \mathcal{P}(T)} \{e_1\}^{gr(f)} = E_f \in \mathcal{E}(\mathcal{S})$,

(ii) $\forall_{E \in \mathcal{E}(\mathcal{S})} \{e_2\}^E = f_E \in \mathcal{P}(T)$, and

(iii) $\forall_{f \in \mathcal{P}(T)} \forall_{E \in \mathcal{E}(\mathcal{S})} (\{e_1\}^{gr(f)} = E \text{ if and only if } \{e_2\}^E = f)$.

where $\{e\}^B$ denotes the function computed by the $e^{\text{th}}$ oracle machine with oracle $B$. We also write $\{e\}^B = A$ for a set $A$ if $\{e\}^B$ is a characteristic function of $A$, and for a function $f: \omega \rightarrow \omega$, $gr(f) = \{< x, f(x) >: x \in \omega\}$. Condition (i) says that the branches of the tree $T$ uniformly produce extensions (via an algorithm with index $e_1$), and condition (ii) says that extensions of $\mathcal{S}$ uniformly produce branches of the tree $T$ (via an algorithm with index $e_2$). Condition (iii) asserts that if $\{e_1\}^{gr(f)} = E_f$ then $f$ is Turing equivalent to $E_f$. In what follows, we shall not explicitly construct the indices $e_1$ and $e_2$ but it will be clear that such indices exist in each case.

**Theorem 2.1** *Given a highly recursive nonmonotonic rule system* $\mathcal{S} =< U, N >$*, there is a highly recursive tree* $T \subseteq 2^{<\omega}$ *such that there is an effective one-to-one degree preserving correspondence between* $\mathcal{E}(\mathcal{S})$ *and* $\mathcal{P}(T)$*.*

Proof: First of all, we can assume that $U = \omega$. For if $U \subset \omega$, we simply consider the system $< \omega, N >$. Ther is no harm done by this assumption since if $\varphi \in \omega \setminus U$, then $\varphi$ is not a conclusion of any rule $r$ in $N$, so that the set of minimal derivations of $\varphi$, $Dr_\varphi$, is empty. If $\varphi \in U$, then the set of minimal derivations for $\varphi$ with respect to $< \omega, N >$ is the same as the set of minimal derivations for $\varphi$ with respect to $< U, N >$. Thus, since $U$ is a recursive set, it easily follows that $< \omega, N >$ is a highly recursive nonmonotonic rule system. Moreover, since $\varphi \in \omega \setminus U$ is also not a premise or a restraint in any rule in $N$, it follows that $E$ is an extension of $< \omega, N >$ if and only if $E$ is an extension of $< U, N >$. Thus assume that $U = \omega$ and let $Dr_i$ denote the finite set of $\prec$-minimal derivations of $i$. Let $n(i)$ denote the largest $j$ such that $j$ occurs in either a premise, or a restraint or is the conclusion of some rule in a derivation in $Dr_i$. By assumption the map assigning to $i$ the value $can(Dr_i)$ is recursive, so that the map $i \mapsto n(i)$ is also a recursive function. The import of $n(i)$ is as follows. For any $E \subseteq \omega$, to decide if $i \in C_E(\emptyset)$, we only need to know $E$ up to $n(i)$. That is, since only those $j \leq n(i)$ can be involved in any minimal derivations of $i$, it will be the case that if $E, F \subseteq \omega$ and $E \cap \{j : j \leq n(i)\} = F \cap \{j : j \leq n(i)\}$, then

$i \in C_E(\emptyset)$ if and only if $i \in C_F(\emptyset)$. Moreover, if we know $E \cap \{j : j \leq n(i)\}$, then we can effectively decide if $i \in C_E(\emptyset)$.

We shall build a recursive tree $T \subseteq 2^{<\omega}$ such that $f \in \mathcal{P}(T)$ if and only if $f = \chi(E)$ for some $E \in \mathcal{E}(<\omega, N>)$. That is, $f$ is a characteristic function of an extension. Note that any recursive tree $T \subseteq 2^{<\omega}$ is automatically highly recursive so $T$ will be the highly recursive tree required by our theorem. Our idea is to start with the full binary tree $B_\omega = 2^{<\omega}$, and then prune it to get $T$. We think of each $\sigma = <\sigma_1, \ldots, \sigma_k>$ in $B_\omega$ as specifying a finite set $S_\sigma = \{i - 1 : \sigma_i = 1\}$. We put $\sigma$ into $T$ if and only if for all $i \leq k = lh(\sigma)$ with $n(i) \leq k-1$ the following conditions (a) and (b) are satisfied:

(a) If $i \in S_\sigma$, then there is a derivation $p = << \varphi_0, r_0, can(g_0)>, \ldots, <\varphi_m, r_m, can(g_m)>>$ in $Dr_i$ as in (1) such that $\varphi_m = i$ and $g_m \subseteq \{1, \ldots, k\} \setminus S_\sigma$.

(b) If $i \notin S_\sigma$, then there is no such derivation $p \in Dr_i$.

Note that because the maps $i \mapsto can(Dr_i)$ and $i \mapsto n(i)$ are recursive, we can effectively decide if $\sigma \in T$. Moreover, it is easy to see that $\sigma \notin T$ and $\sigma \subseteq \tau$ then $\tau \notin T$ so that $T$ is a recursive tree.

Now suppose that $E \subseteq \omega$ and $\chi(E)$ is its characteristic function. If $E$ is not an extension of $<\omega, N>$, then $E \neq C_E(\emptyset)$ so there exists some $i$ such that either $i \in E \setminus C_E(\emptyset)$ or $i \in C_E(\emptyset) \setminus E$. Let $\sigma = <\chi(E)(0), \ldots, \chi(E)(n(i))>$ so that $S_\sigma = E \cap \{0, \ldots, n(i)\}$. If $i \in E \setminus C_E(\emptyset)$, then $\sigma$ fails to satisfy criterion (a) of our definition for $\sigma$ to be in $T$. Similarly, if $\sigma \in C_E(\emptyset) \setminus E$, then $\sigma$ fails to satisfy condition (b), thus $\sigma \notin T$ and $\chi(E) \notin \mathcal{P}(T)$. If $E$ is an extension of $<\omega, N>$, then it is easy to see that every $\sigma$ of the form $<\chi(E)(0), \ldots, \chi(E)(n)>$ does meet both criteria to be in $T$. Hence $\mathcal{P}(T) = \{\chi(E) \colon E$ is an extension of $< \omega, N >\}$ as desired. $\square$

We can derive several immediate consequences about the degrees of extensions in highly recursive nonmonotonic rule systems from Theorem 2.1 based on results of [Jockusch and Soare, 1972a]. For any set $A \subseteq \omega$, as usual let $A' = \{e \colon \{e\}^A(e)$ is defined$\}$ denote the jump of $A$ and $0'$ denote the jump of the empty set $\emptyset$. We write $A \leq_T B$ if $A$ is Turing reducible to $B$ and $A \equiv_T B$ if $A \leq_T B$ and $B \leq_T B$. We say that $A$ is *low* if $A' \equiv_T 0'$. Thus $A$ is low if the jump of $A$ is as small as possible with respect to Turing degrees.

**Corollary 2.2** *Let $\mathcal{S} = < U, N >$ be a highly recursive nonmonotonic rule system such that $\mathcal{E}(\mathcal{S}) \neq \emptyset$. Then*

*(i) There exists an extension $E$ of $\mathcal{S}$ such that $E$ is low and*

*(ii) If $\mathcal{S}$ has only finitely many extensions, then every extension $E$ of $\mathcal{S}$ is recursive.*

Proof: (i) The Jockusch–Soare Basis Theorem for recursively bounded $\Pi^0_1$-classes ([Jockusch and Soare, 1972a]) says that every not empty, recursively bounded $\Pi^0_1$-class $C$ contains a function $f$ such that $f' = 0'$. Thus given $\mathcal{S}$, we can construct a highly recursive tree $T \subseteq 2^{<\omega}$ such that $\mathcal{P}(T) = \{\chi(E) \colon E \in \mathcal{E}(\mathcal{S})\}$. Since $\mathcal{P}(T)$ is a recursively bounded $\Pi^0_1$-class, there exists an $E \in \mathcal{E}(\mathcal{S})$ such that $E' \equiv_T \chi(e') \equiv_T 0'$.

For (ii), we use a similar argument plus the fact, also due to Jockusch and Soare [1972a], that if a recursively bounded $\Pi^0_1$-class $C$ has only finitely many elements, then every $f \in C$ is recursive. $\square$

We shall discuss now applications of the highly recursive rule systems to studies of particular cases of examples considered in Section 5, part I.

## 2.2 Highly Recursive Marriage Problems

Consider the Marriage problem investigated in Section 5, part I. We say that a society $\mathcal{S} =< B, G, K >$ in which every boy knows only finitely many girls is *highly recursive* if $B$ and $G$ are recursive subsets of $\omega$, $K$ is a recursive relation, and there is a recursive procedure which, given any $b \in B$, produces a canonical index of the finite set of girls known by $b$. If, in addition, each girl $g \in G$ knows only finitely many boys in $B$ and there is a recursive procedure which, given any $g \in G$, produces a canonical index of the finite set of boys known by $g$, then we say that $\mathcal{S}$ is *symmetrically highly recursive.* Now, it is easy to see that if $\mathcal{S}$ is a highly recursive society and we identify $Mbg$ with its code $c(Mbg) =< b, g >$, then $< U(\mathcal{S}), N(\mathcal{S}) >$ is a recursive nonmonotonic rule system. However, as it stands, $< U(\mathcal{S}), N(\mathcal{S}) >$ is not a highly recursive rule system because of the rules of the form (6), part I, which allow for infinitely many minimal derivations of $\varphi$. For suppose that $b_1 \neq b_2$, $G_{b_1} = \{g_1, \ldots, g_k\}$ is the set of girls known by $b_1$, $G_{b_2} = \{g'_1, \ldots, g'_l\}$ is the set of girls known by $b_2$, and $g_1 = g'_1 = g$. Then the following is a minimal derivation for any $\varphi \in U(\mathcal{S}) - \{Mb_1g, Mb_2g\}$.

$$<< Mb_1g, \frac{: Mb_1g_2, \ldots, Mb_1g_k}{Mb_1g}, \{Mb_1g_i : i = 2, \ldots, k\} > \tag{2}$$

$$< Mb_2g, \frac{: Mb_2g'_2, \ldots, Mb_2g'_l}{Mb_2g}, \{Mb_1g_i, Mb_2g'_j : i = 2, \ldots, k, j = 2, \ldots, l\} >$$

$$< \varphi, \frac{Mbg_1g, Mb_2g :}{\varphi}, \{Mb_1g_i, Mb_2g'_j : i = 2, \ldots, k, j = 2, \ldots, l\} >>$$

Thus, if we have infinitely many $b_1, b_2$ and $g$ with $Mb_1g, Mb_2g \in U(\mathcal{S})$, then there will be infinitely many minimal derivations of $\varphi$ for any $\varphi$. However, if $\mathcal{S}$ is symmetrically highly recursive, then a slight modification of the rules (6), part I, will produce a highly recursive nonmonotonic rule system with the same extensions. That is, suppose $\mathcal{S} =< B, G, K >$ is a symmetrically highly recursive society which has a proper marriage. Let $U(\mathcal{S}) = \{Mbg: b \in B, g \in G, \text{ and } < b, g > \in K\}$ as before. Now suppose $b_1 \neq b_2$ are boys who know the same girl. Then clearly one of the boys $b_1$ and $b_2$ must know at least two girls, since otherwise there can be no proper marriage for $\mathcal{S}$. Since $\mathcal{S}$ is highly recursive, $B_2 = \{b \in B$: $b$ knows at least two girls $\}$ is a recursive set. Now consider rules of the form

$$\frac{Mb_1g, Mb_2g:}{Mb_3g'} \tag{3}$$

for all $b_1, b_2 \in B, g \in G$ where $b_3 = max(\{b_1, b_2\} \cap B_2)$ and $g' \neq g$.

Let $\overline{U(\mathcal{S})} = U(\mathcal{S})$ and let $\overline{N(\mathcal{S})}$ consists of the rules of the form (5), part I, and (3). Then we have the following

**Theorem 2.3** *Let $\mathcal{S} =< B, G, K >$ be a symmetrically highly recursive society such that $\mathcal{S}$ has a proper marriage. Then*

*(i) $< \overline{U(\mathcal{S})}, \overline{N(\mathcal{S})} >$ is a highly recursive nonmonotonic rule system and*

*(ii) $E$ is an extension of $< \overline{U(\mathcal{S})}, \overline{N(\mathcal{S})} >$ if $M_E = \{< b, g >: Mbg \in E\}$ is a proper marriage of $\mathcal{S}$.*

Proof: Clearly $< \overline{U(\mathcal{S})}, \overline{N(\mathcal{S})} >$ is a recursive nonmonotonic rule system. To see that $< \overline{U(\mathcal{S})}, \overline{N(\mathcal{S})} >$ is locally finite and highly recursive we must analyze the minimal proof schemes for $< \overline{U(\mathcal{S})}, \overline{N(\mathcal{S})} >$. Let $B = \{b_0, b_1 \ldots\}$ be the increasing enumeration. We shall prove by induction on $k$ that each $Mb_k g$ is the conclusion of only finitely many minimal proof schemes and that we can find all such minimal proof schemes. So assume that we have a minimal derivation of $Mb_0 g$, say

$p = << \varphi_0, r_0, can(G_0) >, \ldots, < \varphi_m, r_m, can(G_m) >>$, as described in (1). Now, either $r_m$ is of the form

$$\frac{: Mb_0 g_1, \ldots, \widehat{Mb_0 g_k}, \ldots, Mb_0 g_n}{Mb_0 g_k} \tag{4}$$

where $g = g_k$, in which case the minimality of $p$ forces $m = 0$ or $r_m$ is of the form (3) in which case $r_m$ must be of the form

$$\frac{Mb_0 g', Mb_i g' :}{Mb_0 g} \tag{5}$$

where $b_i$ knows a single girl $g'$ and $g' \neq g$. But then $<< \varphi_0, r_0, can(G_0) >, \ldots, < \varphi_{m-1}, r_{m-1}, can(G_{m-1}) >>$ must be a subsequence of an interweaving of minimal proof schemes for $Mb_0 g'$ and $Mb_i g'$. But note that since $b_i$ knows only one girl, $Mb_i g'$ is never a conclusion of any rule of the form (3). Thus $Mb_i g'$ has a single proof, namely as the conclusion of a single axiom $\frac{:}{Mb_i g'}$. Moreover, since $\mathcal{S}$ has a proper marriage, there cannot be two boys in $B \setminus B_2$ who know $g'$ so that $b_i$ is completely determined by $g'$. Thus if we delete those $\varphi_k$ where $r_k = \frac{:}{Mb_i g'}$, we must be left with a minimal proof scheme for $Mb_0 g'$ in which $Mb_0 g$ does not occur as the conclusion of a rule. But now we can repeat the argument. That is, either $Mb_0 g'$ is derived by a single application of a rule of the form (4), or $Mb_0 g'$ is derived from a rule of the form

$$\frac{Mb_0 g'', Mb_i g'' :}{Mb_0 g'}$$

where $g'' \neq g'$ and $b_j$ knows only $g''$. Then once again we can strip off the last element of the sequence plus the entry corresponding to an axiom $\frac{:}{Mb_j g''}$ and we will be left with a minimal derivation of $Mb_0g''$ in which neither $Mb_0g'$ nor $Mb_0g$ is the conclusion of any rule. Since $b_0$ knows only finitely many girls, it is easy to see that there can be only finitely many minimal proof schemes for $Mb_0g$. Moreover, since $\mathcal{S}$ is symmetrically highly recursive, for each girl $g$ known by anybody in $B$, we can decide if there is a boy in $B \setminus B_2$ whom $g$ knows. Then it should be clear from our analysis that from the set of boys $b^\star \in B \setminus B_2$ who know girls known by $b$, we can effectively put together all minimal proof schemes for $Mb_0g$. Thus we can find the canonical index of the set of all possible minimal proof schemes $p$ such that $cln(p) = Mb_0g$.

Now assume by induction that for all $j < k$, there are only finitely many minimal proof schemes $p$ with $cln(p) = Mb_jg$ for any $g$ and that we can effectively find the canonical index of all such proof schemes. Now suppose that $p =<< \varphi_0, r_0, can(G_0) >, \ldots, < \varphi_m, r_m, can(G_m) >>$ is a minimal proof scheme with $cln(p) = Mb_kg$ for some $g$. In this case we have three possibilities,

(i) $r_m = \frac{: Mb_k g_1, \ldots, \widehat{Mb_k g_l}, \ldots, Mb_k g_n}{Mb_k g_l}$

with $g_l = g$,

(ii) $r_m = \frac{Mb_k g', Mb_i g' :}{Mb_k g}$

where $g' = g$ and $b_i$ knows only $g'$, or

(iii) $r_m = \frac{Mb_k g', Mb_i g' :}{Mb_k g}$

where $g' \neq g$, $b_i$ knows more than one girl and hence $i < k$.

In case (i), $m = 0$. In case (ii), we can get a shorter proof scheme which proves $Mb_kg'$ and which does not involve $Mb_kg$ as a conclusion, just as we did for $b_0$. In case (iii) we must again conclude that $<< \varphi_0, r_0, can(G_0) >, \ldots, < \varphi_{m-1}, r_{m-1}, can(G_{m-1}) >>$ must be a subsequence of an interweaving of minimal proof schemes for $Mb_kg'$ and $Mb_ig'$. By induction, there are only finitely many proof schemes for $Mb_ig'$. Moreover, we can thin our present proof scheme to a minimal proof scheme for $Mb_kg'$ which does not involve $Mb_kg$ as the conclusion of any rule. Then we can apply the same analysis over again and in cases (ii) and (iii) we can again produce a minimal derivation of some $Mb_kg''$ in which neither $Mb_kg$ nor $Mb_kg'$ appears as the conclusion of any rule. Continuing in this way we see that, since $b_k$ knows only finitely many girls, and in each case where we use either rules of the form (ii) or (iii) there are only finitely many choices for $b_i$ and $g'$ and only finitely many minimal derivations of $Mb_ig'$, there can be only finitely many proof schemes for $Mb_kg$. Moreover, using (a) the fact that $\mathcal{S}$ is symmetrically highly recursive and (b) our inductive hypothesis, one can see that we can effectively produce the canonical index of the set of all minimal proof schemes $p$ with $cln(p) = Mb_kg$. Thus $< \overline{U(\mathcal{S})}, \overline{N(\mathcal{S})} >$ is a highly recursive nonmonotonic rule system.

Next, suppose that $E$ is an extension of $< \overline{U(\mathcal{S})}, \overline{N(\mathcal{S})} >$ . Then we claim that it can never be the case that a derivation of $\varphi \in C_E(\emptyset)$ can employ a rule of the form (3). That is, suppose that there is a derivation $p =<< \varphi_0, r_0, can(G_0) >, \ldots, < \varphi_m, r_m, can(G_m) >>$ (as in (1)) where $G_m \cap E = \emptyset$ and $r_m = \frac{Mb_ig', Mb_kg' :}{Mb_kg}$. Then one can see from our analysis of minimal proof schemes that at some point in the derivation $p$, we must derive $Mb_kg_j$ for some $g_j$ by using a rule of the form $r_j = \frac{:Mb_kg_1,\ldots,\widehat{Mb_kg_j},\ldots,Mb_kg_n}{Mb_kg_j}$. Moreover, $b_k$ must know at least two girls because otherwise $Mb_kg$ could not be the conclusion of a rule of the form (3). Since $E$ is an extension, we conclude that $Mb_kg, Mb_kg' \in E$. But since $g \neq g'$, either $Mb_kg$ or $Mb_kg'$ would block the application of the rule $r_j$. Thus there can be no such derivation $p$. Now we argue exactly as we did in Theorem 5.1, part I, that the rules of the form (5), part I, ensure that for each $b \in B$, there must be exactly one girl $g$ such that $Mbg \in E$. Thus $M_E$ is defined on all $B$. Since we can never use a rule of the form (3) in a derivation from $E$, we can never have a $g \in G$ and $b_1 \neq b_2$ in $B$, with $Mb_1g$ and $Mb_2g$ in $E$. Thus $M_E$ is one-to-one. Finally, by construction, $Mbg \in \overline{U(\mathcal{S})}$ implies $< b, g > \in K$ so that $M_E$ is a proper marriage.

As concerns the converse implication, that is that proper marriages generate extensions of $< \overline{U(\mathcal{S})}, \overline{N(\mathcal{S})} >$, we argue exactly as we did in Theorem 5.1, part I. $\Box$.

The same modification can be applied to the symmetric marriage problem. That is, suppose that $\mathcal{S} =< B, G, K >$ is a symmetrical highly recursive society. Let $\overline{U_{sym}(\mathcal{S})} = \overline{U(\mathcal{S})}$, and $\overline{N_{sym}(\mathcal{S})}$ be all the rules of form (5), part I, (3), or (6), part I, Then we have the following.

**Theorem 2.4** *Let $\mathcal{S} =< B, G, K >$ be a symmetrically recursive society such that $\mathcal{S}$ has a proper symmetric marriage. Then*

*(i) $< \overline{U_{sym}(\mathcal{S})}, \overline{N_{sym}(\mathcal{S})} >$ is a highly recursive nonmonotonic rule system and*

*(ii) $E$ is an extension of $< \overline{U_{sym}(\mathcal{S})}, \overline{N_{sym}(\mathcal{S})} >$ if and only if the mapping $M_E = \{< b, g >: Mbg \in E\}$ is a proper marriage of $\mathcal{S}$.*

Proof: For (i), we can use essentially the same proof as we did for Theorem 2.3 (i). The only difference is that we now have one more possible way to derive $Mbg$, namely via a rule of the form (7), part I.

$$\frac{: Mb_1g, \ldots, \widehat{Mb_k}g, \ldots, Mb_ng}{Mb_kg} \tag{6}$$

where $b_k = b$. But if we use a rule as in (6) to derive $Mbg$, then the minimal proof scheme is just $< Mbg, r, can(\{Mbg_1, \ldots, \widehat{Mb_k}g, \ldots, Mb_ng\}) >$.

However, since $b$ knows only finitely many girls, there are only finitely many rules of the form (6) which can be used to derive $Mbg$. Since we can effectively find the set of girls known by $b$, we can effectively find the set of rules of the form (6) which can be used to derive $Mbg$. It is then easy to see that despite these extra possibilities for deriving $Mbg$, we can use the same argument as in Theorem 2.3 to show that $< \overline{U_{sym}(\mathcal{S})}, \overline{N_{sym}(\mathcal{S})} >$ is locally finite and highly recursive.

For (ii), we must establish that if $E$ is an extension of $< \overline{U_{sym}(\mathcal{S})}, \overline{N_{sym}(\mathcal{S})} >$ then we can never use a rule of form (3). That is, suppose that there is a derivation $p =<< \varphi_0, r_0, can(G_0) >, \ldots, < \varphi_m, r_m, can(G_m) >>$ (as in (1)) where $G_m \cap E = \emptyset$ and $r_m = \frac{Mb_ig', Mb_kg' :}{Mb_kg}$. Then the same analysis as used in Theorem 2.3 will allow us to show that at some point in the derivation we must use a rule of the form

$$r = \frac{: Mb_kg_1, \ldots, \widehat{Mb_k}g_j, \ldots, Mb_kg_n}{Mb_kg_j} \tag{7}$$

or of the form:

$$r = \frac{: Mb_1g_j, \ldots, \widehat{Mb_k}g_j, \ldots, Mb_ng_j}{Mb_kg_j} \tag{8}$$

where $g' = g_j \neq g$ or for some $l < k$, we used a rule of the form (7)

$$r = \frac{Mb_l g'', Mb_k g'':}{Mb_k g_j}. \tag{9}$$

and $g'' \neq g_j$, $g'' \neq g$.

But note that we can not use (7) because of the assumption that $g' \neq g$ and $Mb_k g'$ and $Mb_k g$ can be derived from $E$ and hence are in $E$ if $E$ is an extension. But then one of $Mb_k g'$ or $Mb_k g$ would block the application of (7) for $C_E(\emptyset)$. Similarly (8) is also blocked. That is, it must be the case that $b_l \neq b_k$ and $Mb_l g_j$ and $Mb_k g_j$ are in $E$. Hence one of $Mb_l g_j$ and $Mb_k g_j$ would block (8). Of course, if we use rule (9) in the derivation, then we can repeat our analysis on a shorter derivation. In this way we can show by induction that there is such $p$. Since we never use a rule of the form (3) in a derivation for $C_E(\emptyset) = E$, we can argue just as in Theorem 2.3 that $M_E$ must be a proper marriage. Moreover, it is easy to see that rules of the form (7), part I, force that for each girl $g$, there must be at least one boy $b$ such that $Mbg \in E$. Thus $M_E$ maps $B$ onto $G$ and $M_E$ is a proper symmetric marriage.

The argument that $M_E$ a proper symmetric marriage implies that $E$ is an extension is similar to the argument in Theorem 5.1, part I, and will be left to the reader. $\square$

There are similar modifications which are required for the remaining examples of Section 5, part I. In what follows we shall briefly describe what is required to make the rule system highly recursive in each case and state the results without proof.

## 2.3 Proper $k$-colorings of graphs for $k \geq 2$

A locally finite graph is said to be *highly recursive* if $V$ and $\{< x, y >: \{x, y\} \in E\}$ are recursive subsets of $\omega$ and there is an effective procedure which, given $x \in V$ produces $can(Nb(x))$.

If $\mathcal{G}$ is a highly recursive graph and we identify $Cxi$ with its code $< x, i >$, then the nonmonotonic rule system $< U(\mathcal{G}), N(\mathcal{G}) >$ of Section 5.2, part I, is recursive but not highly recursive. $< U(\mathcal{G}), N(\mathcal{G}) >$ is not highly recursive because the rules of the form (9), part I, allow for infinitely many minimal proof schemes $p$ with $cln(p) = \varphi$ for any $\varphi \in U(\mathcal{S})$. We replace the rules of the form (9), part I, by the following set of rules.

$$\frac{Cxi, Cyi:}{Czj}. \tag{10}$$

for all $x, y$, and $i$ such that $\{x, y\} \in E$, where $z = max(x, y)$ and $j \in \{1, \ldots, k\} \setminus \{i\}$.

We let $\overline{U(\mathcal{G})} = U(\mathcal{G})$ and $\overline{N(\mathcal{G})}$ be the set of all rules of the form (8), part I, and (10). Then, by a proof which is very similar to that of Theorem 2.3, we can prove the following.

**Theorem 2.5** *Let $k \geq 2$ and $\mathcal{G} =< V, E >$ be a highly recursive graph. Then*

*(i) $< \overline{U(\mathcal{G})}, \overline{N(\mathcal{G})} >$ is a highly recursive nonmonotonic rule system and*

*(ii) A subset $E \subseteq \overline{U(\mathcal{G})}$ is an extension of $< \overline{U(\mathcal{G})}, \overline{N(\mathcal{G})} >$ if and only if $C_E = \{< x, i >: Cxi \in E\}$ is a proper $k$-coloring of $\mathcal{G}$.*

**Example 2.1** *Chain Covers of Partially Ordered Sets.*

A partially ordered set $\mathcal{P} =< D, \leq_D>$ is *recursive* if $D$ is a recursive subset of $\omega$ and $\leq_D$ is a binary recursive relation. If $\mathcal{P}$ is a recursive partially ordered set and we identify $Cxi$ with its code $< x, i >$ then the nonmonotonic rule system $< U(\mathcal{P}), N(\mathcal{P}) >$ of Section 5.3, part I, is recursive but not highly recursive (if the width of $\mathcal{P}$ is at least 2). $< U(\mathcal{P}), N(\mathcal{P}) >$ is not highly recursive because the rules of the form (11), part I, allow for infinitely many minimal proof schemes with $cln(p) = \varphi$ for all $\varphi \in U(\mathcal{P})$.

Therefore we replace the rules (11), part I, by the following set of rules.

$$\frac{Cxi, Cyi:}{Czj} \tag{11}$$

for all $x, y$, and $i$ such that $x \mid y$ and $j \in \{1, \ldots, w\} \setminus \{i\}$, $z = max(x, y)$.

Then just as in example 2.3 we get

**Theorem 2.6** *Let $w \geq 2$ and let $\mathcal{P} =< D, \leq_D>$ be a recursive partially ordered set of width $w$. Then*

*(i) $< \overline{U(\mathcal{P})}, \overline{N(\mathcal{P})} >$ is a highly recursive nonmonotonic rule system, and*

*(ii) A subset $E \subseteq \overline{U(\mathcal{P})}$ is an extension of $< \overline{U(\mathcal{P})}, \overline{N(\mathcal{P})} >$ if and only if $< C_1, \ldots, C_w >$, (where for $i = 1, \ldots, w$, $C_i = \{x \in D: Cxi \in E\}$), is a chain cover of $\mathcal{P}$.*

## 2.4 Recursion-theoretic results for extensions

Before giving other examples, we pause to explain that the examples for the symmetric marriage problem and $k$-colorings of graphs are especially significant for coding up recursively bounded $\Pi^0_1$-classes. Manaster and Rosenstein [1972] showed that for any highly recursive tree $T$, there is a highly recursive society $\mathcal{S} =< B, G, K >$ for which there is an effective one-to-one degree preserving correspondence between the proper symmetric marriages of $\mathcal{S}$ and the set of infinite paths through $T$. Similarly, Remmel [1986] showed that for any $k \geq 3$ and any highly recursive tree $T$, there is a highly recursive $k$-colorable graph $\mathcal{G} =< V, E >$ such that, up to a permutation of colors, there is an effective one-to-one degree preserving correspondence between the $k$-colorings of $\mathcal{G}$ and the set of infinite paths through $T$. Since any recursively bounded class $C$ is of the form $\mathcal{P}(T)$ for some highly recursive tree $T$, the results of Manaster and Rosenstein, and Remmel combined with Theorems 2.4 and 2.5 yield the following.

**Theorem 2.7** *Let $C$ be any recursively bounded $\Pi^0_1$-class. Then there is a highly recursive nonmonotonic rule system $< U, N >$ and an effective one-to-one degree preserving correspondence between the elements of $C$ and the set of all extensions of $< U, N >$.*

Theorem 2.7 now allows us to transfer many results about possible degrees of elements of recursively bounded $\Pi^0_1$-classes to results about degrees of extensions of highly recursive nonmonotonic rule systems. Below we shall list a few examples of such results.

**Corollary 2.8** *There is a highly recursive nonmonotonic rule system $< U, N >$ such that $< U, N >$ has $2^{\aleph_0}$ extensions but no recursive extensions.*

**Corollary 2.9** *There is a highly recursive nonmonotonic rule system $< U, N >$ such that $< U, N >$ has $2^{\aleph_0}$ extensions and any two extensions $E_1 \neq E_2$ of $< U, N >$ are Turing incomparable.*

**Corollary 2.10** *If* $\mathbf{a}$ *is any Turing degree that* $\mathbf{0} <_T \mathbf{a} \leq_T \mathbf{0}'$*, then there is a highly recursive nonmonotonic rule system $< U, N >$ such that $< U, N >$ has $2^{\aleph_0}$ extensions but no recursive extensions and $< U, N >$ has an extension of degree* $\mathbf{a}$*. (Here* $\mathbf{0}$ *is the degree of recursive sets.)*

**Corollary 2.11** *If* $\mathbf{a}$ *is any Turing degree that* $\mathbf{0} <_T \mathbf{a} \leq_T \mathbf{0}'$*, then there is a highly recursive nonmonotonic rule system $< U, N >$ such that $< U, N >$ has $\aleph_0$ extensions, $< U, N >$ has an extension $E$ of degree* $\mathbf{a}$ *and if $E' \neq E$ is an extension of $< U, N >$, then $E'$ is recursive.*

**Corollary 2.12** *There is a highly recursive nonmonotonic rule system $< U, N >$ such that $< U, N >$ has $2^{\aleph_0}$ extensions and if* $\mathbf{a}$ *is the degree of any extension $E$ of $< U, N >$ and* $\mathbf{b}$ *is any recursively enumerable degree such that* $\mathbf{a} <_T \mathbf{b}$*, then* $\mathbf{b} \equiv_T 0'$*.*

**Corollary 2.13** *If* $\mathbf{a}$ *is any recursively enumerable Turing degree, then there is a highly recursive nonmonotonic rule system $< U, N >$ such that $< U, N >$ has $2^{\aleph_0}$ extensions and the set of recursively enumerable degrees* $\mathbf{b}$ *which contain an extension of $< U, N >$ is precisely the set of all recursively enumerable degrees* $\mathbf{b} \geq_T \mathbf{a}$*.*

We note that all of the above results follow from Theorem 2.7 plus the corresponding results for recursively bounded $\Pi^0_1$-classes due to Jockusch and Soare [1972a]

[1972b] with the exception of Corollary 2.12 which follows from the corresponding result for recursively bounded $\Pi_1^0$-classes due to Jockusch and McLaughlin [1969].

Next we give a construction of a rule system $< U, N >$ whose extensions directly code infinite paths through a binary tree $T$ and hence provides us with a more direct route to Theorem 2.7 which avoids using the results of Manaster and Rosenstein [1972] or Remmel [1986].

**Example 2.2** *Paths through binary trees.*

Let $\mathcal{T}$ be a recursive binary tree contained in $2^{<\omega}$. Let $U(\mathcal{T}) = \{P_i, \overline{P_i}: i \in \omega\}$. Our idea is to have a set $\pi$ such that $| \pi \cap \{P_i, \overline{P_i}\} |= 1$ for all $i$ correspond to a path $f_\pi: \omega \rightarrow \omega$ through the complete binary tree $B_\omega = 2^{<\omega}$ where

$$x = \begin{cases} 1 & \text{if } P_i \in \pi \\ 0 & \text{if } \overline{P_i} \in \pi \end{cases}$$

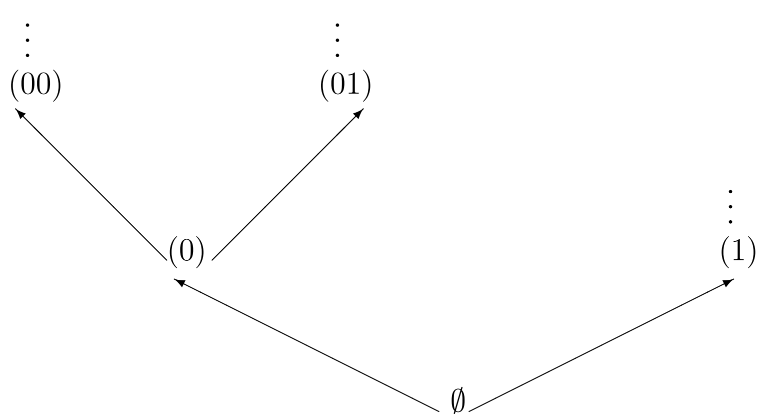


Figure 1.

Thus, picturing $B_\omega$ as in Figure 1, $P_i \in \pi$ says that we branch right at level $i$, and $\overline{P_i} \in \pi$ says that we branch left at the level $i$. Now, for any node $\sigma =< \sigma(0), \dots, \sigma(n) >$, let $\vec{P_\sigma} = \{\sigma(P_0), \dots, \sigma(P_n)\}$ where

$$\sigma(P_i) = \begin{cases} P_i & \text{if } \sigma(i) = 1 \\ \overline{P_i} & \text{if } \sigma(i) = 0 \end{cases}$$

We say that $\sigma =< \sigma(0), \dots, \sigma(n) >$ is a terminal node of $\mathcal{T}$ if $\sigma \in \mathcal{T}$ and both $< \sigma(0), \dots, \sigma(n), 0 > \notin \mathcal{T}$ and $< \sigma(0), \dots, \sigma(n), 1 > \notin \mathcal{T}$ .

Then we consider the following set of rules.

$$\frac{: P_i}{\overline{P}_i} \quad \frac{: \overline{P}_i}{P_i} \tag{12}$$

$$(a) \qquad \frac{\sigma(P_0), \dots, \sigma(P_n):}{P_n} \tag{13}$$

for all $\sigma$ which are terminal nodes of $\mathcal{T}$ where $\sigma(P_n) = \overline{P}_n$

$$\text{(b)} \qquad \frac{\sigma(P_0), \dots, \sigma(P_n):}{\overline{P}_n}$$

for all $\sigma$ which are terminal nodes of $\mathcal{T}$ where $\sigma(P_n) = P_n$.

Let $N(\mathcal{T})$ consist of all rules of the forms (12) or (13). Then we have the following (if we identify $P_i$ with its code $2i$ and $\overline{P}_i$ with its code $2i + 1$).

**Theorem 2.14** *Let $\mathcal{T} \subseteq 2^{<\omega}$ be a recursive tree. Then*

*(i) $< U(\mathcal{T}), N(\mathcal{T}) >$ is a highly recursive nonmonotonic rule system and*

*(ii) $E$ is an extension of $< U(\mathcal{T}), N(\mathcal{T}) >$ if and only if the map $f_E: \omega \to \omega$ defined by*

$$f_E(i) = \begin{cases} 1 & \text{if } P_i \in E \\ 0 & \text{if } \overline{P}_i \in E \end{cases}$$

*is an infinite path through $\mathcal{T}$.*

Proof: We shall show by induction on $i$ that there are only finitely many minimal proof schemes for $P_i$ or $\overline{P}_i$ and that we can effectively find the canonical index of the set of all such minimal proof schemes. Note that $P_0$ is the conclusion of at most two rules, namely $R_0 = \frac{:\overline{P}_0}{P_0}$ or $R_1 = \frac{\overline{P}_0:}{P_0}$ if (0) is a terminal node of $\mathcal{T}$.

Thus if $p =<< \varphi_0, r_0, can(G_0) >, \ldots, < \varphi_m, r_m, can(G_m) >>$ is a minimal proof scheme with $cln(p) = P_0$, then either $r_m = R_0$ in which case $m = 0$ or $r_m = R_1$ in which case $p' =<< \varphi_0, r_0, can(G_0) >, \ldots, < \varphi_{m-1}, r_{m-1}, can(G_{m-1}) >>$ is a minimal proof scheme for $\overline{P}_0$ which does not include $P_0$ as the conclusion of any $r_i$. But if $P_0$ is not a conclusion of any $r_i$, $P_0$ cannot be a premise of any $r_i$. Hence we are left with just one rule that has $\overline{P}_0$ as a conclusion that does not involve $P_0$ as a premise, namely, $R_3 = \frac{:P_0}{\overline{P}_0}$. But this means that if $r_m = R_1$ then $p' =<< \overline{P}_0, \frac{:P_0}{\overline{P}_0}, can(\{P_0\}) >>$ and $p =<< \overline{P}_0, \frac{:P_0}{\overline{P}_0}, can(\{P_0\}) >, < \overline{P}_0, \frac{\overline{P}_0:}{P_0}, can(\{P_0\}) >>$ which is not possible if $p$ is a minimal proof scheme as in (1). Thus there can be exactly one minimal proof scheme for $P_0$, namely $p =<< P_0, \frac{:\overline{P}_0}{P_0}, can(\{\overline{P}_0\}) >>$.

In fact we shall show that, in general, there is precisely one minimal proof scheme for $P_i$ or $\overline{P}_i$. Assume by induction that for all $j < i$, there is only one minimal proof scheme $p$ with $cln(p) = P_j$, namely $p =<< P_j, \frac{:\overline{P}_j}{P_j}, can(\{\overline{P}_j\}) >>$ and one minimal proof scheme $\overline{p}$ with $cln(\overline{p}) = \overline{P}_j$, namely $\overline{p} =<< \overline{P}_j, \frac{:P_j}{\overline{P}_j}, can(\{P_j\}) >>$. Now suppose $p =<< \varphi_0, r_0, can(G_0) >, \ldots, < \varphi_m, r_m, can(G_m) >>$ is a minimal proof scheme with $cln(p) = P_i$. Then either $r_m = \frac{:\overline{P}_i}{P_i}$ in which case $m = 0$ or $r_m = \frac{\sigma(P_0), \ldots, \sigma(P_i):}{P_i}$ where $\sigma =< \sigma(0), \ldots, \sigma(i) >$ is a terminal node of $\mathcal{T}$ and $\sigma(P_i) =$

$\overline{P}_i$. In the latter case, $p' =<< \varphi_0, r_0, can(G_0) >, \ldots, < \varphi_{m-1}, r_{m-1}, can(G_{m-1}) >>$ must be some interweaving of minimal proof schemes for $\sigma(P_0), \ldots, \sigma(P_i) = \overline{P}_i$. Moreover $p'$ cannot involve $P_i$ as a conclusion or a premise of any rule. But it is easy to see that the only rule which has $\overline{P}_i$ as a conclusion and does not involve $P_i$ as a premise is $\frac{:P_i}{\overline{P}_i}$. But this means for some $j < m$, $r_j = \frac{:P_i}{\overline{P}_i}$ and hence $G_j \supseteq \{P_i\}$. But then $G_j \supseteq \{P_i\}$ which would violate the fact that $p$ is a minimal proof scheme. Thus in fact there is a single minimal proof scheme for $P_i$ namely $<< P_i, \frac{:\overline{P}_i}{P_i}, can(\{\overline{P}_i\}) >>$. A similar argument shows that the only minimal proof scheme for $\overline{P}_i$ is $<< \overline{P}_i, \frac{:P_i}{\overline{P}_i}, can(\{P_i\}) >>$. Thus $< U(\mathcal{T}), N(\mathcal{T}) >$ is a highly recursive nonmonotonic rule system if $\mathcal{T}$ is a recursive tree.

For (ii), suppose that $E$ is an extension of $< U(\mathcal{T}), N(\mathcal{T}) >$. Now our analysis of minimal proof schemes shows that we can only use rules of the form (12) in minimal derivations of $C_E(\emptyset)$. It then easily follows that for any $i$, precisely one of $P_i$ and $\overline{P}_i$ must be in $E$. Thus $f_E$ is an infinite path through $B_\omega$.

But note that if $\sigma =< \sigma(0), \ldots, \sigma(n) >=< f_E(0), \ldots, f_E(n) >$ is a node in $\mathcal{T}$, then it cannot be that $\sigma$ is a terminal node of $\mathcal{T}$ since otherwise we could use the rules of form (13) to show that both $P_n$ and $\overline{P}_n$ are in $E$. Then it is easy to show by induction that $< f_E(0), \ldots, f_E(n) >\in \mathcal{T}$ for all $n$ and hence $f_E \in \mathcal{P}(\mathcal{T})$.

Now, if $f_E \in \mathcal{P}(\mathcal{T})$, then it is easy to see that the rules in (12) ensure $E \subseteq C_E(\emptyset)$. Moreover one can prove by induction on the length of a derivation that we can never apply a rule of form (13) to produce anything in $C_E(\emptyset)$. It then follows that $C_E(\emptyset) \subseteq E$ and hence $E$ is an extension. $\square$

We can use now Theorem 2.14 to give a more direct proof of Theorem 2.7. That is, if $C$ is a recursively bounded $\Pi^0_1$-class, let $\mathcal{T}$ be a highly recursive tree included in $\omega^{<\omega}$ such that $C = \mathcal{P}(\mathcal{T})$. It is then easy to construct a recursive binary tree $\mathcal{T}^\star \subseteq 2^{<\omega}$ such that there is an effective one-to-one degree preserving correspondence between $\mathcal{P}(\mathcal{T})$ and $\mathcal{P}(\mathcal{T}^\star)$. The idea is to replace each $k$-ary branching node by using a binary tree of height $k$ and having the lexicographically $k$ first nodes at level $k$ correspond to the $k$ successors of $\eta$. See Figure 2 for an example of this replacement.

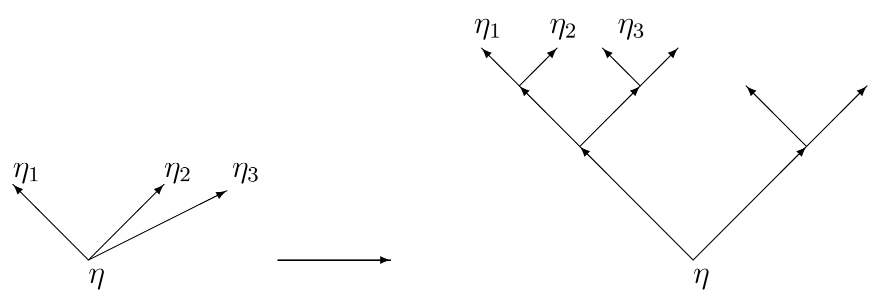


Figure 2

It is not difficult to see that if we do a node-by-node replacement in this fashion we will produce a recursive binary tree $\mathcal{T}^\star$ with the desired properties. Then we can use $< U(\mathcal{T}), N(\mathcal{T}) >= \mathcal{S}$ where $\mathcal{T} = \mathcal{T}^\star$ for the highly recursive nonmonotonic rule system such that there is an effective one-to-one degree preserving correspondence between $C$ and $\mathcal{E}(\mathcal{S})$.

Given Theorems 2.1 and 2.7, it is natural to ask if there are analogous results for locally finite nonmonotonic rule systems which are recursive, but not highly recursive. The answer is "yes". That is we say that a tree $T \subseteq \omega^{<\omega}$ is *highly recursive* in $\mathbf{0}'$ if $T$ is recursive in $\mathbf{0}'$, $T$ is finitely branching, and there is a procedure which is recursive in $\mathbf{0}'$ and which, given any node $\eta \in T$, will produce the canonical index of the set of immediate successors of $\eta$ in $T$. Then the analogues of Theorems 2.1 and 2.7 hold for recursive nonmonotonic rule systems if we replace highly recursive trees by trees which are highly recursive in $0'$.

Moreover, by relativizating to the code of the collection of rules $< U, N >$ we are able to deal with the case of an *arbitrary* locally finite nonmonotonic system $\mathcal{S}$. The distinction between the form of function that computes the canonical index of the collection of proof schemes for elements of $U$ remains: if this function is recursive in (the code of) $< U, N >$, then the tree $\mathcal{T}$ whose branches code extensions of $< U, N >$ is recursive in (the code of) $< U, N >$; otherwise it is recursive in its jump.

These results will be proved in a subsequent paper.

## 2.5 Some applications to Logical Systems

The results of Sections 2.1 and 2.4 can be interpreted using Sections 4.5, part I, and 4.6, part I, as (new) results about default logic and logic programming. The relationship between stable semantics for logic programs and default logic, and the results of Section 4.6, part I, show the relevance of proof schemes to the construction of stable models for logic programs. As far as we know, programs with the local finiteness property have not been previously discussed in the literature, although this covers most practical programs. The definition of proof scheme with a "forbidden" set of atoms (corresponding to the definition of support of a proof scheme above) is perfectly natural and can be lifted from definition (1) in an obvious fashion. The ordering $\prec$ has the same meaning as before. This way we get a natural concept of a locally finite (propositional) program. When the program $P$ involves variables we interpret $P$ as the collection of its Herbrand constant substitutions. In particular this gives rise to a definition of locally finite program. The rule systems that we wrote in Sections 2.1 and 2.4 can be rewritten following *reverse* translations of Section 4.6, part I, (notice that we deal there only with atoms!), that is, the rule $\frac{q_1,\ldots,q_n:r_1,\ldots,r_m}{p}$ is translated to: $p \leftarrow q_1, \ldots, q_n, \neg r_1, \ldots, \neg r_m$. From Proposition 4.2, part I, it then follows that we get stable models from extensions, and it is easy to see that the concept of proof scheme is preserved, locally finite systems generate locally finite programs. Then, in an analogous manner, we can introduce the notion of a highly recursive program as one that is recursive, locally finite, and for which a function assigning to $p$ the code of its finite collection of $\prec$-minimal proof schemes is recursive. Let $Stab(P)$ be the collection of stable models of the program $P$. We then get

**Theorem 2.15** *Given a highly recursive program $P$ there is a highly recursive tree $T \subseteq 2^{<\omega}$ and an effective one-to-one degree preserving correspondence between $Stab(P)$ and $\mathcal{P}(T)$.*

Exactly the same lifting may be done for default logic. We leave the details to the reader.

So the results of Jockusch and Soare [1972a] apply both to logic programming and to default logic, and we get a series of results in the recursion theory of stable models of logic programs by lifting Corollary 2.2, Theorem 2.7, Corollaries 2.8, 2.9, 2.10, 2.11, 2.12, 2.13, and Theorem 2.14.

It is appropriate to compare the results of this section with those of [Apt and Blair , 1990]. They construct, for a given natural number $n \geq 1$, a stratified finite program $P$ (in particular its Herbrand expansion is a recursive propositional program) whose perfect model is a complete $\Sigma^0_n$ set of natural numbers. Since the perfect model is stable, and stratified programs possess a unique stable model (as pointed by [Gelfond and Lifschitz, 1988]), the collection $Stab(P)$ is a one element class. Then this is a $\Pi^0_2$-class, whose only element is a $\Sigma^0_n$ set. Our results show that it is impossible to find a recursive program possessing a unique stable model which is $\Pi^1_1$-complete because the unique element of an arithmetical singleton class in $2^\omega$ must be hyperarithmetic.

# 3 Semantical issues and descriptive characterization of various sets closed under rules

Let $< U, N >$ be a deductive system and assume that $|U| = \omega$. Without loss of generality we may identify the set $U$ with the set $\omega$ of natural numbers, and $N$, which consists of finite objects, with a subset of $\omega$.

Let us recall that we wish to characterize three classes: minimal sets closed under $N$, weak extensions, and extensions of $< U, N >$. We shall provide a semantic characterization of these concepts. These characterizations use the infinitary logic we now introduce.

Logic $\mathcal{L}_S$ is defined as the closure of a collection of atoms of the form "$\varphi \in S$" ($\varphi$ ranging over $U$) under negation, arbitrary denumerable conjunctions and arbitrary denumerable disjunctions.

Given $T \subseteq U$, and a formula $\varphi$ of $\mathcal{L}_S$, define the satisfaction relation $T \models \varphi$ by induction as follows:

(1) $T \models \alpha \in S$ if and only if $\alpha \in T$.

(2) $T \models \neg\psi$ if and only if $not(T \models \psi)$.

(3) $T \models \bigwedge_{i \in J} \psi_i$ if and only if for all $i \in J$, $T \models \psi_i$.

(4) $T \models \bigvee_{i \in J} \psi_i$ if and only if there exists an $i \in J$ such that $T \models \psi_i$.

The connectives $\Rightarrow$ and $\Leftrightarrow$ are abbreviations in the usual way.

Associate with a rule:

$$r = \frac{\alpha_1, \ldots \alpha_n : \beta_1, \ldots, \beta_m}{\varphi} \tag{14}$$

a finitary formula of $\mathcal{L}_S$,

$$t(r) = [\alpha_1 \in S \wedge \ldots \wedge \alpha_n \in S \wedge \neg(\beta_1 \in S) \wedge \ldots \wedge \neg(\beta_m \in S)] \Rightarrow \varphi \in S \tag{15}$$

The conclusion $\varphi$ is denoted by $c(r)$.

**Proposition 3.1** *A subset $T$ of $U$ is deductively closed if and only if for all $r \in N$, $T \models t(r)$.*

Generalizing *Clark's completion* from logic programming, we define Clark's completion of a deductive system $< U, N >$. This is a theory in $\mathcal{L}_S$ (possibly infinitary).

To define it, assume that $r$ is a rule of the form (14). Set

$$A(r) = \alpha_1 \in S \wedge \ldots \wedge \alpha_n \in S \wedge \neg(\beta_1 \in S) \wedge \ldots \wedge \neg(\beta_m \in S). \tag{16}$$

Then $t(r)$ is $A_r \Rightarrow (c(r) \in S)$. Now, given $\alpha \in U$, let $F_\alpha$ be the formula of $\mathcal{L}_S$:

$$\alpha \in S \Leftrightarrow \bigvee\{A_r\colon r \in N \wedge c(r) = \alpha \in S\} \tag{17}$$

Then $F_\alpha$ says that $\alpha$ belongs to $T$ exactly if it is supported by a formula of the form $A_r$ for some $r \in N$.

The formulas $F_\alpha$ can be used to characterize weak extensions.

**Theorem 3.2** *A collection $T \subseteq U$ is a weak extension of $< U, N >$ if and only if for all $\alpha \in U$, $T \models F_\alpha$.*

Proof: From Proposition 3.7, part I, we know that $T$ is a weak extension of $< U, N >$ if and only if

$$T = \{\psi\colon \text{ for some rule } r \in N \text{of the form (14)}, \psi = c(r) \wedge\ \alpha_1 \in T \wedge\ \ldots \wedge$$
$$\alpha_m \in T \wedge\ \beta_1 \notin T \wedge\ \ldots \wedge\ \beta_m \notin T\}.$$

Inspection of the definition of satisfaction shows that this is equivalent to

$$T = \{\psi\colon \text{ for some rule } \ r \in N, \psi = c(r) \wedge\ T \models A_r\} \tag{18}$$

Let $r \in N$, $\psi = c(r)$.

Case 1: $\psi \in T$. Then, for some $r \in N$, $\psi = c(r)$, and $T \models A_r$. But then $T \models \bigvee\{A_r\colon \psi = c(r)\}$. Thus $T \models \psi \in S \Leftrightarrow \bigvee\{A_r\colon \psi = c(r)\}$.

Case 2: $\psi \notin T$. Then, by equation (18), for all $r$ such that $\psi = c(r)$, $T \models \neg A_r$. Hence $T \models \bigwedge\{\neg A_r\colon \psi = c(r)\}$, that is $T \models \neg \bigvee\{A_r\colon \psi = c(r)\}$.

As $T \models \neg\psi \in S$, we get that $T \models \psi \in S \Leftrightarrow \bigvee\{A_r\colon \psi = c(r)\}$. Thus we have proved that for all $r \in N$, $T \models F_{c(r)}$.

Conversely, assume that for all $r \in N$, $T \models F_{c(r)}$. We need to prove two inclusions:

(a) $T \subseteq \{\psi\colon$ for some rule $r \in N, \psi = c(r)$ and $T \models A_r\}$

(b) $\{\psi\colon$ for some rule $r \in N, \psi = c(r)$ and $T \models A_r\} \subseteq T$

(a) Suppose that $\psi \in T$. Then $\psi$ is a conclusion of a rule in $N$. Since $T \models F_{c(r)}$, and $T \models \psi \in S$, for some rule $r \in N$, $T \models A_r$.

(b) Conversely, if $\psi \in \{\psi\colon$For some rule $r \in N, \psi = c(r)$ and $T \models A_r\}$, then $T \models \bigvee\{A_r\colon \psi = c(r)\}$. Thus, as $T \models F_\psi$, $T \models \psi \in S$, that is $\psi \in T$. $\square$

We continue to identify $U$ with the set of natural numbers, $\omega$. The collection of all subsets of $U$ is identified with $2^\omega$. This is the Cantor space. Then:

**Proposition 3.3** *For every formula $\Phi \in \mathcal{L}_S$, $\{T\colon T \models \Phi\}$ is a Borel subclass of $2^\omega$ in the Cantor topology.*

Propositions 3.2 and 3.3, yield, using standard descriptive set theory (see [Kuratowski and Mostowski, 1977]):

**Corollary 3.4** *Let $< U, N >$ be a nonmonotone rule system, $U = \omega$.*

*(a) The collection $W$ of weak extensions of $< U, N >$ is a Borel subclass of $2^\omega$, and consequently*

*(b) $|W|$ is finite, or $|W| = \omega$ or $|W| = 2^{\aleph_0}$.*

When $< U, N >$ is recursive, then the formula $\bigvee\{A_r : \psi = c(r)\}$ is representable as a recursively enumerable set of natural numbers. From this there follows

**Proposition 3.5** *If $< U, N >$ is recursive, then the collection of weak extensions of $< U, N >$ is a $\Pi^0_2$ subclass of $2^\omega$.*

The collection of all extensions of a nonmonotone rule system also possesses a model-theoretical characterization. Using the idea behind proof schemes, we introduce an infinitary description of provability. Fix $< U, N >$.

**Proposition 3.6** *For every $\psi \in U$, there exists a formula $pr_\psi \in \mathcal{L}_S$ such that for every $T \subseteq U$, $T \models pr_\psi$ if and only if $\psi$ possesses a $T$-derivation. (Note that $pr_\psi$ depends on $N$)*

Proof: We proceed as in Section 2, except that now we cannot be sure that the formula we are about to write is finite. We consider all the proof schemes for $\psi$ and for each such scheme $p$ write a formula $k(p)$, where $k(p)$ is the conjunction:

$\neg(\alpha_1 \in S) \wedge \ldots \wedge \neg(\alpha_s \in S)$, and $\{\alpha_1, \ldots, \alpha_s\}$ is the support of the proof scheme $p$.

Now, we define $pr_\psi$ as $\bigvee\{k(p) : p$ is a proof scheme for $\psi\}$.

Now we prove the promised equivalence.

(a) Assume that $\psi$ possesses a $T$-derivation. Then this derivation gives rise to proof scheme $p$ whose rules are used in the derivation. This implies that $T \models k(p)$ and hence $T \models pr_\psi$.

(b) Conversely, if $T \models pr_\psi$, then for some proof scheme $p$, $T \models k(p)$. Then the proof scheme $p$ provides us with the $T$-derivation of $\psi$. □

**Corollary 3.7** *Let $< U, N >$ be a nonmonotonic rule system. Then $T \subseteq U$ is an extension of $< U, N >$ if and only if:*

*(1) For all $\psi \in T$, $T \models pr_\psi$, and*

*(2) For all $\psi \notin T$, $T \models \neg pr_\psi$.*

Proof: Since $T$ is an extension of the nonmonotonic rule system $< U, N >$, $T$ consists of precisely those elements $\psi \in U$ which possess a $T$-derivation. Then proposition 3.6 gives precisely (1) and (2). $\square$

**Corollary 3.8** *Let $< U, N >$ be a deductive system where $U = \omega$.*

*(a) The collection $E$ of extensions of $< U, N >$ is a $\Pi_2^{0,N}$ subclass of $2^\omega$, and consequently*

*(b) $|E|$ is finite, or $|E| = \omega$ or $|E| = 2^{\aleph_0}$.*

In a later paper we will show that the class of extensions is not $\Pi_1^{0,N}$.

Let us assume that $U = \omega$. If $r = \frac{\alpha_1, \ldots, \alpha_n : \beta_1, \ldots, \beta_r}{\gamma}$ is a rule, then a subset $W \subseteq \omega$ is closed under the rule $r$ if and only if $W$ satisfies the implication

$$\alpha_1 \in S \wedge \ldots \wedge \alpha_n \in S \Rightarrow \beta_1 \in S \vee \ldots \vee \beta_r \in S \vee \gamma \in S.$$

Consequently $S$ is *not* closed under $r$ if and only if $S$ belongs to basic neighbourhood in the Cantor topology determined by two finite sequences (elements that need to be "in"), $\langle \alpha_1, \ldots, \alpha_n \rangle$, and (the elements that need to be "out") $\langle \beta_1, \ldots, \beta_r, \gamma \rangle$. Since every rule possesses the conclusion, $\omega$ is always closed under rules in $N$. This, however, is the only restriction on closed sets in question. We have the following characterization of closed sets in the Cantor topology:

**Theorem 3.9** *Let $\mathcal{X} \subseteq 2^\omega$. Let $\omega \in \mathcal{X}$. $\mathcal{X}$ is closed in the Cantor topology if and only if there exists a collection of rules $N$ such that for every $S \subseteq \omega$, $S$ is deductively closed in $\langle \omega, N \rangle \Leftrightarrow \chi(S) \in \mathcal{X}$.*

Finally, we turn our attention to the minimal deductively closed sets of $< U, N >$. Again we deal with the case when $U = \omega$.

**Proposition 3.10** *The collection of minimal deductively closed sets for $< \omega, N >$ is a $\mathbf{\Pi}_2^{0,N}$ subclass of the Cantor space.*

The proof of this result is based on a characterization of inclusion-minimal elements of closed sets in Cantor space due to W. Just [1990], and on standard descriptive set-theoretic and recursion-theoretic techniques. First of all, notice that Theorem 3.9 implies that the minimal closed sets in $< \omega, N >$ are exactly the inclusion-minimal sets in a certain subset of $\mathcal{P}(\omega)$. Since there is a natural $1-1$ correspondence between $\mathcal{P}(\omega)$ and $2^\omega$, we can identify subsets of $\omega$ with their characteristic functions. Thus Theorem 3.9 says that sets closed under rules in $< \omega, N >$ form a closed subset in $2^\omega$. Closed sets in $2^\omega$ are characterized as the set of all branches through a tree in $2^{<\omega}$. A *tree* is a collection of finite binary sequences closed under initial segments. If $T$ is a tree, then $[T]$ is the collection of all infinite branches through $T$. In addition to usual partial ordering $\subseteq$ among finite sequences, that is, extension, we consider an additional partial ordering $\preceq$ on finite binary sequences defined as follows:

$$t \preceq s \text{ if and only if } lh(t) = lh(s) \wedge \forall_{n<lh(s)} t(n) \leq s(n).$$

$s \prec t$ if $s \preceq t$ and $s \neq t$.

Thus $s \preceq t$ says that the "partial set" coded by $s$ is included in the "partial set" encoded by $t$. We write $s \parallel t$, where $s, t$ are finite or infinite binary sequences, if there exist $n_1, n_2$ such that $s(n_1) = 1$ and $t(n_1) = 0$ and $s(n_2) = 0$ and $t(n_2) = 1$. Given $\mathcal{X} \subseteq \mathcal{P}(\omega)$, $B(\mathcal{X})$ is the family of inclusion-minimal elements of $\mathcal{X}$. In general, $B(\mathcal{X})$ may be empty, even if $\mathcal{X}$ is non-empty. However, if $\mathcal{X}$ is closed and nonempty, then $B(\mathcal{X})$ is nonempty. Given a tree $T$, define a set $J(T)$ of functions as follows:

$$J(T) = \{f \in 2^\omega \colon f \in [T] \wedge \forall_k \exists_n (n > k \wedge \forall_{s \in T}(lh(s) = n \wedge s\mid_k \prec f\mid_k \Rightarrow s \parallel f)\}$$

Then:

**Proposition 3.11** (Just) *Let $\mathcal{X}$ be closed in Cantor topology. Let $\mathcal{X} = [T]$. Then $B(\mathcal{X}) = J(T)$.*

Proof: If $X \in B(\mathcal{X})$, $k \in X$, then let $T_{X,k}$ be the tree consisting of initial segments of sequences $s$ satisfying the following condition:

$$s \in T \wedge s\mid_k = \chi(X)\mid_k \wedge s(k) = 0 \wedge s \preceq \chi(X)\mid_{lh(s)}$$

Then $X \in B(\mathcal{X})$ precisely if for all $k$, $T_{X,k}$ is finite. Hence $T_{X,k}$ is of finite height. Let $m_k$ be the height of $T_{X,k}$. Then $n = max\{m_i \colon i \leq k\}$ is $n$ witnessing to $X$ belong to $J(T)$.

Conversly, if $X \in J(T)$, $Y \in [T]$, and $Y \subset X$, $k = min(X \setminus Y)$, then $\chi(Y)\mid_{k+1} \prec \chi(X)\mid_{k+1}$. Since $X \in J(T)$, there is $n$ such that $\chi(Y)\mid_n \parallel \chi(X)\mid_n$, contradicting $Y \subseteq X$. $\square$

Now we are in a position to prove Proposition 3.10.

Proof of Proposition 3.10: As proved in Proposition 3.9, the family $\mathcal{C}$ of sets deductively closed in $<\omega, N>$ is closed in Cantor topology, in fact is $\Pi_1^{0,N}$. Clearly, the family $\mathcal{M}$ of minimal deductively closed sets is equal to $B(\mathcal{C})$. Using Proposition 3.11, we just need to evaluate the form of $J(T)$. As the last universal quantifier in the formula defining $J(T)$ ranges over a finite set, it is easily seen to be $\Pi_2^{0,N}$. $\Box$

When $N$ is recursive, the family of minimal deductively closed sets is, consequently, $\Pi_2^0$.

## 3.1 Applications to Default Logic and Logic Programming

Recall that in Section 4.5, part I, we introduced a translation of default logic theories into nonomonotonic rule systems. We proved that this translation is faithful; that is, if $< D, W >$ is a default theory and $< U, N >$ is its translation, then $S$ is a default extension of $< D, W >$ if and only if $S$ is an extension of $< U, N >$. Similarly, $S$ is a weak default extension of $< D, W >$ if and only if $S$ is a weak extension of $< U, N >$ ([Marek and Truszczyński, 1989]).

Proposition 3.2 and Corollary 3.7 are semantic characterizations of weak extensions and extensions of nonmonotonic rule systems. These remarks immediately imply:

**Proposition 3.12** *Let $< D, W >$ be a default theory. Let $< U, N >$ be its translation. Let $S$ be a subset of $U$ satisfying the translation of $< D, W >$. Then:*

*(1) $S$ is a weak default extension of $< D, W >$ if, and only if, for all $\varphi \in \mathcal{L}$, $S \models F_\varphi$.*

*(2) $S$ is a default extension of $< D, W >$if, and only if,*

*(i) for all* $\vartheta \in S$, $S \models pr_\vartheta$.

*(ii) for all* $\vartheta \notin S$, $S \models \neg pr_\vartheta$.

Etherington ([Etherington, 1987]) characterized default extensions by means of "most preferred models". We use a different device here. First, we imbed the language $\mathcal{L}$ into a new language $\mathcal{L}_S$. This language $\mathcal{L}_S$ possesses a new *atom* for every *formula* of $\mathcal{L}$. Thus $\mathcal{L}_S$ is a much richer language. Second, formulas of $\mathcal{L}$ are translated as atoms of $\mathcal{L}_S$. The relationships between formulas of $\mathcal{L}$ are enforced in $\mathcal{L}_S$ by means of translations of rules. Default rules of $\mathcal{L}$ are translated to corresponding finitary clauses of $\mathcal{L}_S$. Checking satisfiability for these clauses reduces to checking satisfiability of logically simpler formulas of $\mathcal{L}_S$. Some of the simpler formulas needing to be checked are not images of formulas of $\mathcal{L}$ under the translation. Our semantic characterization of extensions and weak extensions uses formulas of $\mathcal{L}_S$ which are not images of formulas of $\mathcal{L}$ under the translation. Some formulas used are properly infinitary. This is a reflection of the infinitary character of the concepts of extension and weak extension.

There is an important area of application for $\mathcal{L}_S$ which goes beyond mere characterization. Because a set of default rules is represented by a set of formulas of a language $\mathcal{L}_S$, we can use the natural deductive structure of $L_{\omega_1,\omega}$ to define what it means for a collection of default rules to *entail* another default rule. This concept of entailment may be formalized in various ways, depending on the structures under investigation; that is, depending on whether the structures are weak extensions,

extensions, sets closed under rules, or something else. The general procedure for defining entailment is to say that a collection $D$ of defaults entails a default rule $d$ if and only if every structure satisfying the translation of $D$ satisfies the translation of $d$. We shall investigate these relationships in a sequel. Note that when the theory $< D, W >$ is finite, its nonmonotonic translation is finitary. Also the characterization formulas $pr_\psi$ are finitary. The reason for this is that, in addition to rules in $D$, we adopt all the rules of ordinary logic as monotone rules. The proof schemes of ordinary logic give infinitely many monotonic rules. Even though there are infinitely many proof schemes, the collection of formulas of form $k(p)$ is finite! This yields the finitary algorithm described in [Marek and Nerode, 1990].

Our translation of propositional logic programs as nonmonotonic rule systems provides an infinitary characterization of the stable models of logic programs. This is important because the definition of stable model of logic program as introduced in [Gelfond and Lifschitz, 1988] is merely operational, while ours is declarative. Let $P$ be a logic program. Let $\Pi$ be its propositional version. That is, let $\Pi$ be the collection of all the Herbrand substitutions of $P$. Let $H$ be the Herbrand base of $P$ and $M \subseteq H$. Gelfond and Lifschitz [1988] gave an algorithm for testing whether $M$ is a stable structure for $P$. They proved that a stable structure is a minimal model for $P$. It is clear that this definition is purely operational. Using the infinitary language $\mathcal{L}_S$ we give a purely declarative, but infinitary, description of stability.

**Proposition 3.13** *Let $P$ be a logic program. Let $\Pi$ be its propositional version. Let $H$ be the Herbrand base of $P$. Let $< H, T >$ be the translation of $\Pi$ as described in*

*Section 4.6, part I.*

*Then $M \subseteq H$ is a stable model of $P$ if and only if*

*(i) for every $\vartheta \in M$, $M \models pr_\vartheta$.*

*(ii) for every $\vartheta \notin M$, $M \models \neg pr_\vartheta$.*

Finally, let us mention an obvious corollary.

**Proposition 3.14** *Let $P$ be finite, or denumerable, general logic program. If we enumerate all grounded atoms of the language and identify subsets of the Herbrand base with points of the Cantor space, $2^\omega$, then the collection of supported models of $P$ is a Borel subclass of $2^\omega$ and the collection of stable models of $P$ is a Borel subclass of $2^\omega$.*

Analogous properties hold for extensions of arbitrary denumerable default theories. The results of Section 5, part I, provide numerous refinements of Proposition 3.14.

The interpretations of default theories and of general logic programs as rule systems provide us with results for minimal sets of formulas closed under defaults, and about minimal models of logic programs.

**Proposition 3.15** *(1) If $< W, D >$ is a recursive default theory, then the family of minimal sets closed under defaults forms a $\Pi^0_2$ set.*

*(2) If $P$ is a finite (or recursive infinite) logic program in a recursive language, then the family of minimal Herbrand models of $P$ forms a $\Pi^0_2$ set.*

Proof: Directly from Proposition 3.10. $\Box$

# 4 Computing extensions, weak extensions, and minimal deductively closed sets

Three classes of structures associated with $< U, N >$ are investigated in this paper: deductively closed sets, weak extensions, and extensions. We can give algorithms for computing these structures for many common cases. First we discuss the case when $N$ consists of monotonic rules only. In that case $T$ is a monotonic operator. The following fact is due to Knaster and Tarski. It solves this case.

**Proposition 4.1** *Let $N$ consist of monotonic (that is rules without restraints) rules. Then:*

*(1) The operator $T$ is monotonic.*

*(2) There exists a least deductively closed set $S_0$ for $< U, M >$. $S_0$ coincides with the least prefixpoint for $T$, which is also the least fixpoint for $T$.*

*(3) $T$ possesses a largest fixpoint. (Thus a largest weak extension for $< U, N >$ exists.)*

*(4) $< U, N >$ possesses exactly one extension, this $S_0$.*

If we admit nonmonotonic rules with restraints the situation changes dramatically. All of properties (1)-(4) of Proposition 4.1 may fail.

**Example 4.1** *Let $U = \{\alpha, \beta\}$, $N = \{\frac{:\beta}{\alpha}, \frac{:\alpha}{\beta}\}$. The associated operator $T$ is nonmonotonic. $S_1 = \{\alpha\}$, $S_2 = \{\beta\}$ are all the minimal closed sets, all the weak extensions, and all the extensions. Thus (1)-(4) can fail in the nonmonotonic case.*

Testing whether or not $S$ is an extension, weak extension or deductively closed set can be carried out if $N$ is finite. In case $N$ is infinite it is sometimes possible to find a test. For default logic with a finite number of default rules including all the monotonic rules of classical propositional logic, see [Marek and Nerode, 1990].

We give three algorithms which test, for a subset $S \subseteq U$, whether or not $S$ is closed under the rules, whether or not $S$ is a weak extension and, finally, whether or not $S$ is an extension of $< U, N >$.

**Algorithm 1**

Input: A system $< U, N >$ and a subset $S \subseteq U$,

Output: A decision whether or not $S$ is deductively closed in $< U, N >$.

Method: For every rule $r = \frac{\alpha_1,\ldots,\alpha_n:\beta_1,\ldots,\beta_m}{\omega}$, test if $r$ is $S$-applicable, that is, whether $\alpha_1, \ldots, \alpha_m \in S$, $\beta_1, \ldots, \beta_m \notin S$. Mark all the conclusions of $S$-applicable rules. Test if all the marked objects belong to $S$. If so, return "yes", otherwise return "no".

**Algorithm 2**

Input: A system $< U, N >$ and a subset $S \subseteq U$,

Output: A decision whether $S$ is a weak extension of $< U, N >$.

Method: For every rule $r = \frac{\alpha_1,\ldots,\alpha_n:\beta_1,\ldots,\beta_m}{\omega}$, test if $r$ is $S$-applicable, that is, whether $\alpha_1, \ldots, \alpha_m \in S$, $\beta_1, \ldots, \beta_m \notin S$. Mark all the conclusions of $S$-applicable rules. Test if $S$ coincides with the collection of marked objects. If so, return "yes", otherwise return "no".

**Algorithm 3**

Input: A system $< U, N >$ and a subset $S \subseteq U$,

Output: A decision whether $S$ is an extension of $< U, N >$.

Method: For every rule $r = \frac{\alpha_1,\ldots,\alpha_n:\beta_1,\ldots,\beta_m}{\omega}$, test if $r$ is $S$-applicable that is whether $\alpha_1, \ldots, \alpha_m \in S$, $\beta_1, \ldots, \beta_m \notin S$. Eliminate all non-$S$-applicable rules. In the remaining rules eliminate all restraints, getting a monotonic system $< U, M_S >$. Compute the closure $C$ of the empty set $\emptyset$ with respect to the monotonic collection $M_S$. Test if $C$ coincides with $S$. If so, return "yes", otherwise return "no".

**Theorem 4.2** *Algorithms 1,2, and 3 test correctly whether or not $S$ is, respectively, a deductively closed set, a weak extension or an extension of $< U, N >$.*

Proof: The correctness of algorithms 1 and 2 follows from Proposition 3.6, part I, in which we provided a characterization of deductively closed sets and of weak extensions as, respectively, prefixpoints and fixpoints of the associated operator $T$. Correctness of algorithm 3 follows from Proposition 3.10, part I, where we proved the adequacy of the procedure of algorithm 3 for the construction of all extensions. $\Box$

One question not tested by algorithms 1, 2 and 3, is whether $S$ is minimal. Of course, algorithm 3, if successful, tests minimality as well, since every extension is a minimal deductively closed set. In other cases minimality is not automatically ensured. In this case, the subsets of $S$ must be tested as well.

The above algorithms test whether or not $S$ is, respectively, deductively closed, a weak extension or an extension, but do not provide a systematic method of constructing such $S$. We shall deal with this problem presently. We observe that both extensions and weak extensions of a system $< U, N >$ consist of conclusions of rules in $N$. Consequently, we need to consider subsets $S$ of the set of conclusions. In principle this is exponential in the cardinality of $N$. This method cannot be improved much since, as shown in [Marek and Truszczyński, 1988], the problem of finding an extension for collections of rules of the form $\frac{:p}{q}$ is $NP$-complete.

Now we deal with testing membership in the least fixpoint of a monotonic rule system. We need this as an auxiliary procedure for algorithm 3. Let $< U, M >$ be a monotonic system, with both $U$ and $M$ countable, and let $\vartheta \in U$. We describe two methods of testing whether $\vartheta$ belongs to the closure of $\emptyset$ under the rules of $M$, that is whether $\vartheta$ is in the least fixpoint of the associated monotone operator $T$. The objects considered here are "marked membership formulas" $T(\alpha \in S)$. (Formulas of the form $\alpha \in S$ are considered in Section 3) Recall that $\alpha$ is an axiom if $\alpha$ is the conclusion of a premiseless rule.

For the first method of testing membership, we introduce a storage space where we initially put all the formulas of form $T(\alpha \in S)$ for $\alpha$ an axiom, and $F(\vartheta \in S)$. Then, systematically for each rule $R = \frac{\alpha_1, \ldots, \alpha_n}{\omega}$ in $M$ which is still unmarked, we test if $T(\alpha_1 \in S), \ldots, T(\alpha_n \in S)$ are all in the storage. If so, we put $T(\omega \in S)$ into the storage, mark $r$ as used. As soon as $T(\vartheta \in S)$ appears in the storage, we close the storage and return that $\vartheta$ belongs to the least fixpoint of $T$.

**Proposition 4.3** *(van Emden, Kowalski) The procedure outlined above tests whether or not $\vartheta$ belongs to the least fixpoint of $T$.*

This procedure has disadvantages. It essentially generates the whole of least fixpoint, until the desired object is discovered. We describe a second method arising from tableaux.

Let Ord be the class of ordinals. We restrict our attention to the case when two restrictions are satisfied.

(1) There exists a function $f\colon U \to \mathrm{Ord}$ satisfying this condition: whenever $r \in M$, $r = \frac{\alpha_1,\dots,\alpha_n}{\vartheta}$, then for all $j \leq n$, $f(\alpha_i) < f(\vartheta)$.

(2) For every $\vartheta \in U$, there are only finitely many rules with conclusion $\vartheta$.

Systems $< U, M >$ satisfying condition (1) are called *ranked* and those satisfying (2) are called *quasi-finite*.

Let $< U, M >$ be such a system. Define a tableau procedure for $< U, M >$ as follows. At the root of the tableau we put the formulas $T(\alpha \in S)$ for all axioms $\alpha$, and also the formula $F(\vartheta \in S)$. Now we describe the tableau development rules. For a formula of form $F(\varphi \in S)$ on a non-closed branch $b$ that we need to extend, and for an unused rule $r = \frac{\alpha_1,\dots,\alpha_n}{\varphi}$, split the branch $b$ into $n$ successors, putting on each of those, respectively, $F(\alpha_i \in S)$. Mark the rule $r$ as "used for the branch $b$", and now, for every extended branch, test whether that branch contains a pair $T(\alpha_i \in S)$ and $F(\alpha_i \in S)$. Close each such branch.

Notice that different rule systems may generate operators with same least fixpoints. It may happen that one of the systems will be ranked and the other not. For instance, cut-elimination theorems can be interpreted as transformations of a non-ranked system to a ranked, quasi-finite system with the same associated operator.

**Theorem 4.4** *Let $< U, M >$ be a monotonic, ranked and quasi-finite system. Then an element $\vartheta \in U$ belongs to the least fixpoint of the associated operator $T$ if and only if every tableau for $\vartheta$ has all branches closed.*

The proof requires some lemmas.

**Lemma 4.5** *Let $< U, M >$ be a monotonic, ranked, and finite system. Then an element $\vartheta \in U$ belongs to the least fixpoint of the associated operator $T$ if and only if every tableau for $\vartheta$ has all branches closed.*

Proof: By induction on the rank of $\vartheta$. Our assumption will be that the property holds for all the elements of smaller rank, and for all systems differing from $< U, M >$ by having fewer rules with the conclusion $\vartheta$.

So, let the rank of $\vartheta$ be 0. Then $\vartheta \in lfp(T)$ precisely if $\theta$ is an axiom. But if $\vartheta$ is an axiom, then the tableau for $\vartheta$ is closed immediately. If $\vartheta$ is not an axiom, then the tableau for $\vartheta$ is not closed at all. Thus the induction base step of the lemma holds.

Now assume that the property holds for all $\vartheta'$ of rank smaller than $\vartheta$, and also for $\vartheta$ in systems with fewer rules with conclusion $\vartheta$.

So our inductive assumption is that every system differing from $< U, M >$ by having fewer rules has the property in the theorem, and that for all elements of $U$ of rank smaller than rank of $\vartheta$, the theorem holds.

First, assume that $\vartheta \in lfp(T)$. Consider $R$, a tableau for $\vartheta$ and assume that $R$ cannot be further extended. We show that every branch of $R$ is closed. Assume that there is a branch which is not closed. Take any such branch $b$, starting with $T(\alpha \in S)$, then $F(\vartheta \in S)$, then $F(\alpha_i \in S)$. The rule $r$ has been marked "used" at this stage. There are two cases to be considered:

Case (1): $\alpha_i$ belongs to $lfp(T)$. If $b$ is not closed, we eliminate from the tree all the references to the rules with conclusion $\vartheta$ and we get a tableau which is not closed for testing whether $\alpha_i$ belongs to $lfp(T)$.

Case (2): $\alpha_i$ does not belong to the $lfp(T)$. Then the least fixpoint of the system which arises from $< U, N >$ by eliminating the rule $r$ has the same least fixpoint. Now use the inductive assumption. Thus the tableau for $\vartheta$ (in the smaller system) has been closed, and consequently the same happened in the bigger system.

The converse implication, that is, showing that the points outside $lfp(T)$ have a non-closed tableau, is simpler. Again, we proceed by induction. If $\vartheta \notin lfp(T)$, then every rule with the conclusion $\vartheta$ must have a premise outside of $lfp(T)$. Thus we conclude that either we can continue without closing, or simply stop without closing the branch. $\square$

Next, for an element $\alpha \in U$ we define the closure of $\alpha$, $Cl(\alpha)$ as follows:

$Cl(\alpha) = \{\alpha\}$ if $\alpha$ is of rank 0.

$Cl(\alpha) = \{\alpha\} \cup \bigcup \{Cl(\beta_i) : \beta_i$ appears as a premise of some rule with the conclusion $\alpha\}$

We have

**Lemma 4.6** *$Cl(\alpha)$ is finite for every $\alpha \in U$.*

Proof: Since $< U, M >$ is quasi-finite, $Cl(\alpha)$ is the union of a finite number of terms. If, for some $\alpha$, $Cl(\alpha)$ is infinite, then for some $\beta$ with rank smaller than $\alpha$, $Cl(\beta)$ must be infinite. Then by induction, because $< U, M >$ is ranked, it follows that for some $\alpha$ of rank 0, $Cl(\alpha)$ is infinite, which contradicts definition of closure. $\Box$

Now, we are in the position to prove Theorem 4.4.

Proof: Let $\vartheta \in U$. We observe that in all the tableaux for $\vartheta$, only the elements of $Cl(\vartheta)$ appear. Consider the system $< Cl(\vartheta), M_\vartheta >$, where $N_\vartheta$ consists of those rules whose premises and conclusion all belong to $Cl(\vartheta)$. We observe that for $\alpha \in Cl(\vartheta)$ the tableaux with respect to $< U, M >$ and $< Cl(\vartheta), M_\vartheta >$ coincide. Moreover, the tableau development principles are identical. Then we observe the equivalence $\vartheta \in lfp(T_{U,M}) \Leftrightarrow \vartheta \in lfp(T_{cl(\vartheta),M_\vartheta})$. The implication $\Leftarrow$ is obvious, the converse follows from the fact that $T_{U,M}$ is finitary.

Finally, we get the sequence of equivalences:

$$\begin{aligned} \vartheta \in lfp(T_{U,N}) &\Leftrightarrow \vartheta \in lfp(T_{Cl(\vartheta)M_\vartheta}) \\ &\Leftrightarrow \text{ Every tableau for } \vartheta \text{ w.r.t. } < Cl(\vartheta), M_\vartheta > \text{ is closed} \\ &\Leftrightarrow \text{ Every tableau for } \vartheta \text{ w.r.t. } < U, M > \text{ is closed.} \end{aligned}$$

The first equivalence was discussed above, the second follows from Lemma 4.5, and the third one from Lemma 4.6. $\Box$

In principle, using tableaux may lead to an infinite descent, as witnessed by the following example:

**Example 4.2** *Let* $U = \{p_i : i \in \omega\}$*,* $N = \{r_i : i \in \omega\}$*,* $r_i = \frac{p_{i+1}}{p_i}$*. The query* $F(p_i \in S)$ *leads to an infinite descent (in spite of the fact that the least fixpoint of* $T$ *is empty).*

This example shows that unranked systems can lead to non-well-founded tableaux. Even if the system is ranked, if $M$ is not quasi-finite a similar phenomenon may occur.

**Example 4.3** *Let* $U = \{p_i : i \in \omega\}$*,* $N = \{r_i : i \in \omega\}$*.* $r_i = \frac{p_{i+1}}{p_0}$*. Here we also get an infinite descent, but for a different reason: once rule* $r_i$ *fails to get us a contradiction, we try the next one, with the same result.*

Checking whether $< U, M >$ is ranked is a graph-theoretic problem. First a well-known definition: If $G =< V, E >$ is a graph, then a *sorting* of $G$ is a linear ordering $\prec$ of $V$ such that $< a, b > \in E$ implies $a \prec b$. If $\prec$ is a well-ordering then we say that $G$ can be sorted into a well-ordering.

With a monotonic system $< U, M >$ we associate its *dependency graph* $G =< U, E >$ as follows: $< \alpha, \beta > \in E$ if and only if for some rule $r \in M$ $\beta$ is the conclusion of $r$ and $\alpha$ is one of the premises of $r$.

**Proposition 4.7** $< U, M >$ *is ranked if and only if there exists a sorting of its dependency graph into a well-ordering.*

Proof: (1) Sorting $G$ into a well-ordering determines a ranking function in the obvious fashion.

(2) A ranking function $f$ determines a sorting as follows. For each ordinal $\xi$, well-order $U_\xi = \{\alpha : f(\alpha) = \xi\}$ in any fashion. Then order $U = \bigcup U_\xi$ lexicographically. $\Box$

When $| U | < \omega$, every linear ordering of $U$ is a well-ordering. So the existence of ranking function for $< U, M >$ is equivalent to the fact that $G$ can be sorted. This, in turn, is equivalent to the fact that $G$ is acyclic.

# 5 Conclusions

We have proved a number of results on nonmonotonic rule systems. This theory allows us to capture many constructions appearing in the current literature on the logical foundations of artificial intelligence.

Our results provide additional tools tying these constructs with traditional methods of logic and recursion theory.

In a sequel we shall deal with rule systems containing variables in the rules and with predicate logics. We shall prove results related to the properties of recursive systems that are not necessarily highly recursive. We shall also explore connections with $L_{\omega_1,\omega}$.

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