

A theory of nonmonotonic rule systems I

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1 Introduction

In mathematics, a consequence drawn by a deduction from a set of premises can also be drawn by the same deduction from any larger set of premises. The deduction remains a deduction no matter how the axioms are increased. This is monotonic reasoning, much imitated in other, less certain, disciplines. The very nature of monotonic reasoning makes mathematical proofs permanent, independent of new information. Thus it has been since Euclid and Aristotle. Theorems with complete proofs are never withdrawn due to later knowledge. It is little exaggeration to say that mathematicians never reject the completed proofs of their predecessors, except to complain about their constructivity.

Mathematicians build directly on the works of their forebearers stretching back two and a half millenia to Euclid. Our current mathematical reasoning is merely a fleshed out version of Euclid's. Monotonic reasoning marks theoretical mathematics as a discipline. The traditional systems of mathematical logic are monotonic since they simply reflect mathematical usage. Tarski [1956] described a calculus of de-

ductive systems and captured in a simple way the general concept of a monotonic formal system. His formulation includes all logics traditionally studied: intuitionistic, modal, and classical. He did not qualify his definition, as we do, with the adjective “monotone,” because there were no other systems studied at that time.

Minsky [1975] suggested that there is another sort of reasoning which is not monotonic. This is reasoning in which we deduce a statement based on the absence of any evidence against the statement. Such a statement is in the category of beliefs rather than in the category of truths. Modern science offers statistics as a tool for establishing provisional beliefs, but in many instances we have no basis for applying statistics, due to a lack of governing distributions or samples for the problem at hand.

What role does belief play in our affairs? Often we must make sharp “yes or no” decisions between alternative actions. There may be no deductive or statistical base which justifies our choice, or there may be such a base but we may not be able to wait for missing information; it may never materialize anyway. Often all we have as a basis for decision is surmise, that is, deductions from beliefs as well as truths and statistically derived statements. These beliefs are often accepted, and used, as premises for deduction and choice of action due to an unquantified lack of evidence against them.

A philosopher’s much-quoted example is about Tweety. We observe only birds that can fly, and accept the belief that all birds can fly from the absence of evidence for the existence of non-flying birds. We are told that Tweety is a bird, and conclude that Tweety can fly using our belief as premise. Later, we observe that Tweety is

a pet ostrich and clearly can't fly. We reject our previous belief set and conclusions as a basis for decision making, and are forced to choose a new belief set. The new set of beliefs may also include equally uncertain statements, accepted due to a lack of evidence against them. But we blithely draw consequences from the new belief set and make decisions on that basis till contrary evidence on some accepted belief is garnered, at which time we again have to acquire a new set of beliefs.

This has happened in the history of practically every subject except mathematics. The principles of physics, or biology, have been changed with every scientific revolution, even though unreflective practitioners of each age think that final principles have been found. For mathematics, the Dutch mathematician and philosopher L.E.J. Brouwer would have argued that the belief in theorems established by “non-constructive methods” was unjustified, and that a new belief set based on constructive principles should be adopted in its place. Other mainstream mathematicians, such as Hilbert, did not agree with this position. Some philosophers of mathematics living now would argue that, even within classical mathematics, the independence proofs for propositions of set theory, such as the continuum hypothesis or the axiom of choice, indicate there are several incompatible axiomatic systems which, as belief sets, could be the foundation of mathematics.

One can envisage making up nonmonotone logics describing the mathematical nature of belief. The exact result depends on the definition chosen for “lack of evidence against”. McCarthy [1980], initiated the study of nonmonotonicity with his notion of circumscription. With all relation symbols but one, \bar{R} , of a model (the world we

are discussing) held fixed, and given axioms $\varphi(\bar{R})$ relating that \bar{R} to the other (fixed) relations of the model, the belief should be that, lacking further evidence to the contrary, \bar{R} denotes the least relation R , if any, satisfying $\varphi(R)$. If further evidence in the form of an additional axiom $\psi(\bar{R})$ becomes available, then we should believe that \bar{R} denotes the least R satisfying $(\varphi \wedge \psi)(R)$, if any, instead, in a changed belief set.

There are now many different nonmonotonic system, abstracted from different questions in computer science and AI. Among the other systems that have been studied are

Theory of multiple believers of Hintikka, [1962].

Truth Maintenance systems of Doyle, [1979]

Default logic of Reiter, [1980]

Autoepistemic logic of Moore, [1985]

Theory of individual and common knowledge and belief of Halpern and Moses [1984]

Logic programming with negation as failure [Apt, 1988].

This, by no means, exhausts the list. What issues in artificial intelligence or computer science motivate these systems?

Suppose that we build a robot in a “blocks world” to navigate in a room, avoid obstacles and perform simple tasks, such as crossing the room with variable obstacles.

We want the robot to learn principles from experience as to how to cross the room. At any given point, one may imagine that the robot should have a consistent deductively closed set of beliefs which are the current basis for its actions, including such provisional beliefs as “I can always traverse the left edge of the room since there has never been anything in the way there”. But when such a principle is contradicted by new obstacles, the robot has to choose another belief set. So an important problem is to define what a belief set is and how to compute them and how to update them based on new evidence. Moore’s autoepistemic logic [1985], is really a first try at this problem, mostly for propositional logic.

In computers, the operating system and program obey rules which compute how to change state. In the absence of exceptional behaviour, such as an error condition or failure to access resources, there is a system of decision rules (beliefs) computing how to change the state of the machine in this “normal behavior”, or “default” case. But when an exceptional behavior happens, we are thrown to a different set of decision rules for change of state, a different set of “beliefs”. One wants to be able to deduce what is true of the machine in states when it is in a particular such “belief set”. A logic for dealing with one such belief set at a time is Reiter’s default logic.

In databases, facts and rules are stored as entries (the PROLOG model). Often also the database computes and stores conclusions, such as summary statistics or rules or tables computed from the database. These act as a deductive base for the set of current beliefs. When we query the database, we are asking for consequences of this belief set. When we update the database, all old entries that have changed have to

be replaced and every consequence that uses these entries has to be recomputed and changed too. This is the process of replacing an old belief set by a new one. One often makes decisions on the basis of the absence of information in the database as well. A logic appropriate for describing a single such belief set is Doyle's truth-maintenance system [1979]. See also de Kleer [1986]. Also stable models for logic programming with negation as failure ([Gelfond and Lifschitz, 1988]) arise in this way.

We expressed these examples informally in terms of the anthropomorphic notion of belief so as to bring out their common features. The actual nonmonotonic logics have much in common, and a number of translations between them have been proposed ([Konolige, 1988], [Gelfond and Przymusińska, 1986],[Gelfond and Przymusińska, 1989], [Reinfrank and Dressler, 1989],[Marek and Truszczyński, 1989]). They have been investigated principally for propositional logic. Predicate versions suitable for actual applications are, up to now, pretty minimal.

Study of monotonic rule systems can be traced to the work of Post on "production systems" and to work of Tarski on the abstract properties of consequence relations for classical logic systems. The investigation of the nonmonotonic component is of much more recent nature and seems to appear first in the work of Reiter on default logic. Reiter's investigations involved finding a natural extension of classical logic which allows one to handle the negative information.

Independently Clark, and subsequently Apt, Blair and Walker, and also (extending their work) Gelfond and Lifschitz, studied negation as failure in logic programming. It has turned out that these investigations are in a common direction. Mu-

tual relationships were uncovered by Bidoit and Froidevaux, [1988], and Marek and Truszczyński, [1989b], who exhibited the precise nature of the connection between logic programming and default logic. The reevaluation of default extensions in terms of “context-dependent proofs” by Marek and Truszczyński, which has its roots in the Apt, Blair and Walker’s ([1987]) “elementary interpreter”, for which it may serve as a clarifying definition, is a point of departure for the investigations of this paper. Here, drawing on all the research mentioned above for inspiration, we present a coherent unified theory of nonmonotonic formal systems.

At the level of abstraction we achieve, we are finally able to see that nonmonotone systems pervade ordinary mathematical practice. There is no sign of any realization of the existence of such mathematical examples in the previous nonmonotonic logic literature. Perhaps these connections can only be seen by having a common abstract notion. What this commonality does for us is to make available known mathematical techniques from other areas of conventional mathematics for constructing and classifying belief sets (extensions) and, simultaneously, to make evident a common thread among disparate parts of mathematics and disparate nonmonotonic systems from artificial intelligence and computer science.

On the level of Mathematical Philosophy there is a connection worth stating as well. Non-monotone reasoning takes place during the process of discovery of mathematical theorems, when one posits temporarily some proposition on the basis of no evidence against it, and explores the consequences of such a belief until new mathematical facts force their abandonment. These nonmonotone belief sets have their

traces eradicated when final belief sets are achieved and demonstrative proofs are finished and published. The only hint of provisional belief sets left in mathematical papers is in the motivational remarks explaining what obstacles were overcome and by what changes in viewpoint the proof was achieved.

Here is the main definition. A nonmonotone rule system consists of a set U and a set of triples (α, β, γ) called rules. Here $\alpha = (\alpha_1, \dots, \alpha_n)$ is a finite sequence of elements of U , called premises, and $\beta = (\beta_1, \dots, \beta_k)$ is a finite sequence of elements from U , called restraints, and γ is an element of U . This is written, generalizing a notation of default logic, as

$$\frac{\alpha_1 \dots, \alpha_n: \beta_1, \dots, \beta_k}{\gamma}$$

The informal reading is: From $\alpha_1, \dots, \alpha_n$ being established, and β_1, \dots, β_k not being established now or ever, conclude γ . You may substitute “computed” for “established” for an informal reading in many applications. A subset S of U is called *deductively closed* if for every rule of the system, whenever $\alpha_1, \dots, \alpha_n$ are in S and β_1, \dots, β_k are not in S , then γ is in S . There are no variables here, these are not schema, this version is not the one appropriate for nonmonotone predicate logics. Nonmonotonic predicate logic cannot be explicated in a few lines and we defer that to a later paper.

The intersection of all deductively closed sets containing a set I is generally not deductively closed. But the intersection of a descending chain of deductively closed sets is deductively closed, and I may be contained in many minimal deductively closed sets over I . In the context of nonmonotone logic the intersection of all deductively

closed sets containing I is a (non-deductively closed) set, called the set of *secure consequences* of I . These are the propositions that a “skeptical reasoner” would take as beliefs based on I . The most important notion of contemporary nonmonotonic logic is that of *extension*. For a fixed subset S of U , one defines (finite) derivations from I , where all restraints encountered are outside S , and all premises encountered are either conclusions of previous rules or in I . This defines the set $C_S(I)$ of S -consequences of I . Extensions are those S such that $S = C_S(I)$. Extensions are minimal deductively closed sets containing I , but not conversely. Extensions represent the “deductively closed belief sets” that contain I . In these sets, if the negative restraints are all obeyed, we are reduced to monotone reasoning. See Section 3 for the exact definition.

These simple definitions capture the common content of the several theories of nonmonotonicity listed above, and of many mathematical theories as well. For example, the set of all marriages of the marriage problem can be formulated as exactly the set of all extensions in a nonmonotone rule system; similarly for the set of all k -colorings of graphs, the set of chain covers of partial orders, the Stone space of all maximal ideals in a Boolean algebra, etc. Similarly, for a commutative ring with unit there is a nonmonotone rule system such that the deductively closed sets are the prime ideals, the McCoy radical (the set of nilpotents) is the set of secured consequences of $\{0\}$, etc. There are similar nonmonotonic systems associated with virtually every algebraic system for which radicals of some sort have been defined and characterized. These mathematical examples have suggested a whole new set of techniques for finding extensions because of the availability of algorithms already investigated in the

mathematical literature on one or another of these problems, not previously known to be relevant to nonmonotonic logic in the artificial intelligence community. They do not arise from logic, but really from operations research. Finally, in recursion theory, priority constructions can be construed as nonmonotone systems, sets constructed by the priority argument as extensions. These ideas give many constructions of recursively enumerable extensions.

We spend a lot of effort in both this and subsequent papers to answer the following question. Exactly how complicated is the set of extensions of a recursively specified nonmonotonic system, and what is its structure? This is the analogue of the classical logic question, how complicated is the set of complete theories containing a recursively enumerable theory, and what is its structure? In classical logic, this leads to analyzing the character of the set of maximal ideals containing a given recursively enumerable ideal in a recursively presented free Boolean Algebra, a subject in which two of the authors have a lot of experience (see [Remmel, 1989], [Nerode and Remmel, 1985]). The simplest case covering many nonmonotonic systems arising from mathematics is that of “highly recursive” nonmonotone rule systems. There it turns out that extensions can, up to a one-to-one recursive map, be exactly any bounded Π_1^0 class of sets of natural numbers. So even in this case the computational problems are of the same level of difficulty as (say) solving “marriage problem” for highly recursive societies, or finding orderings of recursively presented formally real field ([Metakides and Nerode, 1977]), or finding an abscissa between 0 and 1 where a given recursive continuous function on $[0,1]$ takes a maximum value ([Huang and Nerode, 1985]).

This recursion-theoretic methodology can also be refined to give complexity-theoretic results on the same problems about extensions, as has been done in algebra by Nerode and Remmel in [1987], [1989], and [1990]. Since this is a more delicate matter than the recursion theory, these developments are deferred again to a later paper.

Next, we turn to investigations of the semantics of nonmonotonic rule systems. The fundamental common semantics we have found comes from $L_{\omega_1\omega}$, and generalizes the Clark completion of logic programming. It is perfectly general, and gives systematic semantics and completeness for all the nonmonotonic logics discussed above. Such uniform semantics are new. Some of the subjects never before had a decent semantics. We find semantical representations of extensions, weak extensions and deductively closed sets. This representation requires the creation of an additional infinitary language \mathcal{L}_S which properly encodes not only rules as “first order objects”, but also additional (infinitary) objects which characterize the class of intended structures be they extensions, weak extensions, deductively closed sets, etc. The previously established characterization of default logic, in terms of nonmonotonic rule systems, provides us with a semantics for default logic. This semantics, in opposition to the attempt of Etherington, satisfies Tarski’s conditions. That is, it allows us to introduce for defaults (virtual) negations, conjunctions, etc, and also a natural entailment relation. Computer scientists have a prejudice against $L_{\omega_1\omega}$ since on the surface it is removed from computation. But this is only on the surface – well-founded relations of small recursive ordinal height are the “correct” basis for our algorithms for computing extensions and represent $L_{\omega_1\omega}$ deductions – so perhaps the prejudice against

$L_{\omega_1\omega}$ should be reconsidered.

Finally, we turn to the issue of computing extensions, weak extensions, and sets closed under rules. We provide algorithms for testing whether a given subset of the universe U is an extension, weak extension, etc. Since we have previously provided a reduction algorithm to the monotonic case, an extension of the Truszczyński variant of the Gelfond-Lifschitz procedure for testing stability, we are able to use an analytical tableau method to compute the least fixpoint of a monotonic operator.

This short summary indicates that there is a great wealth of problems and results which naturally arise from nonmonotonic rule systems. Our study delineates the role of deduction schemata. This, in turn, connects our work naturally with studies of inductive definability. Our work indicates that logic programming is less related to predicate calculus, than to inductive definability. This is a paradigm different from Kowalski's. We do not claim that this is the only "correct" position, but we do claim that it leads to a new direction for research.

The predicate logic case is not treated in this paper. It will come out from a schematic version of the theory of this paper analogous to Post production systems. Along with U , the set of all strings over an alphabet, there are typed "metavariables" ranging over specific subsets of U called "types". There are "metastrings" built from the alphabet of U and string variables. Rules are of the same form as before, but use metastrings instead of strings. This point of view gives rise not only to a general theory, but also gives outright syntax, semantics, and completeness for new predicate versions of all the logics mentioned above. It also gives nonmonotone classical, or

intuitionistic, or modal predicate and propositional logics.

2 Monotonic formal systems

Tarski [1956] characterized monotonic formal systems by means of monotonic rules of inference. Such systems include intuitionistic logic, classical logics, modal logics, and many others. Suppose that a nonempty set U is given. In a particular application U may be the collection of all statements or all formulas, or all legal strings of a formal system, or of all atomic statements as in logic programming.

A *monotonic rule of inference* is a tuple $r = \langle P, \varphi \rangle$, where $P = \langle \alpha_1, \dots, \alpha_n \rangle$ is a finite list of objects from U , and φ is an element of U . Such a rule r is usually written in the suggestive form

$$r = \frac{\alpha_1, \dots, \alpha_n}{\varphi} \quad (1)$$

We call $\alpha_1, \dots, \alpha_n$ the premises of r and φ the conclusion of r .

Definition: (a) A monotonic formal system is a pair $\langle U, M \rangle$, where U is a nonempty set and M is a collection of monotonic rules.

(b) A subset $S \subseteq U$ is called *deductively closed* over $\langle U, M \rangle$ if for all rules $r \in M$, $\alpha_1, \dots, \alpha_n \in S$ implies $\varphi \in S$.

The collection $D = D(U, M)$ of deductively closed sets has the following properties:

- (1) $U \in D$
- (2) D is closed under arbitrary intersections.

Consequently, for every $I \subseteq U$ there is the least set $T(I)$ such that $I \subseteq T(I)$ and $T(I)$ is deductively closed. The operation $T(= T_{U,M})$ is monotone, that is, if $I \subseteq J$ then $T(I) \subseteq T(J)$. Moreover,

$$(3) T(I) = \bigcup \{T(J) : J \subseteq I \ \& \ |J| < \omega\}.$$

(Where ω is the cardinality of the set of natural numbers.)

Property (3) reflects the finitary nature of deductive closure and is closely associated with the definition of a deduction.

An *axiom* is a rule without premises, that is, with the list P empty.

A *deduction* of an object $\varphi \in U$ from $I \subseteq U$ is a finite sequence $\langle \varphi_1, \dots, \varphi_m \rangle$ such that $\varphi_m = \varphi$, and for all $i \leq m$, $\varphi_i \in I$, or φ_i is an axiom, or φ_i is the conclusion of a rule $r \in M$ such that premises of r are included in $\{\varphi_1, \dots, \varphi_{i-1}\}$. Then $S(I)$ consists of all elements of U that possess a deduction from I .

Tarski made the easy observation that if a collection D of subsets of U possesses properties (1), (2) and (3) above, then there is a collection of monotone rules M such that D is the set of all deductively closed sets in $\langle U, M \rangle$.

An abstract treatment of monotonic logic programming schemes and general methods of processing queries is discussed in [Blair, Brown, and Subrahmanian, 1989].

3 Nonmonotonic formal systems

Inspired by Reiter [1980], and Apt [1988], we introduce the notion of a nonmonotonic formal system $\langle U, N \rangle$. A *nonmonotonic rule of inference* is a triple $\langle P, G, \varphi \rangle$,

where $P = \{\alpha_1, \dots, \alpha_n\}$, $G = \{\beta_1, \dots, \beta_m\}$ are finite lists of objects from U and $\varphi \in U$. Each such rule is written in form

$$r = \frac{\alpha_1, \dots, \alpha_n; \beta_1, \dots, \beta_m}{\varphi} \quad (2)$$

Here $\{\alpha_1, \dots, \alpha_n\}$ are called the *premises* of rule r , $\{\beta_1, \dots, \beta_m\}$ are called the *restraints* of rule r .

Either P , or G , or both may be empty. If $P = G = \emptyset$ then the rule r is called an *axiom*.

A *nonmonotonic formal system* is a pair $\langle U, N \rangle$, where U is a non-empty set and N is a set of nonmonotonic rules.

Each monotonic formal system can be identified with the nonmonotonic system in which every monotonic rule is given an empty set of restraints.

A subset $S \subseteq U$ is called *deductively closed* if for every rule of N , if all premises $\alpha_1, \dots, \alpha_n$ are in S and all restraints β_1, \dots, β_m are not in S then the conclusion φ belongs to S .

In nonmonotonic systems, deductively closed sets are not generally closed under arbitrary intersections as in the monotone case. Tarski's axioms do not generally hold. But deductively closed sets are closed under intersections of *descending* chains. Since U is deductively closed, by the Kuratowski-Zorn Lemma, any $I \subseteq U$ is contained in at least one *minimal* deductively closed set. The intersection of all the deductively closed sets containing I is called the set of *secured consequences* of I . This set is also the intersection of all minimal deductively closed sets containing I . Deductively

closed sets are thought of as representing possible “points of view”. The intersection of all deductively closed sets containing I represents the common information present in all such “points of view”, containing I . (Generally in the literature, if we assign to a given I a collection \mathcal{M} of subsets of U , then assigning to I the intersection of \mathcal{M} is called the *skeptical reasoning* associated with \mathcal{M} and I .)

Example 3.1 Let $U = \{\alpha, \beta, \gamma\}$.

(a) Consider U with $N_1 = \{\frac{\cdot}{\alpha}, \frac{\alpha:\beta}{\beta}\}$. There is only one minimal deductively closed set $S = \{\alpha, \beta\}$. Then $\{\alpha, \beta\}$ is the set of secured consequences of $\langle U, N_1 \rangle$.

(b) Consider U with $N_2 = \{\frac{\cdot}{\alpha}, \frac{\alpha:\beta}{\gamma}, \frac{\alpha:\gamma}{\beta}\}$,

then there are two minimal deductively closed sets, $S_1 = \{\alpha, \beta\}, S_2 = \{\alpha, \gamma\}$. $\{\alpha\}$ is the set of secured consequences of $\langle U, N_2 \rangle$.

Example 3.1, (b) shows that the set of all secured consequences is not, in general, deductively closed in the nonmonotone case.

Let $\mathcal{P}(U)$ be collection of all subsets of U . Let $\mathcal{X} \subseteq \mathcal{P}(U)$ be given. Call a set X a *hitset* for \mathcal{X} if for all $Y \in \mathcal{X}, X \cap Y \neq \emptyset$. For each rule $r = \frac{\alpha_1, \dots, \alpha_n: \beta_1, \dots, \beta_m}{\varphi}$, define $Y_r = \{\beta_1, \dots, \beta_m, \varphi\}$. For $\langle U, N \rangle$, let $\mathcal{X}_{U,N} = \{Y_r: r \in N\}$. The following is a sufficient condition for X to be a deductively closed set in $\langle U, N \rangle$.

Proposition 3.1 *If X is a hit set for $\mathcal{X}_{U,N}$, then X is deductively closed in $\langle U, N \rangle$.*

The converse to Proposition 3.1 does not hold. If $U = \{\alpha, \beta, \gamma\}, N = \{\frac{\alpha:\beta}{\gamma}\}$, then \emptyset is deductively closed for $\langle U, N \rangle$, but this is not a hit set for $\mathcal{X}_{U,N}$.

When N is finite, it is easy to give an algorithm based on Proposition 3.1 for finding all minimal deductively closed sets. This algorithm is a variant of a “brute force” algorithm based on enumerating all hit sets.

Given a set S and an $I \subseteq U$, an S -deduction of φ from I in $\langle U, N \rangle$ is a finite sequence $\langle \varphi_1, \dots, \varphi_k \rangle$ such that $\varphi_k = \varphi$ and, for all $i \leq k$, each φ_i is in I , or is an axiom, or is the conclusion of a rule $r \in N$ such that all the premises of r are included in $\{\varphi_1, \dots, \varphi_{i-1}\}$ and all restraints of r are in $U \setminus S$ (see [Marek and Truszczyński, 1989], also [Reinfrank and Dressler, 1989]). An S -consequence of I is an element of U occurring in some S -deduction from I . Let $C_S(I)$ be the set of all S -consequences of I in $\langle U, N \rangle$. I is a subset of $C_S(I)$. Note that S enters solely as a restraint on the use of the rules imposed by the restraints in the rules. A single restraint in a rule in N may be in S and therefore prevent the rule from ever being applied in an S -deduction from I , even though all the premises of that rule occur earlier in the deduction. Thus S contributes no members directly to $C_S(I)$, although members of S may turn up in $C_S(I)$ by an application of a rule which happens to have its conclusion in S . For a fixed S , the operator $C_S(\cdot)$ is monotonic. That is, if $I \subseteq J$, then $C_S(I) \subseteq C_S(J)$. Also, $C_S(C_S(I)) = C_S(I)$.

Generally, $C_S(I)$ is not deductively closed in $\langle U, N \rangle$. It is perfectly possible that all the premises of a rule be in $C_S(I)$, the restraints of that rule are outside $C_S(I)$, but a restraint of that rule be in S , preventing the conclusion from being put into $C_S(I)$.

Example 3.2 $U = \{\alpha, \beta, \gamma\}, N = \{\frac{\cdot}{\alpha}, \frac{\alpha:\beta}{\gamma}\}, S = \{\beta\}$. Then $C_S(\emptyset) = \{\alpha\}$ is not

deductively closed.

However, the following holds:

Proposition 3.2 *If $S \subseteq C_S(I)$ then $C_S(I)$ is deductively closed.*

Proof: If all the premises of a rule r are in $C_S(I)$ and all the restraints are outside $C_S(I)$, then all the restraints are also outside S . □

We say that $S \subseteq U$ is *grounded* in I if $S \subseteq C_S(I)$.

We say that $S \subseteq U$ is an *extension* of I if $C_S(I) = S$.

Finally, we say that $S \subseteq U$ is a *weak extension* of I if $C_S(I \cup R) = S$,

where

$$R = \{\varphi: \text{for some } r \in N, r = \frac{\alpha_1, \dots, \alpha_n: \beta_1, \dots, \beta_m}{\varphi}, \alpha_1, \dots, \alpha_n \in S, \beta_1, \dots, \beta_m \notin S\}$$

Thus S is a weak extension if S is generated by I and the conclusions of rules that are applicable. The notion of weak extension is related to Clark's logic program completion and will be investigated below. The notion of groundedness is related to the phenomenon of "reconstruction". S is grounded in I if all elements of S are S -deducible from I (remember that S influences only the negative sides of rules). S is an extension of I if two things happen. First, every element of S is deducible from I , that is, S is grounded in I (this is an analogue of adequacy property in logical calculi). Second, the converse holds: all the S -consequences of I belong to S (this is the analogue of completeness). Thus extensions are analogues for a nonmonotonic systems of the set of all consequences for monotonic systems. Both properties (adequacy and completeness) need to be satisfied - if we want S to be an extension.

The third concept, weak extension, is a closure property. In the process of constructing $C_S(I)$, S is used to generate only negatively as a restraint. But we can relax our requirements and allow deductions that use S also on the positive side. That is, elements of S are not treated as “axioms”, but are used to generate objects from U by also testing the positive side of a rule for membership in S . This concept is closely related with the fixpoints of the operator T_P in logic programming, and Clark’s completion, see [Apt, 1988]. We shall prove a similar representation of weak extensions as fixpoints of an operator associated with a nonmonotonic system.

The notion of an extension is related to that of a minimal deductively closed set.

Lemma 3.3 *If S is an extension of I , then:*

- (1) S is a minimal deductively closed superset of I .
- (2) For every I' such that $I \subseteq I' \subseteq S$, $C_S(I') = S$.

Proof: (1) Clearly, S contains I . The operator C_S is anti-monotonic in S . That is, $S_1 \subseteq S_2$ implies $C_{S_2}(I) \subseteq C_{S_1}(I)$. Thus, if S is an extension of I and $S' \subseteq S$ is deductively closed, then $S' \subseteq S \subseteq C_S(I) \subseteq C_{S'}(I) \subseteq S'$. So $S = S'$.

(2) Assume that $I \subseteq I' \subseteq S$ and that $C_S(I) = S$. Then $C_S(I) \subseteq C_S(I') \subseteq C_S(C_S(I)) \subseteq C_S(I) = S$. □

Proposition 3.4 *The set of extensions of I forms an antichain. That is, if S_1, S_2 are extensions of I and $S_1 \subseteq S_2$, then $S_1 = S_2$.*

Proof: $S_1 \subseteq S_2 \subseteq C_{S_2}(I) \subseteq C_{S_1}(I) = S_1$. □

Proposition 3.5 *An extension of I is a weak extension of I .*

Proof: Assume that S is an extension of I , that is, $S = C_S(I)$. We need to prove that $S = C_S(I \cup R)$, where R is

$$\{\varphi: \text{for some } r \in N, r = \frac{\alpha_1, \dots, \alpha_n: \beta_1, \dots, \beta_m}{\varphi}, \alpha_1, \dots, \alpha_n \in S, \beta_1, \dots, \beta_m \notin S\}$$

Then

$$S \subseteq C_S(I) \subseteq C_S(I \cup R) \subseteq C_S(S) = S,$$

So $S = C_S(I \cup R)$. □

Given $S \subseteq U$, a rule r is called *S-applicable* if all the restraints of r are outside S and all the premises of r are in S . We define $N(S)$ to be the collection of all S -applicable rules.

With a nonmonotonic system $\mathcal{S} = \langle U, N \rangle$ we associate the operator $T = T_{\mathcal{S}}: \mathcal{P}(U) \rightarrow \mathcal{P}(U)$ defined as follows:

$$T_{\mathcal{S}}(I) = \{\varphi \in U: \exists_{r \in N} r = \frac{\alpha_1, \dots, \alpha_n: \beta_1, \dots, \beta_m}{\varphi}, \alpha_1, \dots, \alpha_n \in I, \beta_1, \dots, \beta_m \notin I\}$$

This operator is closely related to the operator T_P as considered in logic programming, see [Apt, 1988]. As in the case of general logic programs, the operator T is not monotone in general. But if \mathcal{S} is a monotone system, that is all rules in N have no restraints, then T is monotone. Operator T computes the collection of conclusions of I -applicable rules.

The following result generalizes one for logic programming.

Proposition 3.6 *Let $\langle U, N \rangle$ be a nonmonotonic rule system. Let T be its associated operator, and let $S \subseteq U$. Then:*

(1) $T(S) \subseteq S$ if and only if S is deductively closed.

(2) $T(S) = S$ if and only if S is a weak extension of \emptyset in $\langle U, N \rangle$.

Proof: (1) restates the definition of deductively closed set.

(2) \Rightarrow . We assume that $T(S) = S$ and then prove that S is a weak extension of \emptyset in $\langle U, N \rangle$. We must show that $S = C_S(R)$ where R is the set

$$\{\varphi: \text{for some } r \in N, r = \frac{\alpha_1, \dots, \alpha_n: \beta_1, \dots, \beta_m}{\varphi}, \alpha_1, \dots, \alpha_n \in S, \beta_1, \dots, \beta_m \notin S\}$$

(That is the collection of conclusions of S -applicable rules). To this end we need to prove two inclusions:

(a) $S \subseteq C_S(R)$. But we know that $S = T(S)$, that is, every $\varphi \in S$ is a conclusion of an S -applicable rule $r = \frac{\alpha_1, \dots, \alpha_n: \beta_1, \dots, \beta_m}{\varphi}$, with $\alpha_1, \dots, \alpha_n \in S, \beta_1, \dots, \beta_m \notin S$. Thus $\varphi \in R \subseteq C_S(R)$.

(b) $C_S(R) \subseteq S$. If $\gamma \in C_S(R)$, then there exists an S -deduction d from R with the conclusion γ . We leave the reader to check by induction on the length of such a deduction d that the conclusion of d is in S .

\Leftarrow . We now assume that S is a weak extension of \emptyset in $\langle U, N \rangle$. That is $C_S(R) = S$, where R is the set

$$\{\varphi: \text{for some } r \in N, r = \frac{\alpha_1, \dots, \alpha_n: \beta_1, \dots, \beta_m}{\varphi}, \alpha_1, \dots, \alpha_n \in S, \beta_1, \dots, \beta_m \notin S\}.$$

We prove that $S = T(S)$.

(a) $T(S) \subseteq S$. If $\alpha \in T(S)$ then by definition α is a conclusion of an S -applicable

rule. Thus $\alpha \in R \subseteq C_S(R) = S$.

(b) $S \subseteq T(S)$. By our assumption, if $\alpha \in S$ then $\alpha \in C_S(R)$. By induction on the length of an S -deduction of α , we leave the reader to prove that α belongs to $T(S)$. \square

We get as an immediate corollary

Proposition 3.7 *Let $S \subseteq U$. Then S is a weak extension of \emptyset in $\langle U, N \rangle$ if and only if the following conditions are met:*

(i) *S is closed under the rules in N . That is, if there is a rule $r \in N$ such that $r = \frac{\alpha_1, \dots, \alpha_n; \beta_1, \dots, \beta_m}{\varphi}$, $\alpha_1, \dots, \alpha_n \in S$, and $\beta_1, \dots, \beta_m \notin S$, then φ belongs to S .*

(ii) *Whenever $\varphi \in S$ then there is a rule $r \in N$ such that $r = \frac{\alpha_1, \dots, \alpha_n; \beta_1, \dots, \beta_m}{\varphi}$, with $\alpha_1, \dots, \alpha_n \in S, \beta_1, \dots, \beta_m \notin S$.*

It is worth noting that deductively closed sets here play the role that Herbrand models of programs play in logic programming. Weak extensions here play a role similar to that of supported models of programs, that is models of Clark's completion, in logic programming. This analogy will become clearer when we discuss the semantic issues (see also Section 3 of part II). Here extensions of rule systems play a role analogous to that of stable models of logic programs in logic programming.

Following the analogy with logic programming, a set S such that $T(S) \subseteq S$ is called a prefixpoint of T . Since T is not, in general, a monotone operator, the Knaster-Tarski theorem cannot be applied. There is no guarantee that T possesses a fixpoint. U is, of course, a prefixpoint of T , so prefixpoints of T always exist. Our remark that deductively closed sets are closed under intersections of descending

families implies the following corollary by the Kuratowski – Zorn lemma:

Corollary 3.8 *For every system $\langle U, N \rangle$, and for every $S \subseteq U$ which is a prefixpoint of T , there is a minimal prefixpoint S' of T , $S' \subseteq S$.*

With each rule r of form (2), we associate a monotonic rule of form (1)

$$r' = \frac{\alpha_1, \dots, \alpha_n}{\varphi} \quad (3)$$

obtained from r by dropping all the restraints. Rule r' is called the *projection* of rule r . Let $N(S)$ be the collection of all S -applicable rules. That is, a rule r belongs to $N(S)$ if all the premises of r belong to S and all restraints of r are outside of S . We write $M(S)$ for the collection of all projections of all rules from $N(S)$. The projection $\langle U, N \rangle|_S$ is the monotone system $\langle U, M(S) \rangle$. Thus $\langle U, N \rangle|_S$ is obtained as follows: First, non- S -applicable rules are eliminated. Then, the restraints are dropped altogether. We have the following characterization theorem:

Theorem 3.9 *A subset $S \subseteq U$ is an extension of I in $\langle U, N \rangle$ if and only if S is the deductive closure of I in $\langle U, N \rangle|_S$.*

Proof: If S is an extension of I , then S is closed under S -deductions. We can define a mapping from S -deductions from I in $\langle U, N \rangle$ to deductions from I in the monotone system $\langle U, M(S) \rangle$, by projecting the rules, that is, by dropping all restraints. By induction on the length of proofs we can show that none of the rules that were applied has been eliminated. Also, this map is surjective. This shows that if $S = C_S(I)$ (in $\langle U, N \rangle$), then also $S = C(I)$ in $\langle U, M(S) \rangle$.

Conversely, every deduction of φ in $\langle U, M(S) \rangle$ can be lifted to an S -derivation of φ in $\langle U, N \rangle$. An inductive argument shows that every S -derivation arises from lifting. This shows that $S = C(I)$ in $\langle U, M(S) \rangle$ and implies that S is an extension of I . \square

Actually, the projection procedure can be relaxed somewhat. Instead of eliminating all the rules that have a premise outside of S or a restraint in S , it is enough to eliminate those that have restraints in S . Indeed, the rule $r = \frac{\alpha_1, \dots, \alpha_n; \beta_1, \dots, \beta_m}{\varphi}$ with an $\alpha_i \notin S$ is either inapplicable or, if it is applicable, then it witnesses the fact that the $C_S(I)$ is not S . Let $N^+(S)$ be the collection of rules in N whose restraints are all outside of S , and let $\langle U, M^+(S) \rangle$ be the projection of $\langle U, N^+(S) \rangle$.

Proposition 3.10 *$S \subseteq U$ is an extension of I if and only if S is the set of all consequences of I in the monotonic system $\langle U, M^+(S) \rangle$.* \square

The “relaxed” procedure for generating suitable collection of monotonic rules is closely connected to the Gelfond-Lifschitz procedure ([1988]) for stable models of logic programs. The “tight” procedure is a generalization of the Truszczyński procedure, [Marek and Truszczyński, 1989b].

Theorem 3.9 tells us how to test if a collection $S \subseteq U$ is an extension of I in $\langle U, N \rangle$. In case U and N are finite this leads to an algorithm.

- (1) Compute $N(S)$.
- (2) Project $N(S)$ by dropping restraints to get $M(S)$.
- (3) Compute the deductive closure of I in $\langle U, M(S) \rangle$, call this T .

(4) Test whether $T = S$.

Finding all the extensions of a given I is a complicated problem. A brute force algorithm is to generate all subsets of U , and test each of them for being an extension using the procedure above. A useful fact for improving this algorithm is:

Proposition 3.11 *If S is an extension of I , then S consists entirely of elements of I and conclusions of certain rules in N .*

Thus an element of U that is not a conclusion of any rule is never in an extension.

A simple construction allows us to consider only extensions of the empty set. In fact, if \mathcal{S} is a nonmonotonic rule system, and $I \subseteq U$, then the system $\mathcal{S}(I)$ arises from \mathcal{S} and I by adding to N all the rules of the form $\frac{\cdot}{\vartheta}$ for all $\vartheta \in I$. We then have:

Proposition 3.12 *T is an extension of I in \mathcal{S} if and only if T is an extension of \emptyset in $\mathcal{S}(I)$.*

Consequently, we shall consider below extensions of \emptyset . We say that T is an extension of \mathcal{S} if T is an extension of \emptyset in \mathcal{S} .

4 Examples and Applications in Logic, Logic Programming, and Commonsense Reasoning

4.1 Classical Implicational Propositional Logic

Here the set U is the collection of all well-formed formulas of propositional logic, over some collection At of atoms with binary connective \Rightarrow and constant \perp . The standard

Lukasiewicz axiomatization is represented as a collection of rules of the form:

$$\begin{array}{c}
 \frac{}{\varphi \Rightarrow (\psi \Rightarrow \varphi)} \\
 \\
 \frac{}{(\varphi \Rightarrow (\psi \Rightarrow \vartheta)) \Rightarrow ((\varphi \Rightarrow \psi) \Rightarrow ((\varphi \Rightarrow \vartheta)))} \\
 \\
 \frac{}{((\varphi \Rightarrow \perp) \Rightarrow (\psi \Rightarrow \perp)) \Rightarrow (((\varphi \Rightarrow \perp) \Rightarrow \psi) \Rightarrow \varphi)} \\
 \\
 \frac{\varphi, \varphi \Rightarrow \psi:}{\psi}
 \end{array}$$

The collection of derivable elements of U is the set of tautologies of propositional logic. To represent the least fixed point of the operator associated with this system by the tableaux method requires the introduction of additional rules and the elimination of modus ponens (the fourth group) as a rule.

4.2 Other Classical Systems

Propositional logic may be represented in other ways as well, for instance in the language with the usual connectives $\neg, \wedge, \vee, \Rightarrow$. The collection U then consists of all well-formed formulas of such a language. The exact set of rules depends on the actual axiomatization adopted. Intuitionistic logic may be similarly represented as a rule system using appropriate axioms. Likewise for modal logics.

4.3 Ordinary Clausal Logic Programming

Here U is defined as follows: First, we have a fixed collection At of atoms. U is the set of all pairs $\langle a, L \rangle$ such that $a \in At$, and L is a finite subset of At ($L = \emptyset$ is allowed). Thus, a clause $p \leftarrow q_1, \dots, q_r$ is represented as a pair $\langle p, \{q_1, \dots, q_r\} \rangle$. A

logic program is a collection consisting of two lists of rules:

- (1) (Specific rules) Rules of form $\frac{\dot{\varphi}}{\varphi}$ for $\varphi \in P$ where $P \subseteq U$.
- (2) (Processing rules) Rules of form:

$$\frac{\langle \alpha, \emptyset \rangle, \langle \beta, L \rangle}{\langle \beta, L \setminus \{\alpha\} \rangle}$$

for every $\alpha \in L$.

The pure PROLOG processing method, creating and processing goals, works here.

It is worth mentioning that, although on surface the same things happen here as in logic programming, in reality more is being computed, namely, all the proper Horn clauses derivable from the clauses in the “specific rules”. It is, however, true that the atoms computed in this process constitute the least model of the program.

4.4 Propositional Logic Programming, monotonic rules case

Here the set U consists of logical atoms. This system does not contain processing rules at all. The specific clauses are represented in a different fashion, namely as rules:

$$\frac{\alpha_1, \dots, \alpha_n}{\varphi}$$

with $n \geq 0$.

The resulting system is monotonic. The collection of objects computed here also consists only of atoms.

4.5 Default logic

Again let U be the collection of all formulas of propositional logic \mathcal{L} . Following Reiter [1980] we introduce a default theory $\langle D, W \rangle$ as a pair where D is a collection of default rules, that is, rules of form

$$\frac{\alpha: M\beta_1, \dots, M\beta_m}{\omega}, \quad (4)$$

(where $\alpha, \beta_1, \dots, \beta_m$ are formulas) and W a collection of formulas of the language \mathcal{L} .

Represent such a default theory as a rule system consisting of three lists:

(i) Elements of $\omega \in W$ are represented as rules:

$$\frac{\vdots}{\omega}$$

(ii) Rules of form (4) are represented as

$$\frac{\alpha: \neg\beta_1, \dots, \neg\beta_m}{\omega}$$

(That is, the restraints of the rule representing a default rule r have an additional negation in front).

(iii) Processing rules of logic. That is, all the monotonic rules of the system of classical logic.

We then have the following proposition:

Proposition 4.1 *A collection $S \subseteq U$ is an extension of a system consisting of rules of type (i), (ii), and (iii) if and only if S is a default extension of $\langle D, W \rangle$.*

Computing all extensions of a default theory can be achieved using the techniques developed in Section 4, part II. Notice that during the computation we need to test

if the rule is S -applicable. This requires checking if $\alpha \in S$, $\neg\beta_1, \dots, \neg\beta_m \notin S$. This can be carried out by the ordinary tableaux method, or any other complete theorem prover. These remarks are the source of the algorithm of [Marek and Nerode, 1990], in which a version specifically tailored to default logic is developed.

4.6 Propositional logic programming, general case

A general logic program is a list of *general clauses*, of the form:

$$p \leftarrow q_1, \dots, q_n, \neg r_1, \dots, \neg r_m$$

We refer to [Gelfond and Lifschitz, 1988] for the definition of a stable model of such a program. That concept is a generalization of the *perfect models* as introduced in [Apt, Blair and Walker, 1987].

Let U be the collection of atoms under consideration. Represent a general clause as a rule:

$$\frac{q_1, \dots, q_n : r_1, \dots, r_m}{p}$$

The translation $tr(P)$ of a program P is the set of translations of its individual clauses.

The following result was proved in [Bidoit and Froidevaux, 1988] and [Marek and Truszczyński, 1989b]:

Proposition 4.2 *A subset $M \subseteq U$ is a stable model of P if and only if M is an extension of $tr(P)$.*

However weak extensions correspond to so-called supported models of P . (cf. Apt [Apt, 1988])

Proposition 4.3 *A subset $M \subseteq U$ is a supported model of P if and only if M is a weak extension of $tr(P)$.*

4.7 Logic programming with classical negation

We now discuss the so-called “logic programming with classical negation” of [Gelfond and Lifschitz, 1989] as a chapter in the theory of nonmonotonic formal systems.

Recall the basic notions introduced in [Gelfond and Lifschitz, 1989]. The collection of objects appearing in heads or bodies of clauses is the set of all literals, that is, atoms or negated atoms. In particular, a negated atom may appear in the head of a clause. Consider first “general Horn” clauses in which literals may appear in arbitrary places. To each set P of such clauses assign its *answer set*, the least collection A of literals satisfying the following two conditions:

- (1) If $a \leftarrow b_1, \dots, b_m$ is in P and $b_1, \dots, b_m \in A$ then $a \in A$.
- (2) If for some atom p , p and $\neg p$ are both in A , then A is the whole collection *Lit* of all literals.

Introduce a collection *Str* of *structural processing* rules over the set $U = Lit$.

These are all monotone rules of the form:

$$\frac{p, \neg p:}{a}$$

for all atoms p and literals a .

Translate the clause: $a \leftarrow b_1, \dots, b_n$ as rule:

$$\frac{b_1, \dots, b_n:}{a}$$

and let $\text{tr}(P)$ be the collection of translations of clauses in P plus the structural rules Str . Then we have

Proposition 4.4 *A subset $A \subseteq \text{Lit}$ is an answer set for P if and only if A is an extension of $\text{tr}(P)$. Since $\text{tr}(P)$ is a set of monotonic rules, such an answer set is the least fixpoint of the (monotonic) operator associated with the translation.*

Gelfond and Lifschitz then introduce general rules. Since the negation used in literals is not the “negation-as-failure” of general logic programming, Gelfond and Lifschitz introduce another negation symbol “*not*” and a general logic clause with classical negation in the form:

$$a \leftarrow b_1, \dots, b_n, \text{not}(c_1), \dots, \text{not}(c_m)$$

Then the *answer set* for a set P of clauses of this form is introduced by merging the operational procedure for the construction of stable models for a program (as introduced in [Gelfond and Lifschitz, 1988]) with the procedure above. They define the answer set for a program with classical negation as follows:

Let $M \subseteq \text{Lit}$ and P be a general program. Define P/M as a collection of clauses lacking *not* obtained as follows:

- (1) If a clause C contains a substring $\text{not}(a)$ where $a \in M$, then eliminate C altogether.
- (2) In remaining clauses eliminate all substrings of the form $\text{not}(a)$.

The resulting program P/M lacks the symbol *not*, so the answer set is well defined.

Let M' be the answer set for P/M . We call M an answer set for P precisely when

$M' = M$.

Gelfond and Lifschitz give a computational procedure for finding such answer sets, and subsequently reduce computing them to computing default logic extensions. Here we give a general result showing that the construction of Gelfond and Lifschitz is faithfully represented within nonmonotonic rule systems; here is how. Define U to be *Lit*, and translate the clause:

$$a \leftarrow b_1, \dots, b_n, \text{not}(c_1), \dots, \text{not}(c_m)$$

as the rule:

$$\frac{b_1, \dots, b_n : c_1, \dots, c_m}{a}$$

The translation of the program P then consists of the translations of individual clauses C of P , incremented by the structural rules Str . We get the following result:

Proposition 4.5 *Let P be a general logic program with classical negation and N_P be the translation described above. Then a collection M is an answer set for P if and only if M is an extension for the rule system $\langle U, N_P \rangle$.*

A number of facts proved in [Gelfond and Lifschitz, 1989] then become corollaries of the theory outlined here. Also, our results show that we get the same answer set if we tighten the operational definition of answer set for a general program, and change condition (1) to:

(1') If C contains in the body a literal a which is not preceded by *not* and does not belong to M or it contains a string $\text{not}(b)$ with b belonging to M , then eliminate C

altogether.

((2) is left unaltered.)

5 Representation of Combinatorial and Algebraic Problems as Structures associated with Rule Systems

The next set of examples show that the notions of extension, weak extension, and deductively closed set for nonmonotonic rule systems occur naturally throughout mathematics, particularly in combinatorics. This connection of nonmonotonic theories with everyday mathematics was not previously known.

5.1 The Marriage Problem

A *society*, $\mathcal{S} = \langle B, G, K \rangle$ is a set B of boys, a set G of girls such that $B \cap G = \emptyset$, and a relation $K \subseteq B \times G$, the intended meaning of $\langle b, g \rangle \in K$ being “ b knows g ”. A marriage for a society \mathcal{S} is a map $M: B \rightarrow G$. A marriage M is *proper* if M is one-to-one and for all $b \in B$, $M(b) = g$ implies $K(b, g)$. That is, in a proper marriage each boy marries a girl he knows. A marriage M is *symmetric* if M maps B onto G . In a symmetric marriage, every girl is married.

For finite societies Philip Hall ([1935]) gave a necessary and sufficient condition for the existence of a proper marriage, namely:

(*) For every finite set of boys $B' \subseteq B$, the set of girls that are all known by the boys of B' has cardinality greater or equal than that of B' .

Marshall Hall ([1948]) showed that condition (*) is also a necessary and sufficient condition for the existence of marriages in an infinite society \mathcal{S} as long as each boy knows only finitely many girls. Philip Hall's theorem is a special case of a more general theorem for finding transversals (see [Mirsky, 1971]).

We claim that if $\mathcal{S} = \langle B, G, K \rangle$ is a society satisfying (*) in which each boy knows only finitely many girls, then there is a nonmonotonic rule system $\mathcal{Z} = \langle U(\mathcal{S}), N(\mathcal{S}) \rangle$ such that the collection of extensions of \mathcal{Z} correspond exactly to the set of proper marriages of \mathcal{S} . To this end let us consider a collection of strings $U(\mathcal{S}) = \{Mbg : b \in B, g \in G, \text{ and } K(b, g) \text{ holds}\}$, where M is a new symbol. Then for each boy $b \in B$ if $\{g_1, \dots, g_n\}$ is the set of girls b knows, we add the following set of n rules to $N(\mathcal{S})$.

$$\frac{:Mbg_1, \dots, \widehat{Mbg_k}, \dots, Mbgn}{Mbg_k} \quad (5)$$

where we adopt the convention that for any n -element sequence s_1, \dots, s_n , let $s_1, \dots, \hat{s}_k, \dots, s_n$ be the $n - 1$ -element sequence that results from s_1, \dots, s_n by removing s_k .

For any girl g and any two boys $b_1 \neq b_2$, each of whom knows g , add the following rules to $N(\mathcal{S})$:

$$\frac{Mb_1g, Mb_2g:}{\varphi} \quad (6)$$

for every $\varphi \in U(\mathcal{S})$. Let $N(\mathcal{S})$ consist of all the rules of the form (5) or (6).

Theorem 5.1 *Let $\mathcal{S} = \langle B, G, K \rangle$ be a society satisfying (*), for which each boy knows only finitely many girls. Then E is an extension for $\mathcal{Z} = \langle U(\mathcal{S}), N(\mathcal{S}) \rangle$ if and only if $M_E = \{\langle b, g \rangle : Mb_g \in E\}$ is a proper marriage for \mathcal{S} .*

Proof. (\Rightarrow) First, observe that if $b \in B$ knows only one girl g , then the rule $\frac{\cdot}{Mbg}$ belongs to $N = N(\mathcal{S})$, so that $Mbg \in E$ for any extension E . Let $B_1 \subseteq B$ be the set of boys in B who know only one girl. Since \mathcal{S} satisfies condition (*), we know that there is a proper marriage for \mathcal{S} , so that there can be no girl g such that there exist boys b_1 and b_2 in B_1 for whom Mb_1g and Mb_2g are in $U = U(\mathcal{S})$. Hence if we can apply rules of the form (6), at least one of Mb_1g and Mb_2g is not derived from an axiom. This means that, if $B_1 = B$, then the only extension E of \mathcal{Z} is U and M_U is the unique proper marriage of \mathcal{S} . So assume $B_1 \neq B$. Then we claim that U is not an extension of \mathcal{Z} . Note that if $b \in B \setminus B_1$ then application of all rules of the form (5) for b is blocked by U . It then follows easily from our observation about the applicability of rules of the form (6) that we can never derive both the premises of rules of the form (6) from \emptyset . Thus $C_U(\emptyset) = \{Mbg \in U : b \in B_1\} \neq U$.

Since U is not an extension, for any extension E there cannot be $b_1, b_2 \in B$ and $g \in G$ such that Mb_1g and Mb_2g are in E , since otherwise the rules of the form (6) would show $E \subseteq C_E(\emptyset) \Rightarrow C_E(\emptyset) = U$. This automatically means that the map M_E is one-to-one. Since we can only use rules of the form (5) to derive elements of $C_E(\emptyset)$ for an extension E , it can not be the case that if $b \in B - B_1$, then there exist $g_i \neq g_j$ in G such that Mbg_i and Mbg_j belong to E , for then either Mbg_i or Mbg_j would block the application of any rule of form (5) for b . But then neither Mbg_i nor Mbg_j can be in $C_E(\emptyset)$, so $C_E(\emptyset) \neq E$. Also, if $b \in B$ we have to have at least one Mbg_i in E , otherwise all the rules of form (5) for b apply for E and hence $Mbg_i \in C_E(\emptyset) \setminus E$ for all i . Thus for each $b \in B$, there is exactly one g such that $Mbg \in E$. Thus M_E is

a one-to-one function from B to G . Finally notice that M_E is proper, because by our definition, $Mbg \in U$ implies that $K(b, g)$ holds. Thus if E is an extension, M defines a proper marriage.

(\Leftarrow) Suppose M is a proper marriage for \mathcal{S} and $E = \{Mbg: M(b) = g\}$. Then clearly, $M_E = M$. It is easy to see that rules of the form (6) allow us to derive Mbg precisely when $M(b) = g$. Moreover we can never apply a rule of the form (6) without at some point using rules of the form (5) to derive both premises of some rule of the form (6). Since M is a proper marriage, we can never derive Mb_1g and Mb_2g from rules of the form (5) if $b_1 \neq b_2$. Thus $C_E(\emptyset) = E$ and E is an extension. \square

We should observe that for our $\mathcal{Z} = \langle \mathcal{U}(\mathcal{S}), \mathcal{N}(\mathcal{S}) \rangle$, the only weak extension which is not an extension is $U(\mathcal{S})$. That is, if $E \subseteq U$ and any rule of the form (6) is applicable for E , then

$$R = \left\{ \varphi: \exists_{r \in N} \left(r = \frac{\alpha_1, \dots, \alpha_n: \beta_1, \dots, \beta_m}{\varphi} \wedge \alpha_1, \dots, \alpha_n \in E \wedge \beta_1, \dots, \beta_m \notin E \right) \right\} = U$$

So that $C_E(R) = U$. If no rule of the form (6) is E -applicable, and E is a weak extension, then the same argument used in the proof of Theorem 5.1 shows that M_E must be a proper marriage. Finally, observe that if D is deductively closed and $D \neq U$, then none of the rules of (6) are D -applicable. Moreover, it will be still the case that for each boy $b \in B$, the rules of the form (5) will force at least one $Mbg_i \in D$. However it is possible to have Mbg_i and Mbg_j in D with $g_i \neq g_j$, in which case none of the rules of the form (6) for b force elements into D . Thus deductively closed sets $D \neq U$ correspond to marriages in which each boy marries at least one girl he knows and may marry several girls he knows, but no girl is married to more than

one boy. Consequently, minimal deductively closed sets correspond again to proper marriages.

By expanding our set of rules $N(\mathcal{S})$, we can ensure that extensions correspond to proper symmetric marriages. That is, suppose that $\mathcal{S} = \langle B, G, K \rangle$ is a society in which every boy knows only finitely many girls, and every girl knows only finitely many boys, and there is a symmetric marriage for \mathcal{S} . Let U be defined as before. In addition to all rules of form (5) and (6), add a set of rules for each $g \in G$.

If $g \in G$ and $\{b_1, \dots, b_n\}$ is the set of boys that g knows, then add the following set of rules:

$$\frac{:Mb_1g, \dots, \widehat{Mb_kg}, \dots, Mb_ng}{Mb_kg} \quad (7)$$

Let $N_{Sym}(\mathcal{S})$ be the collection of rules of form (5), (6), and (7) and let $U_{Sym}(\mathcal{S}) = U$.

By a proof which is similar to that of Theorem 5.1, we can prove the following:

Theorem 5.2 *Let $\mathcal{S} = \langle B, G, K \rangle$ be a society such that each boy knows only finitely many girls and each girl knows only finitely many boys, and there is a proper symmetric marriage for \mathcal{S} . Then E is an extension for $\mathcal{Z}_{Sym} = \langle U_{Sym}(\mathcal{S}), N_{Sym}(\mathcal{S}) \rangle$ if and only if $M_E = \{\langle b, g \rangle : Mb_g \in E\}$ is a proper symmetric marriage for \mathcal{S} .*

5.2 Proper k -colorings of graphs

Let $\mathcal{G} = \langle V, E \rangle$ be a graph, where V is the set of vertices of \mathcal{G} , and E is the set of edges of \mathcal{G} . For $x \in V$, $Nb(x) = \{y \in V : \langle x, y \rangle \in E\}$ denotes the set of neighbors of x in \mathcal{G} . We say that \mathcal{G} is locally finite if $Nb(x)$ is finite for all $x \in X$. A map $C: V \rightarrow \{1, \dots, k\}$ is called a *proper k -coloring* if whenever $\langle x, y \rangle \in E$, then

$C(x) \neq C(y)$.

We claim that for any locally finite graph $\mathcal{G} = \langle V, E \rangle$, there is a nonmonotonic rule system $\mathcal{S} = \langle U(\mathcal{G}), N(\mathcal{G}) \rangle$ such that the extensions of \mathcal{S} correspond to the proper k -colorings of \mathcal{G} . Let $U(\mathcal{G}) = \{Cxi : x \in V \wedge i \in \{1, \dots, k\}\}$. Then, for each $x \in V$, we add the following set of rules to $N(\mathcal{G})$.

$$\frac{: Cx1, \dots, \widehat{Cxj}, \dots, Cxk}{Cxj} \quad (8)$$

for $j \in \{1, \dots, k\}$. Also, for each edge $\langle x, y \rangle \in E$ and $i \in \{1, \dots, k\}$ we add the following set of rules to $N(\mathcal{G})$.

$$\frac{Cxi, Cyi:}{\varphi} \quad (9)$$

for all $\varphi \in U(\mathcal{G})$.

By a proof which is very similar to that of Theorem 5.1, we can prove the following

Theorem 5.3 *Let $\mathcal{G} = \langle V, E \rangle$ be a locally finite graph. Then E is an extension for $\langle U(\mathcal{G}), N(\mathcal{G}) \rangle$ if and only if $C_E = \{\langle x, i \rangle : Cxi \in E\}$ is a proper k -coloring of \mathcal{G} .*

Again we can show that the only weak extension of $\langle U(\mathcal{G}), N(\mathcal{G}) \rangle$ which is not an extension is $U(\mathcal{G})$ itself. Also we can show that the deductively closed sets $D \subseteq U(\mathcal{G})$ correspond to the colorings for which each vertex is assigned at least one, but possibly many, colors and if $\langle x, y \rangle \in E$, then it cannot be the case that there is a color i that is assigned to both x and y .

5.3 Chain Covers of Partially Ordered Sets

Let $\mathcal{P} = \langle D, \leq_D \rangle$ be a partially ordered set, that is \leq_D is a binary relation on D such that

- (i) $\forall x \in D, x \leq_D x$,
- (ii) $\forall x, y, z \in D (x \leq_D y \wedge y \leq_D z \Rightarrow x \leq_D z)$,
- (iii) $\forall x, y \in D (x \leq_D y \wedge y \leq_D x \Rightarrow x = y)$.

For $x, y \in D$ we say that x is *incomparable* to y , written $x \mid y$, if neither $x \leq_D y$ nor $y \leq_D x$. We say that a set $A \subseteq D$ is an *antichain* if whenever $x \neq y$ and $x, y \in A$ then $x \mid y$. We define the *width* of \mathcal{P} to be w if the maximum size of an antichain is w . We say that a set $C \subseteq D$ is a *chain* if whenever $x, y \in C$, then either $x \leq_D y$ or $y \leq_D x$.

We say that a set of chains $\{C_1, \dots, C_n\}$ is a *chain cover* of \mathcal{P} or that \mathcal{P} is *covered* by C_1, \dots, C_n if $D = \bigcup_{i=1}^n C_i$. We say that a chain cover of \mathcal{P} is *disjoint* if the chains are pairwise disjoint. Clearly any chain cover for \mathcal{P} can be refined to a disjoint chain cover for \mathcal{P} . Dilworth [1950] proved the following fact for finite partially ordered sets:

Proposition 5.4 *If $\mathcal{P} = \langle D, \leq_D \rangle$ is a partially ordered set of width w , then there are w chains C_1, \dots, C_w in \mathcal{P} which cover \mathcal{P} .*

Dilworth's Theorem can be extended for partially ordered sets of arbitrary cardinality by applying the Compactness Theorem for propositional logic. Then for any partially ordered set $\mathcal{P} = \langle D, \leq_D \rangle$ of width w , we can define a rule system $\langle U(\mathcal{P}), N(\mathcal{P}) \rangle$ such that the extensions of $\langle U(\mathcal{P}), N(\mathcal{P}) \rangle$ correspond to disjoint chain covers

of \mathcal{P} by w chains C_1, \dots, C_w . Note that for disjoint chain covers C_1, \dots, C_w of \mathcal{P} , there is a natural map $\mathcal{C} \rightarrow \{1, \dots, w\}$ defined by $\mathcal{C}(x) = i$ if and only if $x \in C_i$. Thus, defining a disjoint chain cover is just like coloring the elements of D with one of w colors, while ensuring that x and y are colored differently if $x \mid y$. Let $U(\mathcal{P}) = \{Cxi : x \in D \wedge i \in \{1, \dots, w\}\}$. For each $x \in D$, we add the following set of rules to $N(\mathcal{P})$.

$$\frac{:Cxi, \dots, \widehat{Cxi}, \dots, Cxw}{Cxi} \quad (10)$$

for $j \in \{1, \dots, w\}$. For each $x, y \in D$ such that $x \mid y$ and each $i \in \{1, \dots, w\}$ we add the following rules to $N(\mathcal{P})$.

$$\frac{Cxi, Cyi}{\varphi} \quad (11)$$

for all $\varphi \in U(\mathcal{P})$.

By a proof which is very similar to the proof of Theorem 5.1, we can prove the following

Theorem 5.5 *Let $\mathcal{P} = \langle D, \leq_D \rangle$ be a partially ordered set of width w . Then E is an extension for $\langle U(\mathcal{P}), N(\mathcal{P}) \rangle$ if and only if $\langle C_1, \dots, C_w \rangle$ is a disjoint chain cover of \mathcal{P} (where for $i \in \{1, \dots, w\}$, $C_i = \{x \in D : Cxi \in E\}$).*

Again we can show that the only weak extension of $\langle U(\mathcal{P}), N(\mathcal{P}) \rangle$ which is not an extension is $U(\mathcal{P})$. Also we can show that the deductively closed sets $D \neq U(\mathcal{P})$ correspond to assignments in which each $x \in D$ is assigned to at least one, but possibly many, chains, and if $x \mid y$, then x and y are not assigned to the same chain. Thus deductively closed sets $D \neq U(\mathcal{P})$ correspond to arbitrary chain covers of \mathcal{P} .

The above examples are just a few of many examples where extensions correspond to “solutions” of certain natural combinatorial problems. Next, we turn to problems in algebra where extensions naturally occur.

5.4 Maximal Ideals in a Boolean Algebra

Let $\mathcal{B} = \langle B, \wedge, \vee, \neg, 0_{\mathcal{B}}, 1_{\mathcal{B}} \rangle$ be a Boolean Algebra, where \wedge, \vee , and \neg are the operations of meet, join, and complement for \mathcal{B} , and $0_{\mathcal{B}}$ and $1_{\mathcal{B}}$ denote the zero and unit of \mathcal{B} respectively. We let $\leq_{\mathcal{B}}$ denote the order induced by \mathcal{B} . That is, $x \leq_{\mathcal{B}} y$ if and only if $x \wedge y = x$.

$I \subseteq \mathcal{B}$ is an *ideal* of \mathcal{B} if

- (i) $0_{\mathcal{B}} \in I$,
- (ii) $\forall_{x,y \in B} x \leq_{\mathcal{B}} y \wedge y \in I \Rightarrow x \in I$, and
- (iii) $\forall_{x,y \in B} x \in I \wedge y \in I \Rightarrow x \vee y \in I$.

An ideal I is *maximal* if for any $x \in B$ exactly one of x and $\neg x$ is in I .

Given any Boolean Algebra $\mathcal{B} = \langle B, \wedge, \vee, \neg, 0_{\mathcal{B}}, 1_{\mathcal{B}} \rangle$ we can give a nonmonotonic rule system $\langle U(\mathcal{B}), N(\mathcal{B}) \rangle$ such that the extensions of $\langle U(\mathcal{B}), N(\mathcal{B}) \rangle$ correspond exactly to maximal ideals of \mathcal{B} . Let $U(\mathcal{B}) = B$. Then $N(\mathcal{B})$ consists of the following five classes of rules:

$$\frac{\cdot}{0_{\mathcal{B}}} \tag{12}$$

$$\frac{\cdot : x}{\neg x} \tag{13}$$

for all $x \in B$.

$$\frac{x, \neg x:}{y} \tag{14}$$

for all $x, y \in B$.

$$\frac{x:}{y} \tag{15}$$

for all $x, y \in B$ whenever $y \leq_B x$.

$$\frac{x, y:}{x \vee y} \tag{16}$$

for all $x, y \in B$.

Theorem 5.6 *Let $\mathcal{B} = \langle B, \wedge, \vee, \neg, 0_{\mathcal{B}}, 1_{\mathcal{B}} \rangle$ be a Boolean Algebra. Then E is an extension of $\langle U(\mathcal{B}), N(\mathcal{B}) \rangle$ if and only if E is a maximal ideal of \mathcal{B} .*

Proof: (\Rightarrow) Suppose that E is an extension of $\langle U(\mathcal{B}), N(\mathcal{B}) \rangle$. Then since $C_E(\emptyset) = E$, it is easy to see that the rules of form (12), (15) and (16) ensure that E is an ideal of \mathcal{B} . Rules of form (13) ensure that at least one of x and $\neg x$ is in E for all $x \in B$. If for some $x \in B$ both x and $\neg x$ are in E , then rules of the form (14) ensure $E = U(\mathcal{B}) = B$. Thus if $E \neq B$, then exactly one of x and $\neg x$ is in E for each $x \in B$. Hence E is a maximal ideal. Thus we need only to show that $U(\mathcal{B}) = B$ is not an extension. Note that all rules of the form (13) are blocked for B . Thus the only element we can derive to be in $C_B(\emptyset)$ with no premises is $0_{\mathcal{B}}$. But starting with $0_{\mathcal{B}}$, we can not apply rules (15) but only apply rules (15) where $x = 0_{\mathcal{B}}$, or rules (16) where $x = y = 0_{\mathcal{B}}$, and such rules give $0_{\mathcal{B}}$ as conclusion. Then, by an easy induction on the length of proofs, we can show that $C_B(\emptyset) = \{0_{\mathcal{B}}\}$ so that B is not an extension.

(\Leftarrow) Note that if E is a maximal ideal, then rules of the form (13) ensure $E \subseteq C_E(\emptyset)$.

Successive applications of rules (12), (15), and (16) will produce no new elements. Then we can prove by induction on the length of proofs that we can never apply rules of the form (14), hence $C_E(\emptyset) \subseteq E$. \square

Recall that F is a (maximal) filter of \mathcal{B} if and only if $\{x \in B: \neg x \in F\}$ is a (maximal) ideal in \mathcal{B} . Clearly we can set up a nonmonotonic rule system $\langle U_{fil}(\mathcal{B}), N_{fil}(\mathcal{B}) \rangle$ such that extensions of $\langle U_{fil}(\mathcal{B}), N_{fil}(\mathcal{B}) \rangle$ correspond to maximal filters of \mathcal{B} . Now, it is well known that a propositional logic modulo logical equivalence is a Boolean Algebra \mathcal{B} and that the maximal filters correspond to the complete theories. Thus we can also set up a nonmonotonic rule system $\langle U, N \rangle$ whose extensions correspond to complete theories of a propositional logic.

5.5 Complementary Subspaces of Vector Spaces

Let V_∞ be a countably infinite dimensional vector space over a finite field F , and let $B = \{b_0, b_1, \dots\}$ be a basis for V_∞ . If $S \subseteq V_\infty$, we let $(S)^*$ denote the space generated by S . Let $V_n = (\{b_0, \dots, b_n\})^*$ for $n \geq 1$. Given two subspaces A and B of V_∞ , we write $A + B$ for $(A \cup B)^*$ and $A \oplus B$ for $A + B$ if $A \cap B = \{0\}$, where 0 is zero vector of V_∞ .

Now, suppose that W is a subspace of V_∞ . We claim that we can define a nonmonotonic rule system $\langle U, N \rangle = \langle U_W(V_\infty), N_W(V_\infty) \rangle$ so that extensions of $\langle U, N \rangle$ correspond to the complementary subspaces for W that arise from the most natural construction of such spaces. That is, if one were going to construct a subspace A such that $A \oplus W = V_\infty$, a natural way to proceed would be to construct a sequence

of subspaces $A_0 \subseteq A_1 \subseteq \dots$ in stages as follows:

Stage 0 Let $A_0 = \{0\}$.

Stage s+1 Having defined a subspace $A_s \subseteq V_s$ such that $A_s \oplus W_s = V_s$ where $W_n = W \cap V_n$ for $n \geq 1$, we proceed according to one of two cases.

Case 1 W_s is properly included in W_{s+1} .

In this case, it is easy to show that $A_s \oplus W_{s+1} = V_{s+1}$, so we let $A_{s+1} = A_s$.

Case 2 $W_s = W_{s+1}$.

In this case it is easy to show that if we choose $x_{s+1} \in V_{s+1} \setminus V_s$ and $A_{s+1} = (A_s \cup \{x_{s+1}\}^*)$, then $A_{s+1} \subseteq V_{s+1}$ and $A_{s+1} \oplus W_{s+1} = V_{s+1}$.

Then $A = \bigcup_s A_s$ is the desired complementary subspace of W . Note that we can get many different such complementary subspaces depending on the choice of x_{s+1} at each stage in which Case 2 occurs.

We define our nonmonotonic rule system $\langle U, N \rangle = \langle U_W(V_\infty), N_W(V_\infty) \rangle$ as follows. We let $U = V_\infty$. Then we let N consist of the following five classes of rules:

$$\frac{\cdot}{0}, \tag{17}$$

$$\frac{x_1, \dots, x_k \cdot}{\sum_{i=1}^k \lambda_i x_i} \tag{18}$$

for all $x_1, \dots, x_k \in V_\infty$, and $\lambda_1, \dots, \lambda_k \in F$,

$$\frac{x \cdot}{v} \tag{19}$$

for all $x \in W \setminus \{0\}$ and $v \in V_\infty$,

$$\frac{\cdot}{x} \tag{20}$$

where $x \in V_{s_0}$ and s_0 is the largest s such that $W_s = \{0\}$ and $W_{s+1} \neq \{0\}$, and

$$\frac{a_1, \dots, a_n: b_1, \dots, b_m}{x} \quad (21)$$

where $A = \{a_1, \dots, a_n\}$ is a subspace and for some $s > s_0$, $A \oplus W_s = V_s$; $\{0\} \neq W_s = W_{s+1}$, $x \in V_{s+1} \setminus V_s$, and $\{b_1, \dots, b_m\} = V_{s+1} - (A \cup \{x\})^*$.

Theorem 5.7 *Let V_∞ be a countably infinite dimensional vector space over a finite field F , let $B = \{b_1, b_2, \dots\}$ be a basis for V_∞ and let W be a subspace of V_∞ . Then E is an extension of $\langle U, N \rangle = \langle U_W(V_\infty), N_W(V_\infty) \rangle$ if and only if E is a subspace such that $E \oplus W = V_\infty$ and for all $s \geq 1$, $E_s \oplus W_s = V_s$ (where $V_s = \{b_1, \dots, b_s\}^*$, $E_s = E \cap V_s$, and $W_s = W \cap V_s$).*

Proof: (\Rightarrow) Suppose that $E = C_E(\emptyset)$. First observe that the rules of the form (17), (18), and (20) ensure that E is a subspace of V_∞ containing V_{s_0} . Next, observe that the rules (19) ensure that if $E \cap W \neq \{0\}$, then $E = V_\infty$. We claim that $E = V_\infty$ is impossible. For if $E = V_\infty$, then all the rules of the form (21) are blocked by E . The remaining rules are monotonic. So it is easy to prove by induction on the length of deductions that the only $x \in V_\infty$ which can be deduced are the $x \in V_{s_0}$. But then $C_{V_\infty}(\emptyset) = V_{s_0} \neq V_\infty$ so that V_∞ is not an extension. Thus $E \neq V_\infty$, so by the rules (19) we conclude that $E \cap W = \{0\}$. Finally consider the rules of the form (21). We use these rules to prove by induction that for all $s \geq s_0$, $E_s \oplus W_s = V_s$, where $E_s = E \cap V_s$. First, we know $E_{s_0} = V_{s_0}$ and $W_{s_0} = \{0\}$ so that $E_{s_0} \oplus W_{s_0} = V_{s_0}$. Assume that $s \geq s_0$ and $E_s \oplus W_s = V_s$. Then if $W_s \subset W_{s+1}$, we know that $E_s \oplus W_{s+1} = V_{s+1}$. Since $E \cap W = \{0\}$, we must have $E_{s+1} = E \cap W_{s+1} = E_s$. If $W_s = W_{s+1}$, then we can

apply rules of the form (21) to conclude that for some $x \in V_{s+1} \setminus V_s$, $x \in E$. But then $E_{s+1} \supset E_s$ and so $E_{s+1} + W_{s+1} = V_{s+1}$. But again since we know that $E \cap W = \{0\}$, we must conclude that $E_{s+1} \cap W_{s+1} = \{0\}$ so that $E_{s+1} \oplus W_{s+1} = V_{s+1}$.

(\Leftarrow) Suppose that E is a subspace of V_∞ such that $E \oplus W = V_\infty$ and for all s , $E_s \oplus W_s = V_s$. It is easy to see that the rules (20) and (21) allow us to prove by induction on s that $E_s \subseteq C_E(\emptyset)$ for all s . Moreover it is not difficult to show by induction on the length of deductions that if we can deduce that some $x \notin E$ is in $C_E(\emptyset)$, then, at some point in the deduction we must use a rule of the form (21) to derive that some $y \in C_E(\emptyset)$, where $y \notin E$. But all the rules of the form (21) which have a y as a conclusion (where $y \notin E$) are blocked. Thus, we conclude that $C_E(\emptyset) \subseteq E$. Hence $C_E(\emptyset) = E$. \square

We note that the example of subspaces forms a prototype for how nonmonotonic rule systems can be used to mirror stage constructions of substructures in a variety of algebraic situations.

5.6 Prime ideals in a commutative ring \mathcal{R} , the radical and skeptical reasoning

Let $\mathcal{R} = \langle R, +, \cdot, 0, 1 \rangle$ be a commutative ring with unit. An *ideal* I is an additive subgroup of \mathcal{R} such that if $x \in I$ and $y \in R$ then $xy \in I$. I is said to be a *prime ideal* if $I \neq R$ and for all $x, y \in R$, if $xy \in I$, then either $x \in I$ or $y \in I$. The *McCoy radical* of \mathcal{R} is the intersection of all prime ideals of \mathcal{R} . We shall construct a nonmonotonic rule system $\langle U(\mathcal{R}), N(\mathcal{R}) \rangle$ such that the set of extensions of $\langle U(\mathcal{R}), N(\mathcal{R}) \rangle$ is exactly the set of prime ideals of \mathcal{R} . Since prime ideals can be included one in

another, while the extensions are always incomparable, we use a technique which allows us to represent both the ideal and its complement as an extension of a suitably chosen system.

Let $U(\mathcal{R}) = \{x, \bar{x} : x \in R\}$. \bar{x} is a new logical symbol that will be interpreted as indicating that x does not belong to I . $N(\mathcal{R})$ consists of the following eight classes of rules:

$$\frac{:}{0} \quad \frac{1:}{z} \quad (22)$$

for all $z \in U(\mathcal{R})$.

$$\frac{y:}{-y} \quad (23)$$

for all $y \in R$ (here $-y$ is the additive inverse of y)

$$\frac{x, y:}{x + y} \quad (24)$$

for all $x, y \in R$.

$$\frac{x:}{xy} \quad (25)$$

for all $x, y \in R$.

$$\frac{xy, \bar{x}:}{y} \quad \frac{xy, \bar{y}:}{x} \quad (26)$$

for all $x, y \in R$.

$$\frac{xy, \bar{x}, \bar{y}:}{z} \quad (27)$$

for all $x, y \in R$ and $z \in U(\mathcal{R})$.

$$\frac{:x}{\bar{x}} \quad \frac{:\bar{x}}{x} \quad (28)$$

for all $x \in R$.

$$\frac{x, \bar{x} :}{z} \quad (29)$$

for all $x \in R$ and $z \in U(\mathcal{R})$.

Theorem 5.8 *Let $\mathcal{R} = \langle R, +, \cdot, 0, 1 \rangle$ be a commutative ring with a unit. Then E is an extension of $\langle U(\mathcal{R}), N(\mathcal{R}) \rangle$ if and only if there is a prime ideal I of \mathcal{R} such that $E = \{x: x \in I\} \cup \{\bar{x}: x \in R \setminus I\}$.*

Proof: (\Rightarrow) Suppose that E is an extension of $\langle U(\mathcal{R}), N(\mathcal{R}) \rangle$. First we show that $U(\mathcal{R}) = U$ is not an extension of $\langle U(\mathcal{R}), N(\mathcal{R}) \rangle$. That is, it is easy to see that the only axiom is 0 and that derivation of any other element can be blocked. Then, by induction on the length of derivations we can prove that the only element we can generate starting with 0 by applying rules in $N(\mathcal{R})$ which are not blocked by U is 0 itself. Thus $C_U(\emptyset) = \{0\} \neq U$.

Thus $E \neq U$. But then we see that rules of form (28) and (29) will force exactly one of x or \bar{x} to be in E for each $x \in R$. Then let $I_E = \{x: x \in R \wedge x \in E\}$. Clearly rules of the form (22) –(27) force that I_E is a prime ideal. Thus $E = \{x: x \in I_E\} \cup \{\bar{x}: x \in R \setminus I_E\}$.

(\Leftarrow) Now suppose that I is a prime ideal of \mathcal{R} and $E = \{x: x \in I\} \cup \{\bar{x}: x \in R \setminus I\}$. The rules of the form (28) ensure that $E \subseteq C_E(\emptyset)$. Then, a straightforward induction on the length of derivations shows that $C_E(\emptyset) \subseteq E$. Thus E is an extension. \square

Finally, following the intuitions from nonmonotonic logics, if we compute the collection of nonmonotonic consequences of $\langle U(\mathcal{R}), N(\mathcal{R}) \rangle$ that is, the intersection of all extensions of $\langle U(\mathcal{R}), N(\mathcal{R}) \rangle$, then in the above case it corresponds to the radical of \mathcal{R} . Putting it more explicitly we get the collection $\{x: x \in J\} \cup \{\bar{x}: x \in W\}$

where J is the radical of \mathcal{R} and W is the set of units of \mathcal{R} .

6 Conclusions and the Contents of Part II

This paper introduced the concept of a nonmonotonic rule system. We demonstrated that many existing forms of nonmonotonic reasoning due to Reiter (Default Logic), Doyle (Truth Maintenance), Gelfond and Lifschitz (Stable Semantics for Logic Programs), Clark (Negation as Failure), are naturally represented as nonmonotonic rule systems. We demonstrated that when these logics are interpreted as nonmonotonic rule systems, we get uniform treatments for concepts, theorems and algorithms. We also gave purely mathematical examples of nonmonotonic rule systems which show that nonmonotonic rule systems occur widely in mathematical practice.

In Part II we will investigate the recursive complexity of the class of all extensions (or weak extensions or deductively closed sets) of a nonmonotonic rule system. In Section 5 we proved that the set of all marriages for a “marriage problem” is represented exactly as the set of extensions in suitably chosen nonmonotonic rule systems. In Part II we prove a converse for “highly recursive” societies. Similar result is obtained for graph colorings. A more general result, for recursive rule systems, and for general rule systems will be published in Part III of this paper. The connection with “marriage problem” provides a wealth of recursive-theoretic results on extensions of highly recursive nonmonotonic rule systems and therefore many new results for the known nonmonotonic logics mentioned above.

In Part II we give an “infinitary” semantics for nonmonotonic systems in the logic

$\mathcal{L}_{\omega_1, \omega}$ based on infinitary disjunction, conjunction and negation. In the case when the domain U of the nonmonotonic system is ω , using the arithmetical hierarchy of functions as in Rogers [1967], we find that the class of all extensions, weak extensions, closed sets and minimal closed sets. These classes are, respectively, $\Pi_2^{0,N}$ (effectively G_δ), $\Pi_2^{0,N}$, $\Pi_1^{0,N}$ (effectively closed), and $\Pi_2^{0,N}$. We get various results on Reiter's default theories, and Gelfond and Lifschitz stable models of general logic programs.

Finally, we discuss various algorithms of computing extensions, weak extensions, and minimal closed sets for nonmonotonic rule systems.

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