Modal logic for default reasoning

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Abstract

In the paper we introduce a variant of autoepistemic logic that is especially suitable for expressing default reasonings. It is based on the notion of iterative expansion. We show a new way of translating default theories into the language of modal logic under which default extensions correspond exactly to iterative expansions. Iterative expansions have some attractive properties. They are more restrictive than autoepistemic expansions, and, for some classes of theories, than moderately grounded expansions. At the same time iterative expansions avoid several undesirable properties of strongly grounded expansions, for example, they are grounded in the whole set of the agent's initial assumptions and do not depend on their syntactic representation.

Iterative expansions are defined syntactically. We define a semantics which leads to yet another notion of expansion — weak iterative expansion — and we show that there is an important class of theories, that we call $\mathcal{T}$-programs, for which iterative and weak iterative expansions coincide. Thus, for $\mathcal{T}$-programs, iterative expansions can be equivalently defined by semantic means. The question of existence of a semantic definition of iterative expansions for general theories remains open.

1 Introduction

One of the features of commonsense reasoning is the ability to draw conclusions in a situation of incomplete information. For example, if a certain property holds "un-

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der normal circumstances" we are willing to accept that it indeed holds if we do not have any information implying that the situation is "abnormal" However, a larger set of facts, including information that the situation is "abnormal" does not imply that the property holds. That is, a bigger set of assumptions does not imply all the conclusions that are implied by the smaller set of assumptions. This makes classical logic unsuitable as a tool for formally describing commonsense reasonings. Instead, several nonclassical logics were proposed. Most important among them are first order logic with closed world assumption (Reiter [1978]), default logic (Reiter [1980]), circumscription (McCarthy [1980]), inheritance nets with exceptions (Touretzky [1986]), logic programming with negation (Apt et al. [1987]), and autoepistemic logic (Moore [1985]).

The main idea is the same as in the classical logic. Namely, given a set of assumptions I, we want to construct a theory consisting of the consequences of I. However, in addition to classical means of reasoning, in deriving consequences we use inference rules that take into account the absence of information. Informally, if certain facts needed in a particular derivation are unknown, the mechanism makes some assumptions about them and uses these assumptions in the derivation. Usually, there are several possible ways of extending initial knowledge by assumptions about facts not explicitly described. Thus, a given set of assumptions can have several possible sets of consequences (and sometimes none). In other words, a given set of assumptions and a mechanism used for drawing conclusions specify several possible descriptions of the real world (or sometimes do not define any world at all). Our intention is that the set of original assumptions (knowledge base) and a mechanism to deal with incomplete information provide a possibly precise description of the world. To achieve that the modification of the set of initial assumptions by new assumptions added where explicit information was missing should be kept minimal.

Among the many existing formalisms that were proposed as a means of investigating commonsense reasoning Moore's autoepistemic logic [1985] has recently gained in importance. Autoepistemic logic is a modal logic with one modal operator K. The intended interpretation of a modal formula Kp is "p is known" to an agent. For a given set of assumptions I (some of them may involve the modal operator K) Moore defines the notion of an expansion of I (in his paper he uses the term extension instead of expansion). Expansions can be thought of as theories describing states of knowledge grounded in I that might be derived from I by an agent with perfect introspection capabilities. Let us note here that in his paper Moore introduced autoepistemic logic to model belief sets rather than knowledge states of an agent.

The mechanism used to define expansions allows an agent to modify the set of initial assumptions by adding modal facts of the forms $K\phi$ and $\neg K\phi$ that state which facts are known and which are not known to him/her. That makes expansions rather weakly grounded in the initial assumptions. For example, consider the set $I = \{Kp \Rightarrow p\}$. One of the expansions of this set contains p. This is so, because an agent can use in the reasoning formulas that state what he/she knows. Once the agent chooses to assume that p is known, that is assumes Kp, p can be derived which, in turn, justifies this added assumption and yields an expansion. Thus, p

is a consequence of I only because the agent chooses to assume that p is known to him. Because expansions are weakly grounded, they are difficult to relate to theories that serve as consequence sets in other formalisms like default extensions, truth maintenance system extensions, stable models for logic programs, all of which are more strongly grounded in the original assumptions I in the sense that in the reasoning the agent is not allowed to use formulas that state what is known but only the formulas that state what is unknown to him/her.

To find a relationship between default logic and autoepistemic logic Konolige [1988] introduced the notions of moderately and strongly grounded expansions. (The original definition of strongly grounded expansions as given in [Konolige, 1988] is incorrect. A correct version is given in [Konolige, 1989] and independently in [Marek and Truszczyński, 1989a]). Konolige [1988] showed that each strongly grounded expansion is moderately grounded and that each moderately grounded expansion is an expansion in the sense of Moore. His motivation was to find a subclass of thee class of expansions that would exactly correspond to default extensions. Konolige showed that under a certain interpretation of defaults as autoepistemic formulas, extensions of a default theory exactly correspond to strongly grounded expansions of its autoepistemic interpretation.

Konolige's strongly grounded expansions have, however, several counterintuitive properties. They depend on the syntactic representation of initial assumptions. That is, sets of assumptions equivalent in classical logic may have different strongly grounded expansions. In particular sets of formulas $\{p\}$ and $\{Kp \Rightarrow p,\ \neg Kp \Rightarrow p\}$

are equivalent in classical logic whereas the first one has a strongly grounded expansion and the other one hase none. Moreover, a strongly grounded expansion is grounded not in the set of initial assumptions I but in a subset of I, and in addition, different strongly grounded expansions may be grounded in different subsets. This is unnatural, as we expect an agent to design a knowledge base so that all its formulas are relevant to the worlds they define. We discuss these issues again in more detail at the end of Section 2.

The goal of this paper is to introduce and study a new variant of an expansion — an iterative expansion. In contrast to strongly grounded expansions, iterative expansions do not depend on the syntactic representation of the theory and are grounded in the whole set I. The definition is simple and follows the general approach to nonmonotonic logics originated by McDermott and Doyle [1980] and McDermott [1982]. For several important classes of theories the class of iterative expansions is contained in the class of moderately grounded expansions and coincides with the class of strongly grounded expansions. Most importantly, iterative expansions provide a particularly elegant and simple connection to default logic. Thus, the nonmonotonic modal logic based on the notion of iterative expansion provides a natural modal formalization of default logic.

In the paper we investigate properties of iterative expansions. We refer to the modal logic that is based on iterative expansions as *strong autoepistemic logic* (SAL, for short). Fundamental properties of iterative expansions are discussed in Section 2. In particular, the relation of iterative expansions to moderately grounded expansions and strongly grounded expansions is studied. The notion of iterative expansion is syntactic in nature. In the case of general autoepistemic theories we were not able to find an equivalent semantic definition. In Section 3 we show, however, that for a restricted class of theories, iterative expansions can be given a semantics similar to Moore's autoepistemic semantics for expansions. These considerations lead to another subclass of the class of all expansions of a theory. Expansions of this type are called *weak iterative* and are also studied in Section 3.

We also show that iterative expansions provide an elegant connection to default logic. In Section 4 we show an interpretation — different from the one used by Konolige — under which extensions of a default theory correspond exactly to iterative expansions.

We conclude this section with some terminology. The investigations of autoepistemic logic have so far concentrated on the propositional case. Some limited extensions to first order logic were also studied [Konolige, 1988], but despite the efforts of several researchers ([Marek, 1989, Niemelä, 1988a, Niemelä, 1988b]) no general predicate variant of autoepistemic logic has been discovered, yet. In this paper we also deal with only the propositional case. There are several notational conventions that we use throughout the paper. We fix a language $\mathcal{L}$ of propositional calculus and by Lit we denote the set of all literals (i.e. atoms or their negations) from $\mathcal{L}$. By $\mathcal{L}_K$ we denote the extension of $\mathcal{L}$ by the modal operator K. Thus, the autoepistemic formulas we consider are exactly formulas from $\mathcal{L}_K$. By Cn we denote the operator of propositional consequence. We will apply it to theories contained in $\mathcal{L}$ or $\mathcal{L}_K$. It will always follow from the context whether Cn should be treated as the consequence operator in $\mathcal{L}$ or in $\mathcal{L}_K$. For a theory $T \subseteq \mathcal{L}_K$ we define $KT = \{K\phi : \phi \in T\}$, $\neg T = \{\neg\phi : \phi \in T\}$, and $\overline{T} = \{\phi : \phi \notin T\}$. A formula of the form

$$K\phi_1 \wedge \ldots \wedge K\phi_k \wedge \neg K\psi_1 \wedge \ldots \wedge \neg K\psi_r \Rightarrow \omega, \tag{1}$$

where $\phi_i \in \mathcal{L}_K$, $1 \leq i \leq k$, $\psi_i \in \mathcal{L}_K$, $1 \leq i \leq r$, and $\omega \in \mathcal{L}$, is called an *autoepistemic clause* (*ae-clause*, for short). Note that the antecedent of an ae-clause may be missing. Consequently, each formula of $\mathcal{L}$ is an ae-clause. We often write ae-clauses as $A \Rightarrow \omega$, possibly with indices. If all ϕ_i, ψ_i belong to $\mathcal{L}$, than such an ae-clause is called a $\mathcal{K}$*-clause*. A $\mathcal{K}$-clause is called a *program* $\mathcal{K}$*-clause* if all ϕ_i, ψ_i and ω are literals of $\mathcal{L}$. An ae-clause of the form

$$K\phi_1 \wedge \ldots \wedge K\phi_k \wedge \neg KK\psi_1 \wedge \ldots \wedge \neg KK\psi_r \Rightarrow \omega, \tag{2}$$

where $\phi_i \in \mathcal{L}$, $1 \leq i \leq k$, $\psi_i \in \mathcal{L}$, $1 \leq i \leq r$, and $\omega \in \mathcal{L}$ is called a $\mathcal{T}$*-clause*, and if all ϕ_i, ψ_i and ω are literals of $\mathcal{L}$, a *program* $\mathcal{T}$*-clause*. Finally, a collection of program $\mathcal{K}$-clauses (resp. $\mathcal{T}$-clauses) is called a $\mathcal{K}$*-program* (resp. $\mathcal{T}$*-program*).

All modal logics considered in the paper have two inference rules: modus ponens and necessitation ($\phi/K\phi$). Their axioms include all propositional tautologies in $\mathcal{L}_K$ and some axiom schemata that specify properties of the operator K. Among most widely considered are for example (see [Chellas, 1980]):

K $K(\phi \Rightarrow \psi) \Rightarrow (K\phi \Rightarrow K\psi)$,

T $K\phi \Rightarrow \phi$,

4 $K\phi \Rightarrow KK\phi$,

5 $\neg K\phi \Rightarrow K\neg K\phi$.

By K we denote the modal logic based on the axiom K, T denotes the logic based on axioms K and T, $S4$ is the logic based on K, T and 4, $S5$ is based on K, T, 4 and 5, and $K45$ on K, 4 and 5.

To facilitate reading all the proofs were gathered at the end of the paper in the appendix.

2 Iterative expansions

The first important attempt to systematically study nonmonotonic reasoning within modal logic was made by McDermott and Doyle [1980] and was further developed by McDermott [1982]. Let $\mathcal{S}$ be a modal logic. By $Cn_{\mathcal{S}}$ we mean the consequence operator for logic $\mathcal{S}$. McDermott and Doyle described a construction which, for every modal logic $\mathcal{S}$ produces its nonmonotonic variant. They argued that in a nonmonotonic logic corresponding to $\mathcal{S}$, a theory T should be considered as a set of consequences of an initial theory I if and only if T is exactly the set of facts that can be derived from I and all modal facts of the form "$\neg\phi$ is consistent", that is, formulas from $\neg K\overline{T}$. McDermott and Doyle formalized this intuition by a fixed point equation

$$T = Cn_{\mathcal{S}}(I \cup \neg K\overline{T}), \tag{3}$$

and proposed to consider its solutions as candidates for the set of consequences of I. Let us call solutions to this equation $\mathcal{S}$-*expansions*. The operator $Cn_{\mathcal{S}}$ is, of course, monotonic. But T appears on both sides of the equation (3 and the dependence of T on I is no longer monotonic. In particular, a theory can have no $\mathcal{S}$-expansions, exactly one $\mathcal{S}$-expansion, or many $\mathcal{S}$-expansions. We refer to the logic based on (3) as the *nonmonotonic logic* $\mathcal{S}$.

The definition of an $\mathcal{S}$-expansion can be looked at in terms of context-dependent proofs. Let T be a theory, called the *context*. We say that a formula ϕ has in logic $\mathcal{S}$ a T-*proof* (context-dependent proof with context T) from I if ϕ has a proof in logic $\mathcal{S}$ from $I \cup \neg K\overline{T}$. A context T is an $\mathcal{S}$-expansion if T consists of exactly those facts that have a T-proof from I. Thus, in this type of context-dependent reasoning the context is used to modify the initial theory I. After that, we reason just like in a monotonic logic $\mathcal{S}$ and then, we accept the context T as a "consequence" of I precisely if T consists of formulas possessing a T-proof (in $\mathcal{S}$). Clearly the notion of the context-dependent proof can be extended by allowing context not only to have influence on how the initial theory is modified but also on what inference rules are used in the reasoning.

Note that (3) defines a whole family of modal nonmonotonic logics. Choosing different modal logic $\mathcal{S}$ will usually yield a different nonmonotonic logic. McDermott and Doyle [1980] considered the case where $\mathcal{S}$ is a classical propositional logic (note though that they did not allow for the necessitation rule). The resulting nonmonotonic logic had several counterintuitive properties and was abandoned as a possible formalism for describing commonsense reasonings. McDermott [1982] proposed several other candidates for logic $\mathcal{S}$, among them T, $S4$ and $S5$, and studied in detail the case of logic $S5$. However, the resulting nonmonotonic logic also did not turn out to be suitable for commonsense reasoning applications.

These investigations were continued by Moore [1985]. He introduced an apparently different nonmonotonic modal logic through the following fixed point equation:

$$T = Cn(I \cup KT \cup \neg K\overline{T}). \tag{4}$$

Moore called solutions to this equation *expansions* of I. It is easy to see that expansions satisfy the following three properties:

(ST1) $T = Cn(T)$,

(ST2) If $\phi \in T$ then $K\phi \in T$,

(ST3) If $\phi \notin T$ then $\neg K\phi \in T$.

The conditions (ST2) and (ST3) capture the intuition of full introspection of an agent. Think of T as the total knowledge of an agent. Then, if the agent knows ϕ (that is, if $\phi \in T$), then the agent knows that he/she knows ϕ (that is, $K\phi \in T$). Similarly, if the agent does not know ϕ (that is, $\phi \notin T$), then the agent knows that he/she does not know ϕ (that is, $\neg K\phi \in T$). Thus, Moore argued that expansions describe sets of beliefs of an agent with full introspection. Moore called the logic based on expansions *autoepistemic* logic.

The conditions (ST1) – (ST3) were introduced by Stalnaker [1980] who called any theory $T \subseteq \mathcal{L}_K$ which satisfies them *stable*. Intuitively, expansions of I are those stable sets containing I that are "grounded" in I, that is consisting of exactly those facts that the agent can derive from I and facts available to him/her through introspection.

Autoepistemic logic also falls into the category of context-dependent formalisms. The proof means used are the simplest possible, only modus ponens and tautologies of the propositional calculus, but the modification of I is more substantial than in the case of McDermott and Doyle's scheme — not only $\neg K\overline{T}$ but also KT are added to the original theory I. However, the difference between Moore's definition and the scheme of McDermott and Doyle is only superficial. Later, Shvarts [1988b] proved that a consistent theory is an expansion if and only if it is a $K45$-expansion (see also Corollary 3.8 below). Thus, the autoepistemic logic of Moore is just a special case of the general scheme of McDermott and Doyle.

In general, a theory may have multiple expansions or no expansion at all. For objective theories the situation is simpler. It was proved by Moore [1985] that for each subset $S \subseteq \mathcal{L}$ there is exactly one stable set T satisfying $Cn(S) = T \cap \mathcal{L}$. Moreover, this unique stable set is also the unique expansion of S. For a consistent set S, this expansion (denoted in the paper by $E(S)$) can be defined by an iterative process as follows [Marek, 1989]: define $\mathcal{L}_{K,n}$ to be the subset of $\mathcal{L}_K$ consisting of all formulas with K-depth at most n and put

$$E_0(S) = Cn(S) \cap \mathcal{L},$$

$$E_{n+1}(S) = \mathcal{L}_{K,n+1} \cap Cn(E_n(S) \cup KE_n(S) \cup \neg K(\mathcal{L}_{K,n} - E_n(S))),$$

and finally,

$$E(S) = \bigcup_{n=0}^{\infty} E_n(S).$$

It can be shown [Marek, 1989] that for every non-negative integer n,

$$E_n(S) = E(S) \cap \mathcal{L}_{K,n}.$$

As we already mentioned, expansions are rather weakly grounded in the initial theory. In some expansions, a fact p is present simply because the agent assumed that p is known and this assumption allows for the derivation of p (for instance in the case of the theory $\{Kp \Rightarrow p\}$). In other words, Kp is established before p is. Our approach is to introduce a variant of the notion of expansion, that requires that in order to prove Kp, fact p has to be derived earlier and without a reference to Kp. To this end we introduce an operator A by

$$A(S) = Cn(S \cup KS).$$

The operator A is monotone and finitizable. Let $I \subseteq \mathcal{L}_K$ and $T \subseteq \mathcal{L}_K$. Define

$$A_0^T(I) = Cn(I \cup \neg K\overline{T}),$$

$$A_{n+1}^T(I) = A(A_n^T(I)) = Cn(A_n^T(I) \cup KA_n^T(I)),$$

and let

$$A^T(I) = \bigcup_{n=0}^{\infty} A_n^T(I).$$

Observe that $A^T(I)$ is simply the set of all consequences of $I \cup \neg K\overline{T}$ in the logic that consists of all tautologies of the propositional calculus, uses modus ponens and necessitation. We will denote such a logic by $\mathcal{N}$. Thus,

$$A^T(I) = Cn_{\mathcal{N}}(I \cup \neg K\overline{T}). \tag{5}$$

So, instead of considering a rich modal logic like $T, S4, S5$ (McDermott [1982], Shvarts [1988a, 1988b]) or $K45$ (Konolige [1988], Shvarts [1988b]), we consider the simplest possible modal logic with no axiom schemata involving the operator K other than the tautologies of $\mathcal{L}_K$. Thus, in fact, our logic is very close to the logic originally considered by McDermott and Doyle [1980], the only difference being that we use necessitation as an inference rule.

For a theory I, we call its $\mathcal{N}$-expansions *iterative expansions*. The nonmonotonic logic corresponding to logic $\mathcal{N}$ will be called *strong autoepistemic logic* (SAL, for short). In the remainder of this section we present several fundamental properties of the operator $A^T(I)$ and iterative expansions. The first result (established in [Marek and Truszczyński, 1989a]) shows that the class of iterative expansions of I is a subclass of the class of all expansions.

Proposition 2.1 *Let $I \subset \mathcal{L}$. If T is an iterative expansion of I, then T is an expansion of I.*

The converse of this statement does not hold. For example, the theory $\{Kp \Rightarrow p\}$ has two expansions: $E(TAUT)$, where $TAUT$ stands for the set of all tautologies in $\mathcal{L}$, and $E(\{p\})$. Only the first of them is iterative. There is, however, a wide class of theories for which notions of expansion and iterative expansion coincide. Let us call an ae-clause of the form (1) *negatively determined* if $k = 0$, that is if the antecedent of the clause does not contain conjuncts $K\phi$. We have the following theorem.

Theorem 2.2 *Let I consist of negatively determined ae-clauses. Then $T \subseteq \mathcal{L}_K$ is an expansion of I if and only if T is an iterative expansion of I.*

Let us call an ae-clause *positively determined* if the antecedent of the clause does not contain conjuncts $\neg K\phi$, that is $r = 0$ in (1). An important property of theories consisting of positively determined $\mathcal{T}$-clauses is that they have exactly one iterative expansion — possibly inconsistent.

Theorem 2.3 *If a theory I consists of only positively determined $\mathcal{T}$-clauses then I has exactly one iterative expansion T. Moreover, for any stable set T' containing I, $T \cap \mathcal{L} \subseteq T' \cap \mathcal{L}$.*

Theorem 2.3 indicates that there is a strong analogy between positively determined $\mathcal{T}$-clauses in the language $\mathcal{L}_K$ and Horn clauses in logic programming, and between iterative expansions of theories consisting of positively determined clauses and least Herbrand models of Horn programs. The analogy would be even richer if we extended the definition of a Horn clause to allow a negated atom in the head. Then, as in the case of theories consisting of positively determined ae-clauses, an inconsistent set of literals might be the only "answer set" for an extended Horn program.

As we said earlier, expansions are rather weakly grounded in I because, when reasoning, an agent is allowed to use in addition to facts from I all facts in $KT \cup \neg K\overline{T}$. The idea behind the notion of the iterative expansion is to allow in reasoning only the facts in $\neg K\overline{T}$ in addition to the facts in I. Another appealing class of expansions can be defined through the parsimony (or minimality) principle. Let T_1 and T_2 be two stable sets. We say that $T_1 \sqsubseteq T_2$, if $T_1 \cap \mathcal{L} \subseteq T_2 \cap \mathcal{L}$, that is, if the objective part of T_1, that is $T_1 \cap \mathcal{L}$, is contained in the objective part of T_2 ($T_2 \cap \mathcal{L}$). Note that $\sqsubseteq$ is a partial ordering for the class of stable sets. Following Konolige [1988] we say that a stable set T containing I (resp. expansion T of I) is $\sqsubseteq$-*minimal* if no other stable set T' containing I satisfies $T' \sqsubseteq T$. Note, that according to this definition, if T is an expansion of I and no other expansion T' of I satisfies $T' \sqsubseteq T$, it does not necessarily mean that T is $\sqsubseteq$-minimal. The expansion T is $\sqsubseteq$-minimal only if no other **stable** set T' containing I, satisfies $T' \sqsubseteq T$. Minimal expansions and stable sets are attractive candidates for the set of beliefs (or known facts) of an agent with an initial set of beliefs (knowledge) I because they limit the assumptions the agent makes about the world. Konolige [1988] proved that $\sqsubseteq$-minimal expansions coincide exactly with moderately grounded expansions.

Below we show that for an important class of theories iterative expansions are $\sqsubseteq$-minimal stable sets (and hence, $\sqsubseteq$-minimal expansions). Consequently, by Konolige's result, for these classes of theories each iterative expansion is moderately grounded.

Theorem 2.4 *Let $I \subseteq \mathcal{L}_K$ consist of $\mathcal{K}$-clauses. Then, every iterative expansion of I is $\sqsubseteq$-minimal. Consequently, each iterative expansion of I is moderately grounded.*

Let us note that Theorem 2.4 does not hold in general. Consider the theory $I = \{\neg K(\neg Kp) \Rightarrow p\}$. Theory I has two iterative expansions: $E(\emptyset)$ and $E(\{p\})$. Thus, the iterative expansion $E(\{p\})$ is neither a $\sqsubseteq$-minimal expansion nor a $\sqsubseteq$-minimal stable set.

A similar result holds for $\mathcal{T}$-clauses. It is a simple corollary from Theorem 4.2 (Section 4), and the proof follows immediately from the proof of Theorem 4.2 in the appendix.

Theorem 2.5 *Let I consist of $\mathcal{T}$-clauses. Then, every iterative expansion of I is $\sqsubseteq$-minimal. Consequently, each iterative expansion of I is moderately grounded.*

A similar example as before ($I = \{\neg KK(\neg Kp) \Rightarrow p\}$) shows that the converse fails in this case, too. Consider now two theories $I_1 = \{\neg Kp \Rightarrow q,\ Kp \Rightarrow p\}$ and $I_2 = \{\neg KKp \Rightarrow q,\ Kp \Rightarrow p\}$. Each has two moderately grounded expansions $E(p)$ and $E(q)$ but only one iterative expansion $E(q)$. Thus, for theories consisting of $\mathcal{K}$-clauses and for theories consisting of $\mathcal{T}$-clauses, iterative expansions form a strictly smaller class than the class of moderately grounded expansions.

Logic $\mathcal{N}$ can be also used to define several types of expansions other than iterative expansions. The approach is based on the following result (see [Marek and Truszczyński, 1989a]).

Proposition 2.6 *Let $I \subseteq \mathcal{L}_K$. Let $I' \subseteq I$ and let T be an iterative expansion of I'. If $I \subseteq T$, then T is an iterative expansion of I.*

Different choices for I' thus lead to different classes of expansions. Consider a theory I consisting of ae-clauses. Define

$$GC(I,T) \;=\; \{K\alpha_1 \wedge \ldots \wedge K\alpha_m \wedge \neg K\beta_1 \ldots \wedge \neg K\beta_n \Rightarrow \omega \in I : \\ \alpha_1, \ldots, \alpha_m \in T,\ \beta_1, \ldots, \beta_n \notin T\}.$$

Clauses in $GC(I,T)$ are called *generating*. We have the following proposition.

Proposition 2.7 *If* $T = A^T(GC(I,T))$ *then* T *is an iterative expansion of* I*.*

Iterative expansions satisfying $T = A^T(GC(I,T))$ will be called GC-*iterative*. Construction $GC(I,T)$ was first given in Marek and Truszczynski [1989a], where it was shown that GC-iterative expansions (called there strongly iterative) correspond exactly to extensions of default theories thus giving an alternative description of the relationship between default and autoepistemic logics. As a consequence, it follows that GC-iterative expansions coincide with strongly grounded expansions of Konolige.

Let us discuss now iterative and GC-iterative (strongly grounded) expansions as candidates for the sets of conclusions of an agent with full introspection. First, observe that different GC-iterative expansions of I may be grounded in different subtheories of I. This is unnatural, as each set of consequences of I should be grounded in I in its entirety. The theory I is not arbitrary — it corresponds to the agent's perception of the world. Thus we have to assume that the agent designed I to fit his/her knowledge, and we should avoid modifying it. In addition, since theory $GC(I,T)$ depends on the syntactic representation of the formulas in I, GC-iterative expansions depend on the syntactic representation of the formulas in I. For example, theories $\{Kp \Rightarrow p, \neg Kp \Rightarrow p\}$ and $\{p\}$ are logically equivalent but only the latter has a GC-iterative expansion $E(p)$. This is certainly an unpleasant property of GC-iterative (strongly grounded) expansions. Secondly, and perhaps more importantly, because GC-iterative expansions are grounded in $GC(I,T)$ they are too restrictive. Information that can be obtained from I by means of logic $\mathcal{N}$ (and that naturally should form a part of any set of consequences of I) sometimes will not be derivable from $GC(I,T)$. For example, if $I = \{Kp \Rightarrow q,\ K(Kp \Rightarrow q) \Rightarrow a\}$, then $E(a)$ is a natural expansion of I (it is iterative), yet I has no strongly grounded expansions. In particular, $T = E(a)$ is not a strongly grounded expansion of I because $GC(I,T) = \{K(Kp \Rightarrow q) \Rightarrow a\}$, and does not allow for a derivation of a.

Iterative expansions do not have these disadvantages. They are grounded in the whole set I and do not depend on the syntactic representation of the formulas in I. They form a smaller class than expansions of I and, for theories consisting of $\mathcal{K}$-clauses or $\mathcal{T}$-clauses, smaller than moderately grounded expansions of I. Thus, the notion of iterative expansion is an attractive candidate for the set of consequences of I that an agent with full introspection might derive.

3 Semantical issues

The notion of iterative expansion was defined syntactically. In this section we will define another notion of expansion, through semantical means, and show that in some cases this new class of expansions coincides with iterative expansions. Thus, a semantics will be given to a substantial fragment of SAL.

Let us recall that it was a semantic argument that originally led Moore to the notion of expansion. Let T be a theory in $\mathcal{L}_K$. An *autoepistemic interpretation with respect to* T is any interpretation v of $\mathcal{L}_K$ that satisfies additional conditions:

AI1 if $\phi \in T$, then $v(K\phi) = 1$,

AI2 if $\phi \notin T$, then $v(K\phi) = 0$.

As argued by Moore, this requirement captures the perfect introspection capabilities of an agent given by the stability conditions. Now, we say that I T-entails ϕ (in symbols: $I \models_T \phi$) if and only if ϕ is true in all autoepistemic interpretations with respect to T which satisfy I. Moore defines an expansion to be any theory T such that $T = \{\phi : I \models_T \phi\}$. Since the class of autoepistemic interpretations is quite narrow, expansions of I are only weakly grounded in I.

To make them more strongly grounded, a wider class of interpretations has to be considered. Let us call an interpretation v of $\mathcal{L}_K$ a *weak autoepistemic interpretation with respect to* T if v satisfies the following two conditions:

WAI1 if $\phi \in T$ then $v(\phi \Rightarrow K\phi) = 1$,

WAI2 if $\phi \notin T$ then $v(K\phi) = 0$.

It is clear that every autoepistemic interpretation is a weak autoepistemic interpretation. The converse does not hold in general. The major difference is that for a weak autoepistemic interpretation we may have $v(K\phi) = 0$ for $\phi \in T$, provided $v(\phi) = 0$. The intuition behind this relaxation of condition **AI1** is the following. If v does not assign value 1 to formula $\phi \in T$ then there is no reason why it should assign value 1 to formula $K\phi$.

Let us now define the notion of *strong T-entailment*. We say that I strongly T-

entails ϕ (in symbols: $I \models^T \phi$) if ϕ is true in every weak autoepistemic interpretation with respect to T in which all formulas of I are true. Theory T is a weak iterative expansion of I if

$$T = \{\phi :\, I \models^T \phi\}. \tag{6}$$

Clearly, this semantic approach has a syntactic counterpart. By the definition of $\models^T$, we have

$$\{\phi :\, I \models^T \phi\} = \{\phi :\, I \cup (T \Rightarrow KT) \cup \neg K\overline{T} \models \phi\},$$

where $T \Rightarrow KT$ abbreviates the set $\{\phi \Rightarrow K\phi \,:\, \phi \in T\}$ and $\models$ is the standard propositional entailment. Consequently, by the completeness theorem of propositional calculus we obtain

$$\{\phi :\, I \models^T \phi\} = Cn(I \cup (T \Rightarrow KT) \cup \neg K\overline{T}).$$

Thus, weak iterative expansions can be alternatively defined through the following (syntactic) fixed point equation:

$$T = Cn(I \cup (T \Rightarrow KT) \cup \neg K\overline{T}). \tag{7}$$

It is now evident (by analysis of equation (7)), that if T is a weak iterative expansion then it is a stable set. In addition we have the following fact which justifies the term weak iterative expansion.

Proposition 3.1 *Every weak iterative expansion of I is an expansion of I.*

Next, we investigate the relationship between weak iterative and iterative expansions. Intuitively, the role of formulas in $T \Rightarrow KT$ is to replace the necessitation rule of logic $\mathcal{N}$. Whenever we use necessitation when deriving $K\phi$ from ϕ in the process of constructing an iterative expansion, we could replace it by inserting he formula $\phi \Rightarrow K\phi$ into the proof, right after ϕ, and by applying modus ponens. This shows that the notions of weakly iterative and iterative expansions are deeply connected. Indeed, we have the following result.

Proposition 3.2 *If T is an iterative expansion of I then T is a weak iterative expansion of I.*

The converse implication does not hold in general. Several examples are given at the end of the next section. The reason for that seems to be the fact that modal logic $\mathcal{N}$, like most modal logics, does not satisfy the deduction theorem. That is, $T \cup \{\phi\} \vdash_{\mathcal{N}} \psi$ does not imply $T \vdash_{\mathcal{N}} \phi \Rightarrow \psi$. (For example, let $T = \{\neg Kp \vee \neg Kq, \neg Kp \Rightarrow r\}$, $\phi = q$. Then, $T \cup \{\phi\} \vdash_{\mathcal{N}} r$, but it is easy to see that $T \not\vdash_{\mathcal{N}} q \Rightarrow r$.) This, in particular, implies that deduction in $\mathcal{N}$ cannot be reduced to deduction in propositional calculus. Thus, iterative expansions (solutions to $T = Cn_{\mathcal{N}}(I \cup \neg K\overline{T})$) cannot, in general, be expressed by means of the classical consequence operator Cn.

We shall show now an important class of theories which have the property that iterative expansions are expressible in terms of the operator Cn, and in fact, coincide with weak iterative expansions.

Theorem 3.3 *Let I be a $\mathcal{T}$-program. If $T \subseteq \mathcal{L}_K$ is consistent, then, a theory T is an iterative expansion of I if and only if T is a weak iterative expansion of I.*

We consider now the question of relevance of the restriction to $\mathcal{T}$-programs in

Theorem 3.3. It turns out that Theorem 3.3 cannot be improved.

Example 3.1 (a) Let $I_1 = \{Ka \Rightarrow b,\ K(a \Rightarrow b) \Rightarrow a\}$. Then I_1 possesses two expansions, $E(TAUT)$, and $E(\{a, b\})$. Both of these expansions are weak iterative, the first one is iterative, the other is not. Checking that $E(TAUT)$ is an iterative expansion and that $E(\{a, b\})$ is not an iterative expansion is straightforward. We shall show that $T_1 = E(\{a, b\})$ is a weak iterative expansion of I_1. A careful inspection shows that it is enough to prove that $E_0(\{a, b\}) \subseteq Cn(I_1 \cup (T_1 \Rightarrow KT_1) \cup \neg K\overline{T}_1)$. This is shown as follows: since $a \in T_1$, $a \Rightarrow Ka \in (T_1 \Rightarrow KT_1)$. Since $Ka \Rightarrow b \in I$, $a \Rightarrow b \in Cn(I \cup (T_1 \Rightarrow KT_1)$. But then $a \Rightarrow b \in T_1$ and so $(a \Rightarrow b) \Rightarrow K(a \Rightarrow b) \in (T_1 \Rightarrow KT_1)$. Consequently, $K(a \Rightarrow b) \in (T_1 \Rightarrow KT_1)$. Using the fact that $K(a \Rightarrow b) \Rightarrow a \in I$, we find that $a \in Cn(I \cup (T_1 \Rightarrow KT_1) \cup \neg K\overline{T}_1)$. Consequently, $b \in Cn(I \cup (T_1 \Rightarrow KT_1) \cup \neg K\overline{T}_1)$, too. This example shows that the assumption that the formulas in the antecedent of the implication are of the form Ka with $a \in Lit$ cannot be dropped.

(b) Let $I_2 = \{\neg Kc \Rightarrow b, \neg Ka \Rightarrow c, \neg Kb \Rightarrow a, Ka \Rightarrow b, Kb \Rightarrow c, Kc \Rightarrow a\}$. Then $T_2 = E(\{a, b, c\})$ is the only expansion of I_2, it is a weak iterative expansion of I_2 but it is not an iterative expansion of I_2. The argument is similar to that of (a). It is straightforward to show that T_2 is not an iterative expansion. To show that $T_2 = E(\{a, b, c\})$ is weak iterative, one shows that equivalences $Ka \Leftrightarrow Kb$, $Ka \Leftrightarrow Kc$, $Kc \Leftrightarrow Kb$ are all in $Cn(I \cup (T_2 \Rightarrow KT_2) \cup \neg K\overline{T_2})$. ¿From this one gets that $Ka \Rightarrow c, Kc \Rightarrow b, Kb \Rightarrow a$ all are in $Cn(I \cup (T_2 \Rightarrow KT_2) \cup \neg K\overline{T_2})$ The rest follows easily. This example shows that the assumption that negated formulas in the antecedent have prefix $\neg KK$ cannot be dropped.

(c) Let $I_3 = \{a \vee b, c \Rightarrow a, c \Rightarrow b, Ka \Rightarrow c, Kb \Rightarrow c\}$. As before it is straightforward to show that $T_3 = E(\{a, b, c\})$ is an expansion of I_3 and that it is not an iterative expansion. On the other hand, it is a weak iterative expansion. For having $a \Rightarrow Ka$ and $b \Rightarrow Kb$ allows one to prove $Ka \vee Kb$ (from I_3) and then c. Thus also a and b can be derived. This example shows that the assumption that successors of epistemic implications are literals cannot be dropped.

(d) The assumption that T is consistent cannot be dropped either. Consider theory $I = \{Ka \Rightarrow b, K\neg a \Rightarrow b, Kc \Rightarrow \neg b, K\neg c \Rightarrow \neg b\}$. It is easy to see that $\mathcal{L}_K$ is a weak iterative expansion of I. On the other hand, $\mathcal{L}_K$ is not an iterative expansion of I.

Finally, let us note that consistent weak iterative expansions can be given a fixed point characterization of type (3). Let $\mathcal{S}$ stand for a modal logic that uses necessitation as an inference rule, A for the set of all instances of axioms that are satisfied by $\mathcal{S}$, and let I denote the set of initial assumptions ($I \subseteq \mathcal{L}_K$). We have the following proposition.

Proposition 3.4 *If*

(a) $A \subseteq Cn(I \cup (T \Rightarrow KT) \cup \neg K\overline{T})$, *and*

(b) $(T \Rightarrow KT) \subseteq Cn_{\mathcal{S}}(I \cup \neg K\overline{T})$,

then T is a weak iterative expansion of I if and only if T is an $\mathcal{S}$-expansion of I.

Consider now the following two axiom schemata:

(A1) $K\phi \Rightarrow KK\phi$ (this is simply the axiom schema 4),

(A2) $\neg K \neg K\phi \Rightarrow (\phi \Rightarrow K\phi)$ (notice that **(A2)** is a weaker version of axiom schema 5).

Let us denote by 4^+ the modal logic containing (**A1**) and (**A2**) and closed under necessitation and modus ponens. It is easy to see that if T is consistent, then logic 4^+ satisfies conditions (a) and (b) of Proposition 3.4. Thus, we get the following corollary.

Corollary 3.5 *A consistent theory T is a weak iterative expansion of I if and only if it is an 4^+-expansion of I (that is, a solution to $T = Cn_{4^+}(I \cup \neg K\overline{T})$).*

Similar results hold for expansions. Under the same notation as before, we have the following general result.

Proposition 3.6 *If*

(a) *$A \subseteq Cn(I \cup KT \cup \neg K\overline{T})$, and*

(b) *$KT \subseteq Cn_{\mathcal{S}}(I \cup \neg K\overline{T})$,*

then T is an expansion of I if and only if T is an $\mathcal{S}$-expansion of I.

The proof is similar to that of Proposition 3.4 and is omitted. Proposition 3.6 implies that consistent expansions can be characterized as 5-expansions where by 5 we mean a modal logic satisfying axiom 5: $\neg K \neg K\phi \Rightarrow K\phi$ (Chellas [1980]).

Corollary 3.7 *A consistent theory T is an expansion of I if and only if it is an 5-expansion of I (that is, a solution to $T = Cn_5(I \cup \neg K\overline{T})$).*

Several other logics satisfy the requirements of Proposition 3.6. Examples are logics $K45$, $K5$ and 45. Thus, the following corollary follows.

Corollary 3.8 *Consistent expansions of I are characterized as: $K45$-expansions* (Shvarts [1988b]), *$K5$-expansions, 45-expansions, 5-expansions of I.*

Our results seem to indicate that the presence of axioms K and 4 is incidental and that logics **not** containing axiom K are naturally related to various classes of expansions.

4 Iterative expansions and default extensions

The goal of this section is to study the problem of expressing default extensions by means of iterative expansions. To better motivate the solution we propose here, let us briefly recall the result of [Marek and Truszczyński, 1989a], where extensions are shown to correspond to GC-iterative (strongly grounded) expansions. Assume that we consider Konolige's translation of default logic into autoepistemic logic, that is given default $d = \frac{\alpha:\beta_1,\ldots,\beta_n}{\omega}$ we assign to it a $\mathcal{K}$-clause $tr_{\mathcal{K}}(d)$ given by:

$$tr_{\mathcal{K}}(d) = (K\alpha \wedge \neg K\neg\beta_1 \wedge \ldots \wedge \ldots \neg K\neg\beta_n) \Rightarrow \gamma$$

The default theory (D, W) is then translated to $tr_{\mathcal{K}}(D, W) = W \cup \{tr_{\mathcal{K}}(d) \colon d \in D\}$. Let T be a stable theory containing $I = tr_{\mathcal{K}}(D, W)$. In [Marek and Truszczyński, 1989a] we proved the following:

Theorem 4.1 *A theory $S \subseteq \mathcal{L}$ such that $S = Cn(S)$ is an extension of (D, W) if and only if the unique stable set T such that $S = T \cap \mathcal{L}$ is a GC-iterative (strongly grounded) extension of $tr_{\mathcal{K}}(D, W)$.*

Although it completely characterizes relationship of extensions in default logic and expansion in autoepistemic logic, this theorem is not completely satisfactory as it uses GC-iterative (strongly grounded) expansions. We argued in the previous section that GC-iterative expansions have several undesirable properties.

It would be desirable to find a construction that would not require using GC-iterative (strongly grounded) expansions. A close analysis of Konolige's translation shows that it is not appropriate for representing defaults as modal formulas. In the default $\frac{p:}{q}$, p is interpreted as "p is provable" and in the default $\frac{:\neg p}{q}$, $\neg p$ is interpreted as "p is not in the context", and one is not the negation of the other. However, after the translation, the corresponding autoepistemic expressions are Kp and $\neg Kp$, and one is the negation of the other. Thus the key to our new approach is to use a different interpretation of defaults in autoepistemic logic. We translate now a default $d = \frac{\alpha:\beta_1,\ldots,\beta_n}{\omega}$ to the following $\mathcal{T}$-clause:

$$tr_{\mathcal{T}}(d) = (K\alpha \wedge \neg KK\neg\beta_1 \wedge \ldots \wedge \ldots \neg KK\neg\beta_n) \Rightarrow \omega,$$

and define $tr_{\mathcal{T}}(D,W)$ analogously as $tr_{\mathcal{K}}(D,W)$. Consider again the defaults $\frac{p:}{q}$ and $\frac{:\neg p}{q}$. Now, after the translation, Kp corresponds to "p is provable" and $\neg KKp$ to "p is not in the context". Clearly, $\neg KKp$ is not the negation of Kp. Thus this translation better captures our interpretation of the prerequisite and justification parts of defaults.

It turns out that this new translation allows for an extremely simple and elegant correspondence result between default logic and SAL in which default extensions translate exactly into iterative expansions. The proof we give here is a streamlined variant of the argument of [Marek and Truszczyński, 1989a]. But first, we need to recall some concepts of default logic (see [Reiter, 1980] and [Marek and Truszczyński, 1989a]). Let D be a set of defaults, and let $S \subseteq \mathcal{L}$. For $W \subseteq \mathcal{L}$ we put:

$$R^{D,S}(W) = Cn(W \cup \{c(d) : d \in D,\ p(d) \in W,\ \forall_{\beta \in j(d)}\ \neg\beta \notin S\}).$$

Then, we iterate operator $R^{D,S}$ on W:

$$R_0^{D,S}(W) = Cn(W),$$

$$R_{n+1}^{D,S}(W) = R^{D,S}(R_n^{D,S}(W)),$$

for $n \geq 0$. Thus,

$$R_{n+1}^{D,S}(W) = Cn(R_n^{D,S} \cup \{c(d) : d \in D,\ p(d) \in R_n^{D,S}(W),\ \forall_{\beta \in j(d)}\ \neg\beta \notin S\}).$$

Finally, we put

$$R_\infty^{D,S}(W) = \bigcup_{n=0}^{\infty} R_n^{D,S}(W).$$

Theory S is an extension of a default theory (D, W) if and only if $R_\infty^{D,S}(W) = S$. This definition, although different form the original Reiter's definition, is equivalent to it.

There is an evident similarity between the definitions of extensions and iterative expansions. This is not coincidental as the next theorem shows.

Theorem 4.2 *Let (D, W) be a default theory. A theory $S \subseteq \mathcal{L}$, closed under propositional consequence, is an extension of (D, W) if and only if $S = T \cap \mathcal{L}$ for an iterative expansion T of $tr_{\mathcal{T}}(D, W)$.*

Corollary 4.3 *Let I be a theory consisting of $\mathcal{T}$-clauses. Theory T is an iterative expansion of I if and only if T is a GC-iterative (strongly grounded) expansion of I.*

Theorems 4.2 and 3.3 give a semantics based on weak autoepistemic interpretations, for a substantial fragment of default logic, namely for those default theories (D, W) in which $W \subseteq Lit$ and every default is of the following form:

$$\frac{a_1 \wedge \cdots \wedge a_n;\, b_1, \ldots, b_m}{c},$$

where all a_i, b_i and c are literals.

5 Conclusions

In the paper we introduced a new class of expansions — iterative expansions. Iterative expansions have several advantages over previously studied classes of expansions. The class of iterative expansions of I is properly included in the class of Moore's expansions of I and, for several important classes of theories, in the class of moderately grounded expansions of Konolige. In addition, unlike strongly grounded expansions, they do not depend on the syntactic representation of the initial knowledge I and are grounded in the whole I. Thus, the notion of iterative expansion is an attractive candidate for the set of consequences of I that an agent with full introspection might derive.

We presented a semantics that defines another type of expansion — weak iterative expansion. We showed that for the class of $\mathcal{T}$-programs iterative expansions and weak iterative expansions coincide. This result provides a natural semantics for a nonmonotonic logic $\mathcal{N}$ if we restrict ourselves to $\mathcal{T}$-programs. The question of an existence of a general list-like semantic approach to iterative expansions remains open.

Let us note that in this paper for the first time in the context of commonsense reasonings modal logics were studied that do not satisfy axiom schema K (neither $\mathcal{N}$ nor 4^+ contain K). It is interesting that weak logic such as $\mathcal{N}$, without any axioms to simplify modalities turns out to be useful in formalizing methods of reasoning with incomplete information. As our correspondence result (Theorem 4.2) shows, nonmonotonic logic $\mathcal{N}$ with its iterative expansions can be regarded as a generalization of the default logic of Reiter. Since the logic $\mathcal{N}$ does not satisfy axiom schema K and thus cannot be given a traditional Kripke "possible worlds" semantics (each such semantics satisfies schema K, or at least some of its weaker variants) it provides an explanation of the difficulties with constructing Kripke-like semantics for default logic. As our correspondence result shows, default reasonings are in fact carried out in logic $\mathcal{N}$ and thus cannot be given a standard Kripke possible worlds interpretation. Let us briefly mention, though, that recently Fitting has found a Kripke-like semantics for logic $\mathcal{N}$ that requires infinitely many accessibility relations. Thus also default logic can be given a similar semantics.

6 Appendix — Proofs

In the proofs we will use the following two lemmas. The first was proved in [Marek and Truszczyński, 1989a]. It collects several simple properties of sets $A^T(I)$ and $A^T_n(I)$.

Lemma 6.1 **(a)** *Theory $A^T(I)$ is closed under necessitation, that is, $KA^T(I) \subseteq A^T(I)$.*

(b) *For all* $n \in \mathcal{N}$, $A^T_{n+1}(I) = Cn(I \cup KA^T_n(I) \cup \neg K\overline{T})$.

(c) *For all* $n \in \mathcal{N}$, $A^T_{n+1}(I) = Cn(I \cup A^T_n(I) \cup KA^T_n(I) \cup \neg K\overline{T})$.

(d) *If* T *is stable and* $I \subseteq T$, *then for all* $n \in \mathcal{N}$, $A^T_n(I) \subseteq T$. *Consequently,* $A^T(I) \subseteq T$.

Lemma 6.2 *Let* $T = E(S)$ *for some consistent subset* S *of* $\mathcal{L}$. *If* $S \subseteq A^T(I)$ *then* $T \subseteq A^T(I)$.

Proof. By induction on n we will prove that $E_n(S) \subseteq A^T(I)$, for every nonnegative integer n. For $n = 0$ the inclusion holds by the assumption of the lemma. Assume that $E_n(S) \subseteq A^T(I)$. Since $E_{n+1}(S) = Cn(E_n(S) \cup KE_n(S) \cup \neg K(\mathcal{L}_{K,n} - E_n(S))) \cap \mathcal{L}_{K,n+1}$, we need to show that each of the three sets of generators for $E_{n+1}(S)$ is included in $A^T(I)$. By the induction hypothesis and by the fact that $A^T(I)$ is closed under necessitation, first two sets are included in $A^T(I)$. Hence consider $\phi \in \mathcal{L}_{K,n} - E_n(S)$. Then $\phi \notin E(S)$ since $E_n(S) = E(S) \cap \mathcal{L}_{K,n}$. Consequently $\neg K\phi \in \neg K\overline{E(S)} = \neg K\overline{T}$. But $\neg K\overline{T} \subseteq A^T_0(I) \subseteq A^T(I)$, hence $\neg K(\mathcal{L}_{K,n} - E_n(S)) \subseteq A^T(I)$. □

Proof of Theorem 2.2. Let us assume that $I = \{A_i \Rightarrow \omega_i : i = 1, \ldots, n\}$. Only one implication needs a proof. Assume that T is an expansion of I. To prove the theorem it suffices to show that T is iterative. First, notice that T is inconsistent if and only if I is inconsistent. Thus, if T is inconsistent, then T satisfies $T = Cn_{\mathcal{N}}(I \cup \neg K\overline{T})$, that is, T is an iterat ive expansion of I. Thus, assume now that T is consistent. Then, the theorem characterizing consistent expansions applies to T (see [Marek and Truszczyński, 1988]). It states that there is a subset J of $\{1, \ldots, n\}$ such that $T = E(\{\omega_i : i \in J\})$, and $A_i \in T$ for each $i \in J$. Since each clause

$A_i \Rightarrow \omega_i$ of I is prerequisite-free, it follows that $A_i \in \neg K\overline{T}$, for $i \in J$. Consequently, $\{\omega_i : i \in J\} \subseteq Cn_{\mathcal{N}}(I \cup \neg K\overline{T}) = A^T(I)$. Thus, by Lemma 6.2, $T \subseteq A^T(I)$. Inclusion $A^T(I) \subseteq T$ follows from Lemma 6.1(d). □

Proof of Theorem 2.3. Denote $A_n(I) = A_n(I)^{\mathcal{L}_K}$ and $A(I) = A^{\mathcal{L}_K}(I)$, and observe that $Cn_{\mathcal{N}}(I) = A(I)$. Consider a stable theory T such that $I \subseteq T$. We first prove the following identity:

$$Cn_{\mathcal{N}}(I \cup \neg K\overline{T}) \cap \mathcal{L} = Cn_{\mathcal{N}}(I) \cap \mathcal{L}. \tag{8}$$

The equation (8) is clearly true if T is inconsistent. So, assume that T is consistent. Clearly, it suffices to prove that for every integer $n \geq 0$

$$A_n^T(I) \cap \mathcal{L} \subseteq A_n(I) \cap \mathcal{L}. \tag{9}$$

We proceed by induction on n. First, assume that $n = 0$ and let $\phi \in \mathcal{L}$. Assume that $\phi \notin Cn(I) = A_0(I)$. Then, there is a valuation v of $\mathcal{L}_K$ such that $v(\phi) = 0$ and $v(\psi) = 1$, for each clause $\psi \in I$. Define v' by setting $v'(K\psi) = 0$ for each $\psi \notin T$ and $v'(\psi) = v(\psi)$, otherwise. Since all clauses in I are positively determined, we have $v'(\psi) = 1$, for each $\psi \in I$. Thus, v' evaluates each formula in $I \cup \neg K\overline{T}$ to 1 and ϕ to 0. Consequently, $\phi \notin Cn(I \cup \neg K\overline{T}) = A_0^T(I)$. Thus, we have

$$A_0^T(I) \cap \mathcal{L} = Cn(I \cup \neg K\overline{T}) \cap \mathcal{L} \subseteq Cn(I) \cap \mathcal{L} \subseteq A_0(I) \cap \mathcal{L}.$$

This establishes the basis of the induction. Now, assume that (8) holds for some $n \geq 0$ and consider $\phi \in \mathcal{L}$ such that $\phi \notin A_{n+1}(I)$. Since $A_{n+1}(I) = Cn(I \cup KA_n(I))$ (by Lemma 6.1(b)), there is a valuation v of $\mathcal{L}_K$ such that $v(\phi) = 0$ and $v(\psi) = 1$

for every $\psi \in I \cup KA_n(I)$. Define now valuation v' of $\mathcal{L}_K$ by letting $v'(K\psi) = 1$ if $\psi \in A_n^T(I) \setminus \mathcal{L}$, $v'(K\psi) = 0$ if $\psi \in \overline{T}$, and $v'(\psi) = v(\psi)$ in all other cases.

Note that since T is stable and $I \subseteq T$, it follows that

$$KA_n^T(I) \cup \neg K\overline{T} \subseteq T.$$

Since T is consistent, it follows that

$$KA_n^T(I) \cap K\overline{T} = \emptyset.$$

Thus, v' is well-defined.

Clearly, the induction hypothesis implies that $v'(K\psi) = 1$ for every $\psi \in A_n^T(I)$. Consider a formula $\psi = K\alpha_1 \wedge \ldots \wedge K\alpha_n \Rightarrow \omega \in I$. If for some α_i, $\alpha_i \notin T$, then $v'(K\alpha_i) = 0$ and $v'(\psi) = 1$. So suppose that for every i, $\alpha_i \in T$. Since each $\alpha_i \in \mathcal{L}$, it follows that $v'(K\alpha_i) = v(K\alpha_i)$. Since $v'(\omega) = v(\omega)$, we have that $v'(\psi) = v(\psi) = 1$. Thus, for every $\psi \in I$, $v'(\psi) = 1$. Consequently, $\phi \notin A_n(I)$. This completes the proof of (8).

Now assume that I is $\mathcal{N}$-inconsistent. Then $\mathcal{L}_K$ is obviously the only solution to

$$T = Cn_{\mathcal{N}}(I \cup \neg K\overline{T}). \tag{10}$$

Thus, assume that I is $\mathcal{N}$-consistent. Let T be an iterative expansion of I. Then T is consistent and, by (8), $T \cap \mathcal{L} = Cn_{\mathcal{N}}(I) \cap \mathcal{L}$. Since T is stable, $T = E(Cn_{\mathcal{N}}(I) \cap \mathcal{L})$. Hence, an iterative expansion of I, if exists, is unique.

On the other hand, let $T = E(Cn_{\mathcal{N}}(I) \cap \mathcal{L})$. Then, by Lemma 6.2, $T \subseteq Cn_{\mathcal{N}}(I \cup \neg K\overline{T})$. Conversely, it is easy to see that $I \subseteq E(Cn_{\mathcal{N}}(I) \cap \mathcal{L}) = T$. Since T is stable, it follows that $Cn_{\mathcal{N}}(I \cup \neg K\overline{T}) \subseteq T$. Thus, T is an iterative expansion of I, and by the above argument the unique one. The second part of the assertion follows from the observation that every stable set that contains I contains also $Cn_{\mathcal{N}}(I)$. $\square$

Proof of Theorem 2.4. We first define an auxiliary operator $a^T(I)$ by:

$$a_0^T(I) = Cn(I \cup \neg K(\mathcal{L} \setminus T)),$$

$$a_{n+1}^T(I) = Cn(a_n^T(I) \cup K a_n^T(I)) \ \ (= Cn(I \cup \neg K(\mathcal{L} \setminus T) \cup K a_n^T(I))),$$

$$a^T(I) = \bigcup_{n=0}^{\infty} a_n^T(I).$$

First, we list some simple properties of the operator a^T:

$$a_n^T(I) \subseteq A_n^T(I), \tag{11}$$

$$a^T(I) \subseteq A^T(I). \tag{12}$$

In addition, if $T' \sqsubseteq T$, then

$$a^T(I) \subseteq a^{T'}(I). \tag{13}$$

Finally, we will need the following equality that holds for stable theories T:

$$\mathcal{L} \cap a_n^T(I) = \mathcal{L} \cap A_n^T(I). \tag{14}$$

We prove (14) by induction on n. So, let $n = 0$. By (11) we only need to prove that $\mathcal{L} \cap A_0^T(I) \subseteq \mathcal{L} \cap a_0^T(I)$. So, let $\phi \in \mathcal{L}$ and assume that $\phi \notin a_0^T(I)$. Then, there is a valuation v of $\mathcal{L}_{K,1}$ such that $v(I \cup \neg K(\mathcal{L} \setminus T)) = 1$ and $v(\phi) = 0$. Define v_1 to be a valuation of $\mathcal{L}_K$ satisfying:

- $v_1(\psi) = v(\psi)$, for $\psi \in \mathcal{L}_{K,1}$,

- $v_1(K\psi) = 0$, for $\psi \in \mathcal{L}_K \setminus \mathcal{L}$, $\psi \notin T$.

Clearly, v_1 exists, $v_1(I \cup \neg K\overline{T}) = 1$ and $v_1(\phi) = 0$. Thus, $\phi \notin Cn(I \cup \neg K\overline{T})$.

Now, assume that $n \geq 1$ and that (14) holds for n. To prove (14) for $n+1$ it suffices to prove that

$$\mathcal{L} \cap A^T_{n+1}(I) \subseteq \mathcal{L} \cap a^T_{n+1}(I).$$

Thus, by Lemma 6.1(b) and the definition of the operator a^T_n, it will suffice to prove that

$$\mathcal{L} \cap Cn(I \cup \neg K\overline{T} \cup KA^T_n(I)) \subseteq \mathcal{L} \cap Cn(I \cup \neg K(\mathcal{L} \setminus T) \cup Ka^T_n(I)).$$

Let $\phi \in \mathcal{L}$ and assume that $\phi \notin Cn(I \cup \neg K(\mathcal{L} \setminus T) \cup Ka^T_n(I))$. Then, there is a valuation of $\mathcal{L}_{K,1} \cup Ka^T_n(I)$ such that $v(I \cup \neg K(\mathcal{L} \setminus T) \cup Ka^T_n(I)) = 1$ and $v(\phi) = 0$. Extend v to a valuation v_1 of $\mathcal{L}_K$ satisfying:

- $v_1(\psi) = v(\psi)$, for $\psi \in I \cup \neg K(\mathcal{L} \setminus T) \cup Ka^T_n(I)$,
- $v_1(K\psi) = 0$, for $\psi \in \mathcal{L}_K \setminus \mathcal{L}$, $\psi \notin T$,
- $v_1(K\psi) = 1$, for $\psi \in \mathcal{L}_K \setminus \mathcal{L}$, $\psi \in A^T_n(I)$.

Such a valuation exists since by (11) and stability of T we have $a^T_n(I) \subseteq A^T_n(I) \subseteq T$. Note that if $\psi \in A^T_n(I) \cap \mathcal{L}$ then, by the induction hypothesis $\psi \in a^T_n(I)$. Therefore, $v_1(K\psi) = 1$. Consequently, $v_1(K\psi) = 1$, for every $\psi \in A^T_n(I)$. Thus, $v_1(Cn(I \cup \neg K\overline{T} \cup KA^T_n(I))) = 1$ and $v_1(\phi) = 0$. This implies that $\phi \notin Cn(I \cup \neg K\overline{T} \cup KA^T_n(I))$ and completes the proof of (14).

Now, assume that T is an iterative expansion of I and let T' be any expansion (stable set) of I. Assume that $T' \sqsubseteq T$. Then, by stability of T', $T' \cap \mathcal{L} \supseteq A^{T'}(I) \cap \mathcal{L}$. Thus, by (14), (13) and again (14)

$$T' \cap \mathcal{L} \supseteq a^{T'}(I) \cap \mathcal{L} \supseteq a^{T}(I) \cap \mathcal{L} = A^{T}(I) \cap \mathcal{L}.$$

Since $T = A^T(I)$, we get $T \sqsubseteq T'$. Thus, $T = T'$. $\Box$

Proof of Proposition 2.7. By Proposition 2.6, we need to prove that $I \subseteq T$. Let

$$C = K\alpha_1 \wedge \ldots \wedge K\alpha_m \wedge \neg K\beta_1 \ldots \wedge \neg K\beta_n \Rightarrow \omega$$

be an ae-clause of I not in $GC(I,T)$. Then, by the definition of $GC(I,T)$, some $\alpha_i \notin T$ or some $\beta_i \in T$. In the first case $C \in \neg K\overline{T} \subseteq T$ since $\neg K\alpha_i \in \neg K\overline{T}$, in the second, $C \in T$ since $K\beta_i \in T$. $\Box$

Proof of Proposition 3.1. Let T be a weak iterative expansion of I. Then, $T = Cn(I \cup (T \Rightarrow KT) \cup \neg K\overline{T})$. We want to prove that $T = Cn(I \cup T \cup \neg K\overline{T})$.

(inclusion $\subseteq$) It is enough to prove that for every $\phi \in T$, $\phi \Rightarrow K\phi$ belongs to $Cn(I \cup T \cup \neg K\overline{T})$. This however is obvious since $K\phi \in Cn(I \cup T \cup \neg K\overline{T})$.

(inclusion $\supseteq$) It is sufficient to prove that $KT \subseteq T$. But if $\phi \in T$, then $\phi \Rightarrow K\phi$ belongs to $T \Rightarrow KT$. Our assumption implies that $T \Rightarrow KT \subseteq T$, so $\phi \Rightarrow K\phi \in T$. Hence, by modus ponens $K\phi \in T$. $\Box$

Proof of Proposition 3.2. T is an iterative expansion of I if and only if $T = A^T(I)$, that is, $T = \bigcup_{n=0}^{\infty} A_n^T(I)$. We prove that $T = Cn(I \cup (T \Rightarrow KT) \cup \neg K\overline{T})$.

(inclusion $\supseteq$) Since $T = A^T(I)$, T is stable. Hence $KT \subseteq T$. But $KT \subseteq T$ implies $(T \Rightarrow KT) \subseteq T$. Also $I \cup \neg K\overline{T} \subseteq T$. Consequently, $I \cup (T \Rightarrow KT) \cup \neg K\overline{T} \subseteq T$.

Since T is closed under propositional consequence, $Cn(I \cup (T \Rightarrow KT) \cup \neg K\overline{T}) \subseteq T$.

(inclusion $\subseteq$) Since $T = \bigcup_{n=0}^{\infty} A_n^T(I)$, it suffices to prove that for all $n \in \mathcal{N}$, $A_n^T(I) \subseteq Cn(I \cup (T \Rightarrow KT) \cup \neg K\overline{T})$. Since $A_0^T(I) = Cn(I \cup \neg K\overline{T})$, the base case is obvious. Assume that $A_n^T(I) \subseteq Cn(I \cup (T \Rightarrow KT) \cup \neg K\overline{T})$. Consider a formula $K\phi$, where $\phi \in A_n^T(I)$. Then, $\phi \in Cn(I \cup (T \Rightarrow KT) \cup \neg K\overline{T})$ by the induction hypothesis. Moreover, $\phi \in T$. Thus, $\phi \Rightarrow K\phi \in Cn(I \cup (T \Rightarrow KT) \cup \neg K\overline{T})$. Consequently, $K\phi \in Cn(I \cup (T \Rightarrow KT) \cup \neg K\overline{T})$. Hence, we obtain that $KA_n^T(I) \subseteq Cn(I \cup (T \Rightarrow KT) \cup \neg K\overline{T})$, and $A_{n+1}^T(I) = Cn(A_n^T(I) \cup KA_n^T(I)) \subseteq Cn(I \cup (T \Rightarrow KT) \cup \neg K\overline{T})$. his completes the proof of the inductive step and of the entire proposition. □

Proof of Theorem 3.3. By Proposition 3.2, we only need to prove that if T is a weak iterative expansion of I, then T is an iterative expansion of I. Hence, let T be a weak iterative expansion of I. In particular, by Proposition 3.1 T is an expansion of I. Consequently, T is stable and by Lemma 6.1(d), $A^T(I) \subseteq T$.

Thus, we need to prove that, under our assumptions, the converse inclusion holds as well. Put $S = T \cap Lit$. Since T is consistent, then the theorem characterizing consistent expansions applies (see [Marek and Truszczyński, 1988]). It states that each consistent expansion of a theory consisting of ae-clauses $A_i \Rightarrow \omega_i$, $1 \leq i \leq n$, is of the form $E(\{\omega_i : i \in J\})$ for a suitable set $J \subseteq \{1, \ldots, n\}$. Since the objective parts of program $\mathcal{T}$-clauses are literals, we conclude that $T = E(S)$. In order to prove that $T \subseteq A^T(I)$ we first prove that $S \subseteq A^T(I)$.

We use a construction similar to that of Gelfond and Lifschitz [1988] to reduce I. The T-*reduct* of I, denoted by I/T is obtained from I by applying to each clause

$C \in I$, say of the form

$$C = Ka_1 \wedge \ldots \wedge Ka_k \wedge \neg KKb_1 \wedge \ldots \wedge \neg KKb_r \Rightarrow s,$$

one of the following reduction rules:

(a) If $a_i \notin T$ for some $i \leq k$, then delete C.

(b) If $b_i \in T$ for some $i \leq r$, then delete C.

(c) If neither (a) nor (b) applies (that is, if for all $1 \leq i \leq k$, $a_i \in T$ and for all $1 \leq i \leq r$, $b_i \notin T$), then replace C by the clause $Ka_1 \wedge \ldots \wedge Ka_k \Rightarrow s$.

The resulting theory I/T consists of clauses of the form $Ka_1 \wedge \ldots \wedge Ka_k \Rightarrow s$, where $a_1, \ldots, a_k, s$ are literals, and $a_1, \ldots, a_k \in T$. We shall prove now two basic claims.

Claim 6.3 $S = Lit \cap Cn(I/T \cup \{a \Rightarrow Ka : a \in S\})$. *(In other words, theories T and $Cn(I/T \cup \{a \Rightarrow Ka : a \in S\})$ contain precisely the same literals.)*

Proof of Claim 6.3. First of all notice that $I/T \subseteq Cn(I \cup \neg K\overline{T})$. Indeed, let $C' = Ka_1 \wedge \ldots \wedge Ka_k \Rightarrow s$ be in I/T. There is a clause $C \in I$ from which C' was obtained. Since C was not deleted, it follows that all the formulas of the form Ka_i in the antecedent of C are left intact, and the formulas of the form $\neg KKb_j$ were eliminated. Thus, for every $1 \leq i \leq r$, $b_i \notin T$ and $\neg KKb_i \in \neg K\overline{T}$. Consequently, $C' \in Cn(C \cup \neg K\overline{T}) \subseteq Cn(I \cup \neg K\overline{T})$.

The collection $\{a \Rightarrow Ka : a \in S\}$ is included in $T \Rightarrow KT$. This together with inclusion $I/T \subseteq Cn(I \cup \neg K\overline{T})$ implies that $Cn(I/T \cup \{a \Rightarrow Ka : a \in S\}) \subseteq Cn(I \cup (T \Rightarrow KT) \cup \neg K\overline{T}) = T$. Thus, all literals in $Cn(I/T \cup \{a \Rightarrow Ka : a \in S\})$ belong to

S.

Conversely, assume that $d \in Lit$ and that $d \notin Cn(I/T \cup \{a \Rightarrow Ka\colon a \in S\})$. We will prove that $d \notin T$. Let $\mathcal{L}'$ be the language whose atoms are atoms of $\mathcal{L}$ and these modal atoms $K\psi$ for which $\psi \in S$. All formulas of I/T and of $\{a \Rightarrow Ka\colon a \in S\})$ belong to $\mathcal{L}'$ and $d \in \mathcal{L}'$, as well. Since $d \notin Cn(I/T \cup \{a \Rightarrow Ka\colon a \in S\})$, there is a valuation v of the atoms of language $\mathcal{L}'$ such that v takes value 1 on the whole theory $I/T \cup \{a \Rightarrow Ka\colon a \in S\}$, but $v(d) = 0$.

We extend now the valuation v to a valuation v_1 of the full language $\mathcal{L}_K$. This language has more atoms than $\mathcal{L}'$. Additional atoms are of the form $K\psi$, where $\psi \notin S$. Define now v_1 as follows:

(1) If a is an atom of the language $\mathcal{L}'$, then $v_1(a) = v(a)$.

(2) If a is an atom of $\mathcal{L}_K$ but not of $\mathcal{L}'$ (that is, $a = K\psi$ for $\psi \notin S$), then $v_1(K\psi) = 1$ if and only if $\psi \in T$.

Clearly, v_1 restricted to $\mathcal{L}'$ coincides with v, and so $v_1(d) = 0$. We will prove that $v_1(\psi) = 1$, for each $\psi \in T$, thus showing that $d \notin T$ and completing the proof of the converse inclusion. To this end, we need to prove that v_1 takes value 1 on each of the sets I, $T \Rightarrow KT$, and $\neg K\overline{T}$.

(i) Consider a clause $C \in I$, say C is of the form

$$Ka_1 \wedge \ldots \wedge Ka_k \wedge \neg KKb_1 \wedge \ldots \wedge \neg KKb_r \Rightarrow s.$$

If reduction rule (a) applies to C, then for some $1 \leq i \leq k$, $a_i \notin T$. Since $Ka_i \notin \mathcal{L}'$, we have $v_1(Ka_i) = 0$. Consequently, $v_1(C) = 1$. If reduction rule (b) applies to C, then for some $1 \leq i \leq r$, $b_i \in T$. Thus, $Kb_i \in T$, $v_1(KKb_1) = 1$ and $v_1(\neg KKb_i) = 0$. Hence $v_1(C) = 1$. Finally, if neither (a) nor (b) applies, then all a_i belong to T and all b_j do not belong to T. Then, however, $v_1(C) = v_1(C')$, where $C' = Ka_1 \wedge \ldots \wedge Ka_k \Rightarrow s$. Now, $C' \in \mathcal{L}'$ and so $v_1(C') = v(C')$. But $C' \in I/T$, so $v(C') = 1$. Hence $v_1(C) = 1$. Thus $v_1(C) = 1$ for each $C \in I$.

(ii) Now consider a formula $\phi \Rightarrow K\phi$ for $\phi \in T$. If $\phi \in S$, then since $\phi \Rightarrow K\phi$ is a formula of $\mathcal{L}'$, $v_1(\phi \Rightarrow K\phi) = v(\phi \Rightarrow K\phi) = 1$. Otherwise, $\phi \notin S$ and $v_1(K\phi) = 1$. So, also in this case $v_1(\phi \Rightarrow K\phi) = 1$.

(iii) Finally, if $\phi \notin T$ then $v_1(K\phi) = 0$, thus $v_1(\neg K\phi) = 1$.

Consequently, $v_1(\psi) = 1$ for each $\psi \in T$, and since $v_1(d) = v(d) = 0$, $d \notin T$. This completes the proof of Claim 6.3.

We continue the proof of Theorem 3.3. Now we shall prove that $S \subseteq A^T(I/T)$. To this end, let $Z = A^T(I/T) \cap Lit$. Since $A^T(I/T) \subseteq A^T(I) \subseteq T$, we have $Z \subseteq S$. Assume Z is a proper subset of S and let $d \in S \setminus Z$. By Claim 6.3, $d \in Cn(I/T \cup \{a \Rightarrow Ka\colon a \in S\})$. Since $d \in Cn(I/T \cup \{a \Rightarrow Ka\colon a \in S\})$ and Cn is the classical consequence operation, we can use the deduction theorem for classical logic and get:

$$(\bigwedge_{a \in S} (a \Rightarrow Ka)) \Rightarrow d \in Cn(I/T).$$

The formula $(\bigwedge_{a \in S}(a \Rightarrow Ka)) \Rightarrow d$ can be transformed, by repeated elimination of implication in the antecedent and the distributive law, into a conjunction:

$$\bigwedge_{J \subseteq S}((\bigwedge_{a \in S \setminus J} \neg a \wedge \bigwedge_{a \in J} Ka) \Rightarrow d).$$

Hence, for every $J \subseteq S$ the formula $(\bigwedge_{a \in S \setminus J} \neg a \wedge \bigwedge_{a \in J} Ka) \Rightarrow d$ belongs to $Cn(I/T)$. In particular, since $Z \subseteq S$, $(\bigwedge_{a \in S \setminus Z} \neg a \wedge \bigwedge_{a \in Z} Ka) \Rightarrow d$ belongs to $Cn(I/T)$. After an elementary transformation we find that the clause $(\bigvee_{a \in Z} \neg Ka) \vee (d \vee \bigvee_{a \in S \setminus Z} a)$ belongs to $Cn(I/T)$.

We say that a clause $C' = \phi_1 \vee \ldots \vee \phi_k$ is *weaker* than $C = \xi_1 \vee \ldots \vee \xi_s$ if $\{\xi_1, \ldots, \xi_s\} \subseteq \{\phi_1, \ldots, \phi_k\}$. In this case we also say that C is *stronger* than C'. Let us now represent all the formulas in I/T in the clausal form. We need the following claim.

Claim 6.4 *If a clause C' of the form $\neg Ka_1 \vee \ldots \vee \neg Ka_k \vee s_1 \vee \ldots \vee s_m$ (where $a_1, \ldots, a_k \in S$, and $s_1, \ldots, s_m \in Lit$) belongs to $Cn(I/T)$ but is not a tautology, then for some clause $C \in I/T$, C is stronger than C'.*

Proof of Claim 6.4. Assume otherwise. Then, there is a clause $C' = \neg Ka_1 \vee \ldots \vee \neg Ka_k \vee s_1 \vee \ldots \vee s_m$ in $Cn(I/T)$ for which there is no stronger clause $C \in I/T$.

We define the following partial valuation v of $\mathcal{L}'$:

- $v(Ka_1) = \ldots = v(Ka_k) = 1$.
- $v(s_1) = \ldots = v(s_m) = 0$.

Since C' is not a tautology, this partial valuation v is well defined. Moreover, $v(C') = 0$. We will extend v to a valuation v_1 of $\mathcal{L}'$ such that for all $C \in I/T$, $v_1(C) = 1$. This will contradict $C' \in Cn(I/T)$ and will complete the proof of the claim.

We complete now the proof of Theorem 3.3. To define v_1 proceed as follows:

(1) set $v_1(Ka) = 0$ for all modal atoms of $\mathcal{L}'$ not appearing in C'.

(2) for the unique literal $s \in Lit$ appearing in a clause $B \vee s$ belonging to I/T and such that B is stronger than $\neg Ka_1 \vee \cdots \vee \neg Ka_k$, define $v_1(s) = 1$.

(3) for all other atoms of $\mathcal{L}'$ define v_1 arbitrarily.

We claim that v_1 is well defined. Only rule (2) could cause a problem. First suppose that rule (2) applies and assigns value 1 to a literal s that appears among $s_1, \ldots, s_m$, and thus was earlier assigned value 0. But then, $B \vee s$ is stronger than C', contrary to our assumption. Next suppose, that there are two clauses in I/T to which rule (2) applies, one containing literal s and the other one the negation of s. Say these clauses are $B \vee s$ and $B' \vee \neg s$. But then both s and $\neg$ can be derived in T (since $Ka_1, \ldots, Ka_k \in T$). But this contradicts consistency of T. Thus, v_1 is well defined.

Consider now a clause $C \in I/T$. If one of the literals $\neg Ka$ appearing in C does not appear in C' then $v_1(Ka) = 0$, $v_1(\neg Ka) = 1$ and so $v_1(C) = 1$. If all the literals of the form $\neg Ka$ appearing in C appear in C', then the rule (2) was applicable. Thus, for the unique literal $s \in Lit$ appearing in C we have $v_1(s) = 1$. Hence, $v_1(C) = 1$. Consequently, $v_1(C) = 1$ for every $C \in I/T$.

Finally, note that $v_1(C') = v(C') = 0$. Thus, $C' \notin Cn(I/T)$, a contradiction. This completes the proof of Claim 6.4.

Let us return to the proof of the theorem. As we found earlier, formula

$$(\bigvee_{a \in Z} \neg Ka) \vee (d \vee \bigvee_{a \in S \setminus Z} a) \tag{15}$$

belongs to $Cn(I/T)$. This formula is not a tautology since all the literals of $\mathcal{L}$ that appear in it belong to T (which is consistent). Since formula (15) belongs to $Cn(I/T)$, it follows by Claim 6.4 that there is a clause $C \in I/T$ that is stronger. Say $C = \neg Kb_1 \vee \cdots \neg Kb_t \vee c$. Then, each $b_i \in Z$ and $c \notin Z$ (recall that $d \notin Z$). Since $A^T(I/T)$ is closed under necessitation and $Z \subseteq A^T(I/T)$, we obtain that $Kb_i \in A^T(I/T)$ for every $1 \leq i \leq t$. Since $C \in A^T(I/T)$, it follows that $c \in A^T(I/T)$, a contradiction again.

Hence we proved that $S \subseteq A^T(I/T) \subseteq A^T(I)$. Thus $Cn(S) \subseteq A^T(I)$ that is, $E_0(S) \subseteq A^T(I)$. This, by Lemma 6.2, implies $T \subseteq A^T(I)$ and completes the proof of the theorem. □

Proof of Proposition 3.4. Assume first that $T = Cn(I \cup (T \Rightarrow KT) \cup \neg K\overline{T})$. Then, by (a), $Cn_{\mathcal{S}}(I \cup \neg K\overline{T}) \subseteq Cn(I \cup (T \Rightarrow KT) \cup \neg K\overline{T}) = T$, and, by (b), $T = Cn(I \cup (T \Rightarrow KT) \cup \neg K\overline{T}) \subseteq Cn_{\mathcal{S}}(I \cup \neg K\overline{T})$. Thus, $T = Cn_{\mathcal{S}}(I \cup \neg K\overline{T})$. Conversely, assume that $T = Cn_{\mathcal{S}}(I \cup \neg K\overline{T})$. Then, T is stable. Consequently, $Cn(I \cup (T \Rightarrow KT) \cup \neg K\overline{T}) \subseteq T$ follows. Inclusion $T \subseteq Cn(I \cup (T \Rightarrow KT) \cup \neg K\overline{T})$ follows from (a). □

Proof of Theorem 4.2. In the proof we use the following auxiliary lemma.

Lemma 6.5 *Let (D, W) be a default theory and $I = tr_{\mathcal{T}}(D, W)$. Let T be stable and $S = T \cap \mathcal{L}$. If $I \subseteq T$ then*

(a) *For all* $n \in \mathcal{N}$, $R_n^{D,S}(W) \subseteq A_n^T(I)$,

(b) *For all* $n \in \mathcal{N}$, $A_n^T(I) \cap \mathcal{L} \subseteq R_{n+1}^{D,S}(W)$,

(c) $A^T(I) \cap \mathcal{L} = R_\infty^{D,S}(W)$.

Proof. (a) We proceed by induction on n. Let $n = 0$. Then, since $W \subseteq I$,

$$R_0^{D,S}(W) = Cn(W) \subseteq Cn(I \cup \neg K\overline{T}) = A_0^T(I).$$

Assume now that $R_n^{D,S}(W) \subseteq A_n^T(I)$. By Lemma 6.1(c), it suffices to show that

$$R_{n+1}^{D,S}(W) \subseteq Cn(I \cup A_n^T(I) \cup KA_n^T(I) \neg K\overline{T}).$$

Recall that

$$R_{n+1}^{D,S}(W) = Cn(R_n^{D,S}(W) \cup \{c(d) : \ d \in D, \ p(d) \in R_n^{D,S}(W), \ \forall_{\beta \in j(d)} \ \neg\beta \notin S\}).$$

By the induction hypothesis, $R_n^{D,S}(W) \subseteq A_n^T(I)$. Consider a default $\frac{\alpha:\ \beta_1,\ldots,\beta_p}{\omega}$ from D such that $\alpha \in R_n^{D,S}(W)$ and $\neg\beta_i \notin S$, $1 \leq i \leq p$. Then, $K\alpha \in KA_n^T(I)$ (by the induction hypothesis) and, for $1 \leq i \leq p$, $\neg KK\neg\beta_i \in \neg K\overline{T}$ (recall that $S = T \cap \mathcal{L}$). Since $tr_{\mathcal{T}}(d) \in I$, it follows that $\omega \in Cn(I \cup A_n^T(I) \cup KA_n^T(I) \cup \neg K\overline{T})$.

(b) Again, we proceed by induction on n. We first show that $A_0^T(I) \cap \mathcal{L} \subseteq R_1^{D,S}(W)$. Suppose to the contrary, that there is a formula $\phi \in \mathcal{L}$ such that $\phi \in A_0^T(I) \backslash R_1^{D,S}(W)$. Since $R_1^{D,S}(W)$ is closed under propositional consequence, there exists a valuation v such that $v(\phi) = 0$, and $v(\psi) = 1$, for every $\psi \in R_1^{D,S}(W)$. Extend the valuation v to a valuation v_1 of the whole language $\mathcal{L}_K$ as follows **(1)** $v_1(p) = v(p)$ if p is an atom of $\mathcal{L}$.

(2) For $\psi \in \mathcal{L}$ define $v_1(K\psi) = 1$ if and only if $\psi \in Cn(I \cap \mathcal{L})$.

(3) For $\psi \notin \mathcal{L}$ define: $v_1(K\psi) = 0$ if and only if $\psi \in \overline{T}$.

Since $I \subseteq T$, it follows that for all $\psi \in \overline{T}$, $v_1(K\psi) = 0$. Thus, v_1 takes value 1 on the set $\neg K\overline{T}$. Note also that $I \cap \mathcal{L} = W \subseteq R_1^{D,S}(W)$. Thus, v_1 takes value 1 on the set $I \cap \mathcal{L}$. Now we shall determine the value of v_1 on the set $I \setminus \mathcal{L}$. Consider a formula $\Phi \in I \setminus \mathcal{L}$. Let $\Phi = K\alpha \wedge \neg KK\neg\beta_1 \wedge \ldots \wedge \neg KK\neg\beta_r \Rightarrow \omega$.

(i) If $\alpha \notin Cn(I \cap \mathcal{L})$ then $v_1(K\alpha) = 0$ and so $v_1(\Phi) = 1$.

(ii) If for some $j \leq r$, $\neg\beta_j \in T$, then $K\neg\beta_j \in T$. Consequently, $v_1(\neg KK\neg\beta) = 0$, thus again $v_1(\Phi) = 1$.

(iii) If, finally, $\alpha \in Cn(I \cap \mathcal{L})$ and for all $j \leq r, \neg\beta_j \notin T$, then $\alpha \in W$ and for all $j \leq r, \neg\beta_j \notin S$. Consequently, $\omega \in R_1^{D,S}(W)$. Thus $v(\omega) = v_1(\omega) = 1$. This implies, as before, that $v_1(\Phi) = 1$.

Hence, we proved that v_1 takes value 1 on the whole set $A_0^T(I)$. This implies, in particular, that $v_1(\phi) = 1$, a contradiction, since $v_1(\phi) = v(\phi) = 0$.

We shall perform now the inductive step. Let us assume that

$$A_n^T(I) \cap \mathcal{L} \subseteq R_{n+1}^{D,S}(W).$$

We need to prove that

$$A_{n+1}^T(I) \cap \mathcal{L} \subseteq R_{n+2}^{D,S}(W).$$

To prove this inclusion assume that its fails. Then, there is a formula $\phi \in \mathcal{L}$ such that $\phi \in A_{n+2}^T(I) \setminus R_{n+1}^{D,S}(W)$. Consequently, there is a valuation v of the language $\mathcal{L}$ which takes value 1 on the set $R_{n+1}^{D,S}(W)$ and 0 on ϕ. Extend now the valuation v to a valuation v_2 of the whole language $\mathcal{L}_K$ as follows: **(1)** If p is an atom of $\mathcal{L}$, $v_2(p) = v(p)$.

(2) When $\psi \in \mathcal{L}$, then $v_2(K\psi) = 1$ if and only if $\psi \in R^{D,S}_{n+1}(W)$.

(3) When $\psi \notin \mathcal{L}$, then $v_2(K\psi) = 0$ if and only if $\psi \in \overline{T}$.

Now, by Lemma 6.1(b), $A^T_{n+1}(I) = Cn(I \cup KA_n(I) \cup \neg K\overline{T})$. ¿From part (a) and Lemma 6.1(d), it follows that $R^{D,S}_{n+1}(W) \subseteq A^T(I) \subseteq T$ Thus, v_2 takes value 1 on the theory $\neg K\overline{T}$. We shall prove that v_2 takes value 1 on the set $KA^T_n(I)$. Indeed, if $\psi \in A^T_n(I) \cap \mathcal{L}$ then, by the induction hypothesis, $\psi \in R^{D,S}_{n+1}(W)$, so $v_2(K\psi) = 1$. If $\psi \in A^T_n(I) \setminus \mathcal{L}$ then, since $A^T_n(I) \subseteq T$, $\psi \notin \overline{T}$. Thus $v_2(K\psi) = 1$ in this case, too. Finally, to show that v_2 takes value 1 on I we just need to consider formulas of $I \setminus \mathcal{L}$ (recall that $I \cap \mathcal{L} = W \subseteq R^{D,S}_{n+1}(W)$ and $v_2(\psi) = v(\psi) = 1$ for each $\psi \in R^{D,S}_{n+1}(W)$). Hence, let $\Phi \in I \setminus \mathcal{L}$. Suppose that $\Phi = K\alpha \wedge \neg KK\neg\beta_1 \wedge \ldots \wedge \neg KK\neg\beta_r \Rightarrow \omega \in D(I)$.
(i) If $\alpha \notin R^{D,S}_{n+1}(W)$, then $v_1(K\alpha) = 0$ and so $v_2(\Phi) = 1$.
(ii) If for some $j \leq r, \neg\beta_j \in T$, then $K\neg\beta_j \in T$, so $v_2(KK\neg\beta_j) = 1$. Thus, $v_2(\Phi) = 1$.
(iii) Finally, assume that $\alpha \in R^{D,S}_{n+1}(W)$ and for all $j \leq r$, $\neg\beta_j \in T$. Then, in particular, $\neg\beta_j \notin S$, for all $j \leq r$. Thus, $\omega \in R^{D,S}_{n+2}(W)$. But then $v_2(\omega) = v(\omega) = 1$, therefore $v_2(\Phi) = 1$.
This implies that v_2 takes value 1 on the whole set $A^T_{n+1}(I)$, a contradiction.
Part (c) follows easily from (a) and (b) $\square$

We prove now Theorem 4.2. Let S be an extension of (D, W), and let T be the unique stable set such that $S = T \cap \mathcal{L}$. We need to prove that $T = A^T(I)$, where $I = tr_{\mathcal{T}}(D, W)$. The inclusion $A^T(I) \subseteq T$ follows from Lemma 6.1(d). To show inclusion $T \subseteq A^T(I)$ we we first assume that S is inconsistent. Then , by Lemma 6.5(c),

$A^T(I)$ is inconsistent and $T \subseteq A^T(I)$ follows. Hence, assume that S is consistent. Then $T = E(S)$.

To prove the inclusion $T \subseteq A^T(I)$ we observe that $S \subseteq A^T(I)$ follows from the previous lemma. Thus, $T \subseteq A^T(I)$ follows from Lemma 6.2.

Conversely, if T is an iterative expansion of I, then, again by the previous lemma, we find that $S = T \cap \mathcal{L} = A^T(I) \cap \mathcal{L} = R_\infty^{D,S}(W)$. Thus S is an extension of (D, W). $\square$

Proof of Theorem 2.5. Let $T = E(S)$ be an iterative expansion of I, where S is the objective part of T, that is $S = T \cap \mathcal{L}$. In particular, S is closed under propositional consequence. By Theorem 4.2(a), S is an extension of $tr_{\mathcal{T}}^{-1}(I)$. Consequently, S is a minimal set for $tr_{\mathcal{T}}^{-1}(I)$ (Marek and Truszczyński [1989a], Proposition 4.3.1). Hence, it follows that $E(S)$ is a $\sqsubseteq$-minimal stable set for I (Marek and Truszczyński [1989a], Theorem 4.3.6). $\square$

Proof of Corollary 4.3. First, for a $\mathcal{T}$-clause C, where

$$C \;=\; (K\phi_1 \wedge \ldots \wedge K\phi_k \wedge \neg KK\psi_1 \wedge \ldots \wedge \neg KK\psi_r \Rightarrow \omega),$$

define a $\mathcal{K}$-clause $C_{\mathcal{K}}$ by

$$C \;=\; (K\phi_1 \wedge \ldots \wedge K\phi_k \wedge \neg K\psi_1 \wedge \ldots \wedge \neg K\psi_r \Rightarrow \omega).$$

Let I be a theory consisting of $\mathcal{T}$-clauses. Define $I_{\mathcal{K}} = \{C_{\mathcal{K}} : C \in I\}$. Since for every stable set T, $\psi \in T$ if and only if $K\psi \in T$, it follows that I and $I_{\mathcal{K}}$ have the same GC-iterative expansions.

Now we can prove the assertion. Let (D, W) be a default theory such that $tr_{\mathcal{T}}(D, W) = I$. Then, $tr_{\mathcal{K}}(D, W) = I_{\mathcal{K}}$. By Theorem 4.2, iterative expansions of I correspond exactly to extensions of (D, W) which in turn correspond exactly to GC-iterative expansions of $I_{\mathcal{K}}$ (by Theorem 4.1), and these, in turn, coincide with GC-iterative expansions of I. $\square$

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