On reaching consensus by groups of intelligent agents

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Abstract

We study the problem of reaching the consensus by a group of fully communicating, intelligent agents. Firstly, we study the case of agents which have full information about other agents beliefs. Using previous work ([14, 15]), we establish the properties of systems of agents. We find the relationship of consensus in smaller and larger groups. Subsequently, using the paradigm of "rough" sets and approximations, we study formal properties of an epistemic variant of consensus reaching.

1 Introduction

Investigations concerning a systematic approach to reasoning by one or more intelligent agents have been pursued by both logicians and computer scientists for an extensive period of time. In fact one can safely say that the management of knowledge of an agent or a group of agents is the one of most important motivations for development of various modal logics (e.g. S_4 and S_5 systems, see [1], and the bibliography there). More recently investigations on knowledge transfer in distributed environment, where several intelligent agents communicate have been conducted (e.g. [4, 3]). In most general terms the situation can be described as follows: There is a collection of agents, $A_1, \ldots, A_n$. These agents observe some processes (it may be the same process) and communicate among themselves. Each of these agents is able to reason (both about the reality it observes and about herself). Agents exchange the knowledge among themselves (in a synchronous or asynchronous manner), have complete or incomplete knowledge about other agents assumptions and attempt– or not – to modify the knowledge of other agents.

Within this, very general, scheme various attempts to propose the concept of common knowledge of totality of agents $< A_1, \ldots, A_n >$ are suggested. ([5, 7] and many others). In this paper we propose a scheme which deals with the following situation:

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1) Totality of agents is (partially) ordered, with the intended meaning that $t_1 \sqsubseteq t_2$ intuitively means that "Perception of the agent A_{t_2} is sharper than that of the agent A_{t_1}, but the agent A_{t_2} takes into consideration perceptions of the agent A_{t_1}".

2) The agents observe the same reality and each of them has her own perception of that reality.

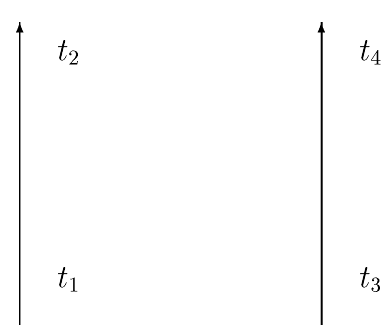
3) Each agent is fully aware of other agents' observations and do not attempt to modify that perception.

4) The totality of agents attempts to reach conclusions via consensus, i.e. agreeing to common statements.

This scheme of reasoning was treated extensively in [13, 14]. We review and extend some results of that work in section 2, and in section 3 we propose an extension of this scheme to epistemic situation. Here we apply an approach suggested by Orlowska ([10, 11]) by using an approximation technique due to Pawlak and others, so called "rough sets" ([9, 8]). In that last paper it was observed that using of approximation technique with the single indiscernibility relation amounts to the consideration of the three-valued Post algebra of truth values. Independently, the idea of using non-classical logics and in particular Post algebras with the partially ordered sets of constants jointly with "rough sets" approximation methods was introduced in [14, 15].

2 Consensus of agents reasoning about single reality

Let us shortly describe a particular case of our scheme. Assume that four agents t_1, t_2, t_3 and t_4 observe a certain reality, say patients, who may be ill. The universe of patients observed by each of the agents is same, consisting of individuals $i_1, \ldots, i_6$. We have a single, unary, predicate letter p interpreted as "having a condition H". The totality of agents $t_1, \ldots, t_4$ is ordered as follows:



This is interpreted in our considerations as follows: Perception of the agent t_2 is sharper than that of the agent t_1 and perception of the agent t_4 is sharper than that of agent t_3.

Moreover agent t_2 takes into considerations *negative* observations of the agent t_1, and, similarly, agent t_4 *negative* information of the agent t_3. There is no relationship assumed between t_1 and t_3, t_4 and t_2 and t_3, t_4 and conversly. Assume now that we have, for each t an interpretation of the predicate p.

$$P_{t_1} = \{i_1, i_2, i_5\}$$
$$P_{t_2} = \{i_1, i_5\}$$
$$P_{t_3} = \{i_2, i_3, i_5\}$$
$$P_{t_4} = \{i_2, i_5\}$$

A simple consensus can be reached about person i_5 having the condition H. But what about the negative information? There are several options that can be taken, "local" and "global" negation. A "local" or Boolean negation assumes taking into account only the agent's observation. An agent assumes $\neg\phi$ if she did not find ϕ. In case of atomic statement she assumes $\neg p(a)$ because she failed to observe $p(a)$. In our approach, the agent, in accepting negative information is more cautious: not only she takes into account her own observations, but also those of the agents who are "less observant" (i.e. $\sqsubseteq$-smaller). Then, the agent t_2 perceives that the property $\neg p$ holds not for i_2, i_3, i_4 and i_6 but rather for i_3, i_4 and i_6 only (whereas the agent t_1 perceives $\neg p$ for i_3, i_4 and i_6).

Now, how is the consensus about *negation* of p reached? This happens, when every agent perceives $\neg p(a)$, i.e. none of the agents perceives $p(a)$. In our example this happens precisely for the case of i_4 and i_6. This is simple enough when we deal with atomic statements (as we did above), but becomes more involved when we deal with more complex statements. For instance it can be shown that (with T finite), the property $\neg\neg p(a)$ is commonly perceived precisely, if no $\sqsubseteq$-*minimal* agent observes $\neg p(a)$.

Firstly, let us describe the syntax of our theory. Consider the language L, first-order predicate calculus without the function or constant symbols, associated with some signature σ. ([2]). Hence, we have a certain number of relational symbols p_j, each having its arity n_j. This first order language is now extended as follows. Let T be a collection of new symbols (these are names of agents), partially ordered by the relation $\sqsubseteq$. We assume here T to be finite. For each agent $t \in T$ we have a propositional constant e_t. In addition, for each $t \in T$ we have a unary operator d_t. The language of perception L^T is defined as an extension of L by propositional constants e_t and modal operators d_t for all $t \in T$.

The intuitive meaning of $d_t\phi$ is: "agent t perceives ϕ". The formulas of L^T not having d_t in front, in particular formulas of L itself, have different meaning, namely, "There is a consensus about ϕ"

We shall give now a semantics for our language and subsequently give a complete axiomatization for our semantics.

The structures for our language are of the following form: We have an universe M, and for each predicate p_j of the language L we have a relation $r_j \subseteq T \times M^{n_j}$. The relations r_j are not arbitrary. They need to satisfy *antimonotonicity property.* This is the following property:

$$(r_j(t, \overline{x}) \wedge s \sqsubseteq t) \Rightarrow r_j(s, \overline{x})$$

The antimonotonicity property means, intuitively, that the observations of t are "sharper", she can eliminate more objects. Consequently, r_j is, in fact, a sequence of relations on the universe M, $r_{j,t} = \{< x_1, \ldots, x_{n_j} >:< t, x_1, \ldots, x_{n_j} >\in r_j\}$. Hence, besides the structure $\mathcal{M} = < M, r_j >_{j\in J}$ we have "local" variants of $\mathcal{M}$, $\mathcal{M}_t$, where $\mathcal{M}_t =< M, r_{j,t} >_{j\in J}$. The antimonotonicity property requires, that the interpretations of predicates in the local structure corresponding to more observant agent are smaller. Hence $\mathcal{M}_s$ is a structure for L, as observed by the agent s. A word of caution: in her perception of reality, the agent s takes into account also the perceptions of agents t, for $t \sqsubseteq s$.

We define formally satisfaction for the formulas $\phi \in L^T$ as follow: Let v be a valuation of variables, i.e. a function from the set Var of variables of the language L to M. We define the relation

$$\mathcal{M} \models \Psi[v]$$

in a roundabout manner. We first define satisfaction for the formulas starting with the operator d_t:

1) $\mathcal{M} \models d_t p_i(x_1, \ldots, x_m)[v]$ iff $< t, v(1), \ldots v(m) >\in r_i$.

2) $\mathcal{M} \models d_t e_s[v]$ iff $t \sqsubseteq s$

3) $\mathcal{M} \models d_t(\phi \vee \psi)[v]$ iff $\mathcal{M} \models d_t(\phi)[v]$ or $\mathcal{M} \models d_t(\psi)[v]$

4) $\mathcal{M} \models d_t(\phi \wedge \psi)[v]$ iff $\mathcal{M} \models d_t(\phi)[v]$ and $\mathcal{M} \models d_t(\psi)[v]$

5) $\mathcal{M} \models d_t(\phi \Rightarrow \psi)[v]$ iff for all $s \sqsubseteq t$, $\mathcal{M} \models d_s(\phi)$[v] implies $\mathcal{M} \models d_s(\psi)[v]$

6) $\mathcal{M} \models d_t(\neg\phi)[v]$ iff for all $s \sqsubseteq t$, not ($\mathcal{M} \models d_s(\phi)[v]$)

7) $\mathcal{M} \models d_s(d_t(\phi))[v]$ iff $\mathcal{M} \models d_t(\phi)[v]$

8) $\mathcal{M} \models d_t(\forall x_i\phi)[v]$ iff for all $a \in M$, $\mathcal{M} \models d_t(\phi)[v(i/a)]$

9) $\mathcal{M} \models d_t(\exists x_i\phi)[v]$ iff there exists $a \in M$, $\mathcal{M} \models d_t(\phi)[v(i/a)]$

Next, we define the satisfaction for all formulas of L^T:

$$\mathcal{M} \models \Psi[v] \text{ if and only if } \mathcal{M} \models d_t\Psi[v] \text{ for all } t \in T.$$

First of all notice that we use the satisfaction relation $\models$ in two meanings; one for formulas of L^T starting with d_t, the other for all the formulas of L^T. The clause 7 implies, however, that for the formulas of form $d_t(\psi)$ these definitions coincide, so no problem is created. Now, let us look at the meaning of the clauses 1-9 in our definition. The clause 1) tells us that in case of atomic statements, each agent follows her observations. Clauses 3 and 4 are self explanatory. Clauses 5 and 6 tell us that the agent evaluates the negative information (as carried by implication or negation) taking into account perceptions of all the agents which are "less informed" (i.e. are $\sqsubseteq$-smaller). Clauses 8 and 9 are tailored to the understanding of quantifiers as generalized conjunctions and alternatives. Clause 2 puts in the relationship of agents ($\sqsubseteq$) into the language. Finally, the condition 7 tells us that all the perceptions are generally known and that the agents do not attempt to modify other agents opinions.

We shall introduce now a complete axiomatization for the semantics $\models$.

0) Axiomatization for intuitionistic logic ([12])

1) $d_t(\Phi \vee \Psi) \Leftrightarrow d_t(\Phi) \vee d_t(\Psi)$

2) $d_t(\Phi \wedge \Psi) \Leftrightarrow d_t(\Phi) \wedge d_t(\Psi)$

3) $d_t(\Phi \Rightarrow \Psi) \Leftrightarrow \bigwedge_{s \sqsubseteq t}(d_s(\Phi) \Rightarrow d_s(\Psi))$

4) $d_t(\neg\Phi) \Leftrightarrow \bigwedge_{s \sqsubseteq t} \neg d_s(\Phi)$.

5) $d_t d_w(\Phi) \Leftrightarrow d_w(\Phi)$.

6) $d_t e_w$ for $t \sqsubseteq w$.

7) $\neg d_t e_w$ for t non-$\sqsubseteq$ w.

8) $d_t(\Phi) \vee \neg d_t(\Phi)$, for every $t \in T$.

9) $\Phi \Leftrightarrow \bigvee_{t \in T}(d_t(\Phi) \wedge e_t)$

In addition to the usual rules of proof of intuitionistic logic ([12]), we need two more rules of proof, for introduction and elimination of the symbols d_s.

$$\frac{\{d_s\phi : s \in T\}}{\phi}$$

$$\frac{\phi}{d_s\phi}$$

We have the following result:

Theorem 1 *(a) The axiomatization (0) – (9), with the rules of proof for intuitionistic logic and the introduction and elimination rules for d_s, forms an adequate and complete axiomatization of formulas of L^T, that are valid in the above semantics.*
(b) For theories $Th \subseteq L^T$ semantic entailment generated by our semantics and provability by means of axioms 1-9 (with the rules of proof as listed above) coincide.

The proof of this theorem is algebraical in its nature and analogous to the completeness argument of [15]. At this moment it is natural to see why we had to adopt the axioms of intuitionistic and not of classical logic. First of all note that classical tautologies are not valid under the above method of reaching consensus. To see this, let us look again at the above example.

The sentence $p(i_3) \vee \neg p(i_3)$ is not satisfied in the structure described in our example. It is, of course, not valid intuitionistically. To see why it fails, we proceed with the computation:

$$\mathcal{M} \models p(i_3) \vee \neg p(i_3)$$

if and only if for $i = 1, \ldots, 4$:

$$\mathcal{M} \models d_{t_i}(p(i_3) \vee \neg p(i_3))$$

i.e.

$$\mathcal{M} \models d_{t_i}(p(i_3)) \vee d_{t_i}(\neg p(i_3))$$

It is easy to see that for $i = 4$ this alternative is not true.

We shall state now two imbedding results for the structures under consideration; one "horizontal", that is with T changing, another "vertical", i.e. with M changing. To explain the idea, let us look at the following interpretation: T corresponds to the "committee" of experts attempting consensus. What happens if the group becomes bigger? It turns out that in this case change comes at two levels: firstly, some members may be forced to change some of their opinions. This happens when a new member is included in the panel and that member's opinion has to be taken into account by an "old" member. If this does not happen, the opinions of "old" members does not change, but the general consensus may still change.

With this intuition in mind we introduce the notion of *ideal* in $< T, \sqsubseteq >$ as follows: A subset T_1 of T is an ideal if and only if:

$$\forall_{t \in T_1} \forall_{s \in T} (s \sqsubseteq t \Rightarrow s \in T_1).$$

In our example, there are nine ideals. For instance $\{t_1, t_2\}$ is an ideal, and $\{t_1, t_3, t_4\}$ is an ideal, whereas $\{t_2, t_3, t_4\}$ is not an ideal. An ideal is principal if it possesses the largest element. Of our two ideals, the first one is principal, the other is not. Ideals are, sometimes called "initial segments", since in case of linearly ordered sets these notions coincide. The collection of all ideals in $\mathcal{T} =< T, \sqsubseteq >$, denoted by $L(\mathcal{T})$ is naturally ordered by inclusion. It is easy to check that it is a lattice ordering, in fact it is a complete lattice. Lattice operations are set-theoretical unions and intersections. There is a natural imbedding of $\mathcal{T}$ into $L(\mathcal{T})$; to every element $t \in T$ we assign principal ideal determined by t, that is the initial segment determined by t. The lattice $L(\mathcal{T})$ is a Heyting lattice. The pseudo-complement of ideal I is the largest ideal disjoint with I (this construction was introduced in [6]).

Now, for $T_1 \subseteq T$ and a structure $\mathcal{M}$ over T define the restriction $\mathcal{M}_1$ as follows. The universe of $\mathcal{M}_1$ is the same as that of $\mathcal{M}$, but the relations are defined by eliminating "snapshots" corresponding to elements of $T - T_1$. Formally:

$$\mathcal{M}_1 =< M, r_{1,j} >$$

where:

$$r_{1,j} = r_j \cap (T_1 \times M^{n_j}).$$

Thus, we leave only that part of $\mathcal{M}$ which pertain to T_1. We have the following result on the connection between $\mathcal{M}$ and $\mathcal{M}_1$.

Theorem 2 *Let T_1 be a nonempty ideal in the poset $\mathcal{T}$ and $\mathcal{M}_1$ be the restriction of $\mathcal{M}$ to T_1. Let ϕ be a formula of the language corresponding to restriction. Then for all $t \in T_1$, and valuation v,*

$$\mathcal{M}_1 \models d_t\phi[v] \text{ if and only if } \mathcal{M} \models d_t\phi[v].$$

The condition that T_1 is an ideal in $\mathcal{T}$ cannot be omitted. In our example the set $T' = \{t_1, t_2, t_4\}$ provides a suitable counterexample.

Besides "horizontal" restriction, that is one in which we deal with a smaller group of agents, we have also a "vertical" one, when the collection of observed objects is larger. Here we have the following result:

Theorem 3 *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two T-systems such that for all $t \in T$ the projection, $\mathcal{M}_{1,t}$ is an elementary subsystem of the corresponding projection of $\mathcal{M}_2$, $\mathcal{M}_{2,t}$. Then the structure $\mathcal{M}_1$ is an elementary substructure of $\mathcal{M}_2$, in particular for all the sentences Φ of the extended language L^T, $\mathcal{M}_1 \models \Phi$ if and only if $\mathcal{M}_2 \models \Phi$.*

3 Consensus with the epistemic operators

In this section we investigate the extension of the language L^T by the "positive knowledge" or necessity operators I_S and knowledge operators K_S. The operators K_S were introduced in case of propositional logic in [11, 10].

Let $s \in T$ be an agent. Let ind_s be an equivalence relation on the universe M observed by that agent. We assign to ind_s the following meaning. From the point of view of the agent s, elements x and y of M, such that $ind_s(x, y)$ are indistinguishable, any reasonable property possessed by x is shared by y. This means that the agent s does not have means which allow her to find some properties that distinguish x and y. The relation ind_s induces on M an interior operation defined as follows: given $X \subseteq M$, $I_s(X)$ is the union of all equivalence classes of ind_s included in X. The properties of the operation I_s were studied in [9, 8]. The relation I_s extends naturally to the Cartesian product of M, M^n by the following: If $\overline{x}, \overline{y} \in M^n$ then:

$$ind_s(\overline{x}, \overline{y}) \text{ iff } \forall_{i \leq n} ind_s(x_i, y_i).$$

Given a poset $\mathcal{T} =< T, \sqsubseteq >$ we assume that for all s and t in T, $s \sqsubseteq t$ implies $ind_s \subseteq ind_t$. This stipulation is justified by our discussion of the perception of agents. When $s \sqsubseteq t$, the perception of t is "sharper"- so there are *less* elements indistinguishable (by t) from a given $x \in M$.

Next, we see that given a group of agents $S \subseteq T$ (remember that we deal with the poset $\mathcal{T} =< T, \sqsubseteq >$ of agents), we can define an associated indiscernibility relation ind_S by setting $ind_S(x, y) = \bigcap_{s \in S} ind_s$. Thus ind_S is the joint capability of all agents from S *together* to discern objects in M. Clearly ind_S is an equivalence relation as well. It is "finer or equal" than all the relations ind_s for $s \in S$. In particular, it follows that for any $t \in T$, the indiscernibility relation associated with the principal ideal $\{s \colon s \sqsubseteq t\}$ is identical with ind_t. This observation indicates that it is enough to consider indiscernibility relations determined by ideals. The interior operation I_S is defined in an analogous fashion.

The interior operator I_S is, of course, monotone: If $X \subseteq Y$, then $I_S(X) \subseteq I_S(Y)$. It generates, however another, non-monotone operation K_S associated with the knowledge of the agent s (or the group of agents S) about X. Define now:

$$K_S(X) = I_S(X) \cup I_S(M - X).$$

The operator K_S has this meaning: Element $x \in M$ has the property $K_S(X)$ if either all elements indistinguishable from x are in X or none of them. Thus the elements of K_s (K_S) are these which are known (from the point of view of s (or group S)) to be in X or outside of X. The remaining elements (i.e. elements of $M - K_S(X)$ are elements which cannot be distinguished both from elements in X and outside of X as well. One can argue that the complement $M - K_s(X)$ is a "grey area" of s with respect to X.

The meaning of the operation K_S becomes clearer when we look at the following construction: Given $X \subseteq M$ and the equivalence relation ind, consider the universe M/ind. A subset $X \subseteq M$ generates the following "three-valued" subset X/ind of M/ind: $[x] \in X/ind$ takes value "truth" if $[x] \subseteq X$, $[x] \in (X/ind)$ takes value "false" if $[x] \cap X = \emptyset$ (for remaining $x \in X$, the logical value of the formula $[x] \in X/ind$ is "unknown"). Then, $K_S(X)$ consists of these x for which the the above definition does not assign value "unknown". If ind_S is a congruence with respect to X, then $K_S(X)$ is the whole set M, otherwise it is not. The above construction can easily be extended to the subsets of M^n.

In our setting, we are interested, of course, in the situation when the subsets X considered in the previous paragraph are themselves definable. But from the point of view of the individual agent s or a group of agents S, their means may be not sufficient to define X. Hence the need to use I_S and K_S.

In accordance to the above, we extend the language L^T, by adjoining new binary predicate symbols $\mathbf{ind}_S(x, y)$ for every $S \in L(\mathcal{T})$ and modal operators I_S and K_S, again for $S \in L(\mathcal{T})$. The intended interpretation of I_S and K_S is as proposed above.

With this intuition we shall extend now the context of the section 1 to the new situation. In view of the results of section 1 and in particular the role of ideals in the definition of the satisfaction relation $\models$, we shall add the modal operators I_S and K_S for $S \in L(\mathcal{T})$. Let us notice, that the relation $\mathbf{ind}_S$ is definable in terms of relations $\mathbf{ind}_s$ for $s \in T$.

The knowledge structure $\mathcal{M}$ will be similar to the previously considered structures except that now we have, for each $S \in L(\mathcal{T})$ an equivalence relations ind_S. These relations satisfy the following two conditions:

a) $S_1 \subseteq S_2 \Rightarrow ind_{S_2} \subseteq ind_{S_1}$

b) Setting $ind_s = ind_{\{t : t \sqsubseteq s\}}$, $ind_S = \bigcap_{s \in S} ind_s$

The condition b implies that $ind_\emptyset$ is the complete relation. Condition a tells us that the more perceptive agent has a finer indiscernibility relation at her disposal. Condition b

tells us what is the relationship between the indiscernibility relation ind_S and the relations ind_s for $s \in S$.

The language L has more symbols. We have, as mentioned above, unary modal operators I_S and K_S for $S \in L(\mathcal{T})$. The language L^T is defined as before, but with respect to richer L now.

The satisfaction relation $\models$ is defined as in Section 1, but additional clauses are necessary now for the relational symbols $\mathbf{ind}_S$ and formulas starting with I_S and K_S. These are as follows:

10a) $\mathcal{M} \models d_t\mathbf{ind}_S(x_1, x_2)[v]$ iff $ind_S(v(1), v(2))$ $(t \in T,\ S \in L(\mathcal{T}),\ S \neq \emptyset)$.

10a) $\mathcal{M} \models d_t\mathbf{ind}_\emptyset(x_1, x_2)[v]$ for every $t \in T$ and valuation v.

11a) $\mathcal{M} \models d_t(I_S(\Phi))[v]$ iff $\mathcal{M} \models d_t((\Phi))$ when $var(\Phi) = \emptyset$,

11b) $\mathcal{M} \models d_t(I_S(\Phi))[v]$ iff for all sequences $u_1, \ldots, u_n$, $ind_S(v(x_1), u_1)$ and ... $ind_S(v(x_n), u_n)$ implies $\mathcal{M} \models d_t((\Phi))[v']$ where $v' = v[x_1/u_1, \ldots x_n/u_n]$ when $var(\Phi) = \{x_1, \ldots x_n\}$.

12a) $\mathcal{M} \models d_t(K_S(\Phi))[v]$ iff $\mathcal{M} \models d_t((\Phi))$ when $var(\Phi) = \emptyset$,

12b) $\mathcal{M} \models d_t(K_S(\Phi))[v]$ iff for all sequences $u_1, \ldots, u_n$, $ind_S(v(x_1), u_1)$ and ... $ind_S(v(x_n), u_n)$ implies $\mathcal{M} \models d_t((\Phi))[v']$ or for all sequences $u_1, \ldots, u_n$, $ind_S(v(x_1), u_1)$ and ...-$ind_S(v(x_n), u_n)$ implies non $\mathcal{M} \models d_t((\Phi))[v']$ (where $v' = v[x_1/u_1, \ldots x_n/u_n]$ when $var(\Phi) = \{x_1, \ldots x_n\}$).

As in Section 1 we extend the definition of satisfaction to all formulas of L^T via consensus condition:

$$\mathcal{M} \models \Psi[v] \text{ if and only if } \mathcal{M} \models d_t\Psi[v] \text{ for all } t \in T$$

and prove that this extension preserves satisfaction (as introduced above) of formulas starting with d_t extend the completeness result of Section 1 to the present context we introduce the following additional axioms:

10) Equivalence relation axioms for each $\mathbf{ind}_S$, $S \in L(\mathcal{T}), S \neq \emptyset$.

11) $d_t\mathbf{ind}_S(x, y) \Leftrightarrow \forall_{S_1 \subseteq S}\mathbf{ind}_{S_1}(x, y)$, $t \in T$, $S, S_1 \in L(\mathcal{T})$.

12) $\forall_{xy}\mathbf{ind}_\emptyset(x, y)$.

13) $I_S\Phi \Leftrightarrow \Phi$, if $var(\Phi) = \emptyset$.

14) $K_S\Phi \Leftrightarrow \Phi$, if $var(\Phi) = \emptyset$.

15) $K_S d_t\Phi \Leftrightarrow I_S d_t\Phi \vee I_S\neg d_t\Phi$.

16) $d_t I_S(\Phi) \Leftrightarrow I_S d_t(\Phi)$.

17) $d_t K_S(\Phi) \Leftrightarrow K_S d_t(\Phi)$.

18) $\forall_{\overline{x},\overline{y}}(\mathbf{ind}_S(x_1, y_1) \wedge \ldots \wedge \mathbf{ind}_S(x_n, y_n) \Rightarrow (d_t\Phi(y_1, \ldots, y_n) \Leftrightarrow I_S d_t \Phi(x_1, \ldots, x_n)))$.

The axiom scheme 11 is used to prove that the operator d_t acts identically on the indiscernibility relation $\mathbf{ind}_S$, that is in our theory we can prove for every $t \in T$ and $S \in L(\mathcal{T})$ that:

$$d_t \mathbf{ind}_S(x, y) \Leftrightarrow \mathbf{ind}_S(x, y).$$

The axiom scheme 11 is also used for proof of antimonotonicity property of relations *ind*. The formulas $\mathbf{ind}_S$ have property of excluded middle, that is the formula $\mathbf{ind}_S(x, y) \vee \neg\mathbf{ind}_S(x, y)$ is provable in the axiomatization (0)-(18).

Let us notice that the properties of knowledge operator, sometimes consider paradoxical persist in our logic. For instance the formula: $(\Phi(x) \Rightarrow \Psi(x)) \wedge K_S\Phi(x) \Rightarrow K_S\Psi(x)$ is not provable in our system.

With the above axiomatization we have the analogues of the results of Section 1.

Theorem 4 *The axiomatization consisting of formulas 0 – 18 is a complete axiom system for theories in the language L^T, that is for an arbitrary theory Th, Th entails Φ semantically if and only if Th together with the axioms 0 – 18 proves Φ using the rules of proof of intuitionistic logic and two additional rules intoduced in Section 1.*

Theorem 5 *If a poset $\mathcal{T}_1$ is an ideal in the poset $\mathcal{T}$, and $\mathcal{M}_{T_1}$ is the restriction of the model $\mathcal{M}$ to T_1, then for every formula Φ of the language corresponding to the restriction, and every valuation v,*

$$\mathcal{M} \models \Phi[v] \text{ if and only if } \mathcal{M}_{T_1} \models \Phi[v].$$

Finally, we have:

Theorem 6 *If $\mathcal{M}_1$ and $\mathcal{M}_2$ are two T-systems satisfying the conditions of the theorem 3 and satisfying the conditions that the interpretations of $\mathbf{ind}_S$ in $\mathcal{M}_1$ are elementary substructures of the interpretations of appropriate relations in $\mathcal{M}_2$, then for all the formulas of the extended language L^T, $\mathcal{M}_1 \models \Phi$ if and only if $\mathcal{M}_2 \models \Phi$.*

The additional condition in the theorem 6 is, in fact, quite strong. In case when the equivalence relations ind_S have property of having finite equivalence classes, it implies that the universe of $\mathcal{M}_1$ is open in every topology generated by ind_S for $S \in L(\mathcal{T})$.

The proof of theorem 4 is algebraical, as in the case of theorem 1 and similar to that of 4.2 in [15]. The argument of theorem 5 requires some preservation and imbedding results: If T_1 is an ideal in $\mathcal{T}$, then there is a natural imbedding of $L(\mathcal{T}_1)$ into $L(\mathcal{T})$. That embedding is not a lattice imbedding (it does not preserve unit) but preserves lattice operations and zero element.

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