Homework 4: CS537, Spring 2016
Due Date: 10:00AM, March 9, 2016

Please show all steps in your work. Please be reminded that you should do your homework independently.

1. (10 points) Prove that the product of two unit lower triangular matrices is also unit lower triangular.

2. (10 points) Let \( A \) be a matrix of tridiagonal form such that \( a_i c_i > 0 \) for \( 1 \leq i \leq n - 1 \). Find the general form of the diagonal matrix \( D = \text{diag}(\alpha_i) \) with \( \alpha_i \neq 0 \) such that \( D^{-1}AD \) is symmetric. What is the general form of \( D^{-1}AD \)?

3. (10 points) Find the \( LU \) factorization, where \( L \) is unit lower triangular for

\[
A = \begin{bmatrix}
1 & 0 & 0 & 1 \\
1 & 1 & 0 & -1 \\
-1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1
\end{bmatrix}
\]

Show all intermediate matrices.

4. (10 points) Using the Jacobi, Gauss-Seidel, and the SOR (\( \omega = 1.4 \)) iterative methods, write and run a code to solve the following linear system to four decimal places of accuracy

\[
\begin{bmatrix}
7 & 3 & -1 & 2 \\
3 & 8 & 1 & -4 \\
-1 & 1 & 4 & -1 \\
2 & -4 & -1 & 6
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} = \begin{bmatrix}
-1 \\
0 \\
-3 \\
1
\end{bmatrix}
\]

Compare the number of iterations in each case and plot a graph showing the change of the residual norms as a function of the number of iterations. The exact solution is \( x = (-1, 1, -1, 1)^T \).

5. (10 points) Solve the following system using Gaussian elimination with scaled partial pivoting. Carry four significant digits.

\[
\begin{align*}
3x_1 + 2x_2 - 5x_3 &= 0 \\
2x_1 - 3x_2 + x_3 &= 0 \\
x_1 + 4x_2 - x_3 &= 4
\end{align*}
\]

6. (10 points) For any \( n \times n \) matrix \( A \), it can be diagonalized as \( A^T A = Q \Lambda Q^{-1} \), where \( Q \) is orthogonal, i.e., \( QQ^T = Q^T Q = I \). \( \Lambda = \text{Diag}[\lambda_1, \lambda_2, \cdots, \lambda_n] \) is a diagonal matrix with \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \). Now if we can decompose the matrix \( A \) as \( A = U \Sigma V \) where \( U \) and \( V \) are orthogonal matrices and \( \Sigma = \text{Diag}[\sigma_1, \sigma_2, \cdots, \sigma_n] \) is a diagonal matrix. Show that \( \sigma_i = +\sqrt{\lambda_i} \), i.e., \( \sigma_i \) are the singular values of the matrix \( A \).