1. (10 points) Find an approximate value of $\int_1^2 \sin x \, dx$ using composite Simpson’s rule with $h = 0.25$. Give a bound on the error. Then calculate the exact value of the integration and compute the exact error to see if the error bound is accurate.

2. (10 points) A numerical integration scheme that is not as well known is the basic Simpson’s $\frac{3}{8}$ rule over three intervals

$$\int_a^{a+3h} f(x) \, dx \approx \frac{3h}{8} [f(a) + 3f(a + h) + 3f(a + 2h) + f(a + 3h)].$$

Estimate the error term for this rule and explain why this rule is not as popular as the Simpson’s rule.

3. (10 points) In the Romberg algorithm, $R(n, 0)$ denotes an estimate of

$$\int_a^b f(x) \, dx,$$

with subintervals of size $h = (b - a)/2^n$. If it were known that

$$\int_a^b f(x) \, dx = R(n, 0) + a_3 h^3 + a_6 h^6 + \cdots$$

how would we have to modify the Romberg algorithm?

4. (10 points) What is a reasonable bound on the error when we use the composite trapezoid rule on

$$\int_0^1 \cos x^3 \, dx$$

taking 201 equally spaced points (including endpoints)?

5. (10 points) Construct a rule of the form

$$\int_{-1}^1 f(x) \, dx \approx \alpha f(-\frac{1}{2}) + \beta f(0) + \gamma f(\frac{1}{2})$$

that is exact for all polynomials of degree $\leq 2$; that is; determine values for $\alpha$, $\beta$, and $\gamma$.

6. (10 points) Determine a formula of the form

$$\int_0^h f(x) \, dx \approx w_0 f(0) + w_1 f(h) + w_2 f''(0) + w_3 f''(h)$$

that is exact for polynomials of as high degree as possible.