Solving Inconsistent Equations

a system of linear equations of the form

$$\sum_{j=0}^{n} a_{kj}\, x_j = b_k \qquad (0 \le k \le m)$$

with $m > n$ is inconsistent, if there is no possible vector $(x_0, x_1, \ldots, x_n)$ to make the residual zero. There is no solution in the conventional sense satisfying this system

it is of some interest in applications to find the vector that minimizes the 2-norm residual

$$\phi(x_0, x_1, \ldots, x_n) = \sum_{k=0}^{m} \left(\sum_{j=0}^{n} a_{kj} x_j - b_k \right)^2$$

we can take the partial derivatives with respect to x_i and set them equal to zero to obtain the normal equations

$$\sum_{j=0}^{n} \left(\sum_{k=0}^{m} a_{ki}\, a_{kj} \right) x_j = \sum_{k=0}^{m} b_k\, a_{ki}$$

for $0 \le i \le n$

Direct Factorizations

The normal equations obtained can be solved by Gaussian elimination and its solution will be a best approximate solution of the original system in the least squares sense

if we write the original linear system as

$$Ax = b$$

it is also possible to directly factor the matrix A as

$$A = Q\,R$$

where Q is an $(m+1) \times (n+1)$ orthogonal matrix satisfying $Q^T Q = I$ and R is an upper triangular $(n+1) \times (n+1)$ matrix satisfying $r_{ii} > 0$ and $r_{ij} = 0$ for $j < i$. We then have

$$R\,x = Q^T\,b$$

which can be solved by a back substitution

Singular Value Decomposition

the QR factorization can be obtained by an algorithm called modified Gram-Schmidt procedure (to orthogonalize the row vectors)

a more involved algorithm needs to compute the Singular Value Decomposition of the matrix A as

$$A = U\,\Sigma\,V^T$$

in which U and V are orthogonal, i.e.,

$$U^T U = I_{m+1} \qquad V^T V = I_{n+1}$$

and Σ is an $(m+1) \times (n+1)$ diagonal matrix having nonnegative entries

$$\Sigma = \begin{pmatrix} \sigma_0 & & & & & \\ & \sigma_1 & & & & \\ & & \ddots & & & \\ & & & \sigma_r & & \\ & & & & 0 & \\ & & & & & \ddots \end{pmatrix}$$

Pseudo Inverse

if A is a nonsingular square matrix with

$$Ax = b$$

we can compute the true inverse of A, i.e., A^{-1} to solve the linear system as

$$x = A^{-1}b$$

if A is an $(m+1) \times (n+1)$ rectangular matrix, we first compute the singular value decomposition of A as in the form of

$$A = U\,\Sigma\,V^T$$

we then "invert" Σ as

$$\Sigma^{-1} = \begin{pmatrix} \sigma_0^{-1} & & & \\ & \sigma_1^{-1} & & \\ & & \ddots & \\ & & & \sigma_r^{-1} \\ & & & 0 \\ & & & \vdots \\ & & & 0 \end{pmatrix}$$

Pseudo Inverse (II)

we can define the pseudo inverse of $\boldsymbol{A}$ as

$$\boldsymbol{A}^{+} = \boldsymbol{V}\,\boldsymbol{\Sigma}^{-1}\,\boldsymbol{U}^{\boldsymbol{T}}$$

and the "solution" of the rectangular linear system is defined to be

$$\boldsymbol{x} = \boldsymbol{A}^{+}\,\boldsymbol{b} = \boldsymbol{V}\,\boldsymbol{\Sigma}^{-1}\,\boldsymbol{U}^{\boldsymbol{T}}\,\boldsymbol{b}$$

it can be shown that this definition of solution does minimize the residual norm of the original inconsistent system

note that QR factorization and Singular Value Decomposition are more expensive to perform, in many cases, than solving the normal equations. In case that the matrix $\boldsymbol{A}$ is a sparse matrix, we may be able to solve it more efficiently using certain iterative methods

forming the normal equation $\boldsymbol{A}^{\boldsymbol{T}}\boldsymbol{A}$ is an option