Interpolating Quadratic Spline

given a set of data

x	t_0	t_1	t_2	$\cdots$	t_n
y	y_0	y_1	y_2	$\cdots$	y_n

a quadratic spline $Q(x)$ can be constructed to interpolate these data, using $t_0, t_1, \ldots, t_n$ as the knots

a quadratic spline consisting of n separate pieces of quadratic functions of the form

$$Q_i(x) = a_i\,x^2 + b_i\,x + c_i \qquad x \in [t_i, t_{i+1}]$$

there are $3n$ coefficients to be determined

the interpolation conditions $Q_i(t_i) = y_i$ and $Q_{i+1}(t_{i+1}) = y_{i+1}$ give $2n$ conditions. The continuity of $Q'(x)$ gives another $n-1$ conditions at the interior knots. The last condition can be specified (arbitrarily) as $Q'(t_0) = 0$

Computational Procedure

construct a piecewise quadratic function

$$Q(x) = \begin{cases} Q_0(x) & x \in [t_0, t_1] \\ Q_1(x) & x \in [t_1, t_2] \\ \vdots & \vdots \\ Q_{n-1}(x) & x \in [t_{n-1}, t_n] \end{cases}$$

which is continuously differentiable on $[t_0, t_n]$ and $Q(t_i) = y_i$ for $0 \leq i \leq n$

the computational procedure is based on a recursive procedure on $Q_i'(t_i)$. Assuming $Q'(t_i) = z_i$, then we have

$$Q_i(x) = \frac{z_{i+1} - z_i}{2(t_{i+1} - t_i)}(x - t_i)^2 + z_i(x - t_i) + y_i$$

this definition can be verified by showing that $Q_i(t_i) = y_i$, $Q_i'(t_i) = z_i$, and $Q_i'(t_{i+1}) = z_{i+1}$. Note that

$$Q_i'(x) = \frac{z_{i+1} - z_i}{t_{i+1} - t_i}(x - t_i) + z_i$$

Computational Procedure

the defined quadratic function $Q_i(x)$ does not necessarily satisfy the condition $Q_i(t_{i+1}) = y_{i+1}$ for $i = 0, 1, \ldots, n-1$. If we apply this condition, we have

$$y_{i+1} = \frac{z_{i+1} - z_i}{2(t_{i+1} - t_i)}(t_{i+1} - t_i)^2 + z_i(t_{i+1} - t_i) + y_i$$

after simplification, we obtain

$$z_{i+1} = -z_i + 2\left(\frac{y_{i+1} - y_i}{t_{i+1} - t_i}\right) \quad (0 \le i \le n-1)$$

from this equation, we can compute the values of $z_0, z_1, \ldots, z_n$ recursively, by assigning an arbitrary value to z_0. Then the second degree spline is written as

$$Q_i(x) = \frac{z_{i+1} - z_i}{2(t_{i+1} - t_i)}(x - t_i)^2 + z_i(x - t_i) + y_i$$

for each interval $[t_i, t_{i+1}]$, $i = 0, 1, \ldots, n-1$.

Subbotin Quadratic Spline

knots are points where the spline function can change its form, nodes are points where the values of the function is specified

it is not necessary that the knots are nodes.

Subbotin suggests to use the midpoints of the knots as the (interpolation) nodes and the two end points. This yields $n+2$ conditions. The continuity of Q and Q' give another $2(n-1)$ conditions, for a total of $3n$ conditions. No free variable is required

let the knots be $a = t_0 < t_1 < \cdots < t_n = b$, we define the interpolation nodes as

$$\begin{cases} \tau_0 = t_0 \qquad \tau_n = t_n \\ \tau_i = \frac{1}{2}(t_i + t_{i+1}) \qquad (1 \leq i \leq n) \end{cases}$$

Subbotin Quadratic Spline (II)

the quadratic spline function $Q(x)$ should satisfy the interpolation conditions

$$Q(\tau_i) = y_i \qquad (0 \le i \le n+1)$$

let $Q_i'(t_i) = z_i$, where $Q_i(x)$ is defined on the interval $[t_i, t_{i+1}]$, which can be written as

$$\begin{aligned} Q_i(x) &= y_{i+1} + \frac{1}{2}(z_{i+1} + z_i)(x - \tau_{i+1}) \\ &\quad + \frac{1}{2h_i}(z_{i+1} - z_i)(x - \tau_{i+1})^2 \end{aligned}$$

where $h_i = t_{i+1} - t_i$. We can verify that

$$Q_i(\tau_{i+1}) = y_{i+1},\ Q_i'(t_i) = z_i,\ Q_i'(t_{i+1}) = z_{i+1}$$

we also want to impose the continuity condition at the knots

$$\lim_{x \to t_i^-} Q_{i-1}(x) = \lim_{x \to t_i^+} Q_i(x) \quad (1 \le i \le n-1)$$

Subbotin Quadratic Spline (III)

these conditions lead to a recursive formula

$h_{i-1}z_{i-1}+3(h_{i-1}+h_i)z_i+h_iz_{i+1} = 8(y_{i+1}-y_i)$
for $i = 1, 2, \ldots, n-1$

we impose the first and last interpolation conditions

$$Q(\tau_0) = y_0 \qquad Q(\tau_{n+1}) = y_{n+1}$$

which yield two more equations

$$\begin{aligned} 3h_0z_0 + h_0z_1 &= 8(y_1 - y_0) \\ h_{n-1}z_{n-1} + 3h_{n-1}z_n &= 8(y_{n+1} - y_n) \end{aligned}$$

now we have $n+1$ equations for the $n+1$ unknowns $z_0, z_1, \ldots, z_n$

we need to solve a linear system of the form $Ax = b$ with A being a tridiagonal matrix

An Example

determine if the function is a quadratic spline

$$Q(x) = \begin{cases} -x^2 & x \leq 0 \\ x & x > 0 \end{cases}$$

the interval can be viewed as $(-\infty, \infty)$ which is not a closed interval

also, the derivative of $Q(x)$ is

$$Q'(x) = \begin{cases} -2x & x \leq 0 \\ 1 & x > 0 \end{cases}$$

hence, $Q'(x)$ is discontinuous at $x = 0$. It follows that $Q(x)$ is not a quadratic spline

how about the function

$$Q(x) = \begin{cases} 0.1x^2 & 0 \leq x \leq 1 \\ 9.3x^2 - 18.4x + 9.2 & 1 \leq x \leq 1.3 \end{cases}$$