CS321-002

Introduction to Numerical Methods

Lecture 6

Approximation by Spline Functions

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Lower Degree Splines

given a set of data, it is possible to construct a polynomial to interpolate these data. The disadvantage of high order polynomials is its oscillation behavior

a spline function is a function that consists of polynomial pieces joined together with certain smoothness conditions

first degree spline function is a polygonal function with linear polynomials joined together to achieve continuity

the points $t_0, t_1, \ldots, t_n$ at which the function changes its character are called knots

note that **knots** do not have to be nodes

Piecewise Linear Polynomial

a piecewise linear function for the spline of degree 1 can be written as

$$S(x) = \begin{cases} S_0(x) & x \in [t_0, t_1] \\ S_1(x) & x \in [t_2, t_2] \\ \vdots & \vdots \\ S_{n-1}(x) & x \in [t_{n-1}, t_n] \end{cases}$$

where

$$S_i(x) = a_i\, x + b_i$$

is a linear polynomial

the knots $t_0, t_1, \ldots, t_n$ and the coefficients a_i, b_i, $i = 0, 1, \ldots, n-1$ have to be known in order to evaluate $S(x)$

first to determine which interval x lies, then evaluate the linear function defined on that interval

Spline of Degree 1

a function $S(x)$ is called a spline of degree 1 if

1. the domain of S is an interval $[a, b]$

2. S is continuous on $[a, b]$

3. there is a partitioning of the interval $a = t_0 < t_1 < \cdots < t_n = b$ such that S is a linear polynomial on each subinterval $[t_i, t_{i+1}]$

for outside part of the interval $[a, b]$, we define $S(x) = S_0(x)$ when $x < a$, and $S(x) = S_{n-1}(x)$ when $x > b$

it is important that the spline of degree 1 be continuous at the knots, i.e., the left limit and the right limit are equal

$$\lim_{x \to s^+} f(x) = \lim_{x \to s^-} f(x) = f(s)$$

An Example

determine if the following function is a first degree spline

$$S(x) = \begin{cases} x & -1 \le x \le 0.5 \\ 0.5x + 2(x - 0.5) & 0.5 \le x \le 2 \\ x + 1.5 & 2 \le x \le 4 \end{cases}$$

each linear function is continuous on the subinterval it is defined. We need to verify if they are continuous at the two interior knots $x = 0.5$ and $x = 2$

$$\lim_{x \to 0.5^-} S(x) = \lim_{x \to 0.5^-} x = 0.5$$

$$\lim_{x \to 0.5^+} S(x) = \lim_{x \to 0.5^+} 0.5x + 2(x - 0.5) = 0.25$$

the function is not a spline of degree 1, as

$$\lim_{x \to 0.5^-} S(x) \neq \lim_{x \to 0.5^+} S(x)$$

i.e., $S(x)$ is discontinuous at the knot $x = 0.5$

Construct Spline of Degree 1

given a data set with $t_0 < t_1 < \cdots < t_n$

x	t_0	t_1	$\cdots$	t_n
y	y_0	y_1	$\cdots$	y_n

a linear polynomial can be constructed using two pairs of neighboring data

first compute the slope of the line as

$$m = \frac{y_{i+1} - y_i}{t_{i+1} - t_i}$$

the straight line equation is given by the point-slope formula as

$$S_i(x) = y_i + m_i(x - t_i)$$

it is easy to see that we have $2n$ degrees of freedom a_i and b_i and $2n$ conditions. So the construction of first degree spline is guaranteed

Modulus of Continuity

the modulus of continuity of a function f is defined as, for $a \leq u \leq v \leq b$

$$\omega(f;h) = \sup\{|f(u) - f(v)| \ : |u - v| \leq h\}$$

the quantity is the largest variation of f over a small interval of size h. It measures how much f can change in such an interval

if f is continuous on $[a, b]$ then

$$\lim_{h \to 0} \omega(f;h) = 0$$

if f is differentiable on $[a, b]$ then

$$|f(u) - f(v)| = |f'(c)(u - v)| \leq M_1|u - v| \leq M_1 h$$

where M_1 is the maximum value of $|f'(x)|$ on (a, b). It follows that

$$\omega(f;h) \leq M_1 \, h$$

Theorems

if p is the first degree polynomial that interpolates a function f at the end points of an interval $[a, b]$, then with $h = b - a$, we have

$$|f(x) - p(x)| \leq \omega(f; h) \qquad (a \leq x \leq b)$$

note that the linear function passes through $[a, b]$ can be written as

$$p(x) = \left(\frac{x-a}{b-a}\right) f(b) + \left(\frac{b-x}{b-a}\right) f(a)$$

hence

$$\begin{aligned} f(x) - p(x) &= \left(\frac{x-a}{b-a}\right) [f(x) - f(b)] \\ &\quad + \left(\frac{b-x}{b-a}\right) [f(x) - f(a)] \end{aligned}$$

note that

$$\left(\frac{x-a}{b-a}\right) f(x) + \left(\frac{b-x}{b-a}\right) f(x) = f(x)$$

the result follows immediately

Accuracy of First Degree Spline

let p be a first degree spline having knots $a = x_0 < x_1 < \cdots < x_n = b$. If p interpolates a function f at these knots, then with $h = \max_i(x_i - x_{i-1})$ we have

$$|f(x) - p(x)| \leq \omega(f;h) \qquad (a \leq x \leq b)$$

if more knots are inserted such that the maximum spacing h goes to zero, the corresponding first degree spline will converge uniformly to f

if f' or f'' exist and are continuous, we have

$$|f(x) - p(x)| \leq M_1 \frac{h}{2} \qquad (a \leq x \leq b)$$

$$|f(x) - p(x)| \leq M_2 \frac{h^2}{8} \qquad (a \leq x \leq b)$$

where M_1 and M_2 are the maximum values of f' and f'' on (a,b), respectively

Second Degree Spline

a function Q is a second degree spline if

1. the domain of Q is an interval $[a, b]$

2. Q and Q' are continuous on $[a, b]$

3. there are points t_i such that $a = t_0 < t_1 < \cdots < t_n = b$ and Q is a polynomial of degree at most 2 on each subinterval $[t_i, t_{i-1}]$

a quadratic spline is a continuously differentiable piecewise quadratic function. It is a linear combination of basic functions $1, x, x^2$. The smoothness condition is stronger than that for the first degree spline

to determine a quadratic spline, we must verify the continuity of both Q and Q' at the knot points