Richardson Extrapolation (I)

first derivative can be approximated as

$$f'(x) = \frac{1}{2h}[f(x+h) - f(x-h)]$$

$$+a_2 h^2 + a_4 h^4 + a_6 h^6 + \cdots$$

in which the constants $a_2, a_4, \ldots$ depend on the higher order derivatives of f and the value of x. When such information is available, it is possible to construct much more accurate approximation schemes

define a function

$$\psi(h) = \frac{1}{2h}[f(x+h) - f(x-h)]$$

which is an approximation to $f'(x)$ with error of order $O(h^2)$. This approximation becomes accurate as $h \to 0$. So we can study the quantity $\lim_{h\to 0} \psi(h)$

Richardson Extrapolation (II)

Richardson extrapolation estimates the value of $\psi(0)$ from some computed values of $\psi(h)$ near 0

$$\begin{aligned}
\psi(h) &= f'(x) - a_2h^2 - a_4h^4 - a_6h^6 - \cdots \\
\psi\left(\frac{h}{2}\right) &= f'(x) - a_2\left(\frac{h}{2}\right)^2 - a_4\left(\frac{h}{2}\right)^4 - a_6\left(\frac{h}{2}\right)^6 - \cdots
\end{aligned}$$

multiply the 2nd equation by 4 and subtract it from the 1st equation

$$\psi(h) - 4\psi\left(\frac{h}{2}\right) = -3f'(x) - \frac{3}{4}a_4h^4 - \frac{15}{16}a_6h^6 - \cdots$$

hence

$$\phi\left(\frac{h}{2}\right) + \frac{1}{3}\left[\psi\left(\frac{h}{2}\right) - \psi(h)\right] = f'(x) + \frac{a_4}{4}h^4 + \frac{5}{16}a_6h^6 + \cdots$$

$f'(x)$ can be computed as accurate as $O(h^4)$

General Approach

a general Richardson extrapolation form

$$\psi(h) = L - \sum_{k=1}^{\infty} a_{2k}\, h^{2k}$$

where we assume that $\psi(h)$ is computable for any $h > 0$ and we want to approximate L as accurately as possible

choose a special sequence $\frac{h}{2^n}$, define

$$D(n,0) = \psi\left(\frac{h}{2^n}\right) \qquad (n \geq 0)$$

then, we have

$$D(n,0) = L + \sum_{k=1}^{\infty} A(k,0) \left(\frac{h}{2^n}\right)^{2k}$$

with $A(k,0) = -a_{2k}$. $D(n,0)$ is a rough approximate of $L = \lim_{x\to 0} \psi(x)$

Richardson Theorem

the extrapolation formula is

$$
\begin{aligned}
D(n,m) &= \frac{4^m}{4^m-1}D(n,m-1) - \\
&\quad \frac{1}{4^m-1}D(n-1,m-1) \quad (1 \le m \le n)
\end{aligned}
$$

Richardson Extrapolation Theorem:

$$D(n,m) = L + \sum_{k=m+1}^{\infty} A(k,m)\left(\frac{h}{2^n}\right)^{2k}$$

for $0 \le m \le n$

the proof of this theorem is based on induction on m, see p. 175 of the Book. Proof will be given in class

note that $D(n,m)$ approximates L at the order of $O(h^{2m})$. The convergence rate is fast

Computational Procedure

Richardson extrapolation computational procedure:

1.) write a procedure to compute $\psi(h)$

2.) decide on suitable values for n and h

3.) for $i = 0, 1, \ldots, n$, compute

$$D(i,0) = \psi(h/2^i)$$

4.) for $0 \le i \le j \le n$, compute

$$D(i,j) = D(i,j-1) + (4^j - 1)^{-1}[D(i,j-1) - D(i-1,j-1)]$$

Using Interpolation Polynomial

we can approximate the function $f(x)$ by a polynomial $p_n(x)$ of order n, such that $p_n(x) \approx f(x)$

to compute $f'(x)$, we use the approximation $f'(x) \approx p'_n(x)$

higher order polynomials are avoided because of oscillation

let p interpolates f at two points, x_0 and x_1

$$p_1(x) = f(x_0) + f[x_0, x_0](x - x_0)$$

the first derivative of $p_1(x)$ is

$$p'_1(x) = f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \approx f'(x)$$

1st and 2nd Order Approx.

let $x_0 = x$ and $x_1 = x + h$, we have

$$f'(x) \approx \frac{1}{h}[f(x+h) - f(x)]$$

this is just the $O(h)$ order sided approximation formula

put $x_0 = x - h$ and $x_1 = x + h$, we have the $O(h^2)$ approximation scheme

$$f'(x) \approx \frac{1}{2h}[f(x+h) - f(x-h)]$$

a three point polynomial interpolation is

$$\begin{aligned} p_2(x) &= f(x_0) + f[x_0, x_1](x - x_0) \\ &\quad + f[x_0, x_1, x_2](x - x_0)(x - x_1) \end{aligned}$$

we have a corrected approximation

$$p_2'(x) = f[x_0, x_1] + f[x_0, x_1, x_2](2x - x_0 - x_1)$$

Second Derivative

if we have first derivative, we can use

$$f''(x) = \frac{1}{2h}[f'(x+h) - f'(x-h)]$$

to approximate the second derivative to $O(h^2)$

a direct approximation would be using Taylor expansion

$$f(x+h) + f(x-h) =$$

$$2f(x) + h^2 f''(x) + 2\left[\frac{1}{4!}h^4 f^{(4)}(x) + \cdots\right]$$

hence, we have

$$f''(x) \approx \frac{1}{h^2}[f(x+h) - 2f(x) + f(x-h)]$$

this approximation is of $O(h^2)$ accuracy