Nested Form

for easy programming and efficient computation, we can write Newton form of the interpolating polynomial in **nested form**

$$\begin{aligned} p(x) &= a_0 + a_1[(x - x_0)] + a_2[(x - x_0)(x - x_1)] \\ &+ a_3[(x - x_0)(x - x_1)(x - x_2)] + \cdots \\ &+ a_n[(x - x_0)(x - x_1) \cdots (x - x_{n-1})] \end{aligned}$$

or, using standard product notations as

$$p(x) = a_0 + \sum_{i=1}^{n} a_i \left[\prod_{j=1}^{i-1} (x - x_j) \right]$$

using successive factorization, the nested form is

$$\begin{aligned} p(x) &= a_0 + (x - x_0)(a_1 + (x - x_1)(a_2 + \cdots \\ &\quad +(x - x_{n-1})a_n)) \cdots) \\ &= (\cdots((a_n(x - x_{n-1}) + a_{n-1})(x - x_{n-2}) \\ &\quad +a_{n-2}) \cdots)(x - x_0) + a_0 \end{aligned}$$

Computation Procedure

to evaluate $p(x)$ for a given x, we start from the innermost parentheses, forming successively some intermediate quantities

$$\begin{aligned}
v_0 &= a_n \\
v_1 &= v_0(x - x_{n-1}) + a_{n-1} \\
v_2 &= v_1(x - x_{n-2}) + a_{n-2} \\
&\vdots \\
v_i &= v_{i-1}(x - x_{n-i}) + a_{n-i} \\
&\vdots \\
v_n &= v_{n-1}(x - x_0) + a_0
\end{aligned}$$

a pseudocode is

real array $(a_i)_{0:n}$, $(x_i)_{0:n}$
integer i, n
real x, v
$v \leftarrow a_n$
for $i = n - 1$ **to** 0 **step** -1 **do**
  $v \leftarrow v(x - x_i) + a_i$
end for

Divided Difference

the coefficients a_i in Newton form of the interpolating need to be computed. A notation is introduced for facilitating such computation

$$a_i = f[x_0, x_1, \ldots, x_k]$$

which is called the **divided difference of order** k for f

Newton form interpolating polynomial is

$$\begin{aligned} p_n(x) &= a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) \\ &\quad + \cdots + a_n(x - x_0)\cdots(x - x_{n-1}) \end{aligned}$$

or written in a compact form

$$p_n(x) = \sum_{i=1}^{n} a_i \prod_{j=0}^{i-1}(x - x_j)$$

with the convention

$$\prod_{j=0}^{-1}(x - x_j) = 1$$

Computing Coefficients a_i

we want $p_n(x_i) = f(x_i)$. So we have

$$\begin{array}{rcl} f(x_0) & = & a_0 \\ f(x_1) & = & a_0 + a_1(x_1 - x_0) \\ f(x_2) & = & a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) \\ \cdots & & \cdots \end{array}$$

the solution of this system is

$$a_0 = f(x_0)$$

$$\begin{aligned} a_1 &= \frac{f(x_1) - a_0}{x_1 - x_0} \\ &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} \end{aligned}$$

the divided difference of order 1 is

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

note that $f[x_0, x_1, \ldots, x_k]$ is the coefficient of x^k in the polynomial p_k of degree $\leq k$

Computing Coefficients – Cont.

$$
\begin{aligned}
a_2 &= \frac{f(x_2) - a_0 - a_1(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)} \\
&= \frac{f(x_2) - f[x_0] - f[x_0, x_1](x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)} \\
&= f[x_0, x_1, x_2]
\end{aligned}
$$

in general, we have

$$
f[x_0, x_1, \ldots, x_k] = \frac{f(x_k) - \Sigma_{i=0}^{k-1} f[x_0, x_1, \ldots, x_i] \, \Pi_{j=0}^{i-1}(x_k - x_j)}{\Pi_{j=0}^{k-1}(x_k - x_j)}
$$

computational algorithm

- set $f[x_0] = f(x_0)$
- for $k = 1, 2, \ldots, n$, compute $f[x_0, x_1, \ldots, x_k]$ using the above equation

Recursive Formula

the divided difference has a recursive formula

$$f[x_0, x_1, \ldots, x_k] =$$

$$\frac{f[x_1, x_2, \ldots, x_k] - f[x_0, x_1, \ldots, x_{k-1}]}{x_k - x_0}$$

Proof:

$f[x_0, x_1, \ldots, x_k]$ is the coefficient of x^k in the polynomial p_k of degree $\leq k$, which interpolates f at $x_0, x_1, \ldots, x_k$

$f[x_1, x_2, \ldots, x_k]$ is the coefficient of x^{k-1} in the polynomial q_{k-1} of degree $\leq (k-1)$, which interpolates f at $x_1, x_2, \ldots, x_k$

$f[x_0, x_1, \ldots, x_{k-1}]$ is the coefficient of x^{k-1} in the polynomial p_{k-1} of degree $\leq (k-1)$, which interpolates f at $x_0, x_1, \ldots, x_{k-1}$

Recursive Formula - Proof

we have

$$p_k(x) = q_{k-1}(x) + \frac{x - x_k}{x_k - x_0}[q_{k-1}(x) - p_{k-1}(x)]$$

to prove this identity, it suffices to show that it holds at $(k + 1)$ different points, since the left-hand side and the right-hand side are polynomials of degree $\leq k$. Note that the left-hand side is $p_k(x_i) = f(x_i)$ for $i = 0, 1, \ldots, k$

check the right-hand side at point x_0

$$q_{k-1}(x_0) + \frac{x_0 - x_k}{x_k - x_0}[q_{k-1}(x_0) - p_{k-1}(x_0)]$$

$$= q_{k-1}(x_0) - [q_{k-1}(x_0) - p_{k-1}(x_0)]$$

$$= p_{k-1}(x_0) = f(x_0)$$

Formula Proof - Cont.

check for points $1 \leq i \leq (k-1)$,

$$q_{k-1}(x_i) + \frac{x_i - x_k}{x_k - x_0}[q_{k-1}(x_i) - p_{k-1}(x_i)]$$

$$= \; f(x_i) + \frac{x_i - x_k}{x_k - x_0}[f(x_i) - f(x_i)] = f(x_i)$$

check the right-hand side at point x_k

$$q_{k-1}(x_k) + \frac{x_k - x_k}{x_k - x_0}[q_{k-1}(x_k) - p_{k-1}(x_k)]$$
$$= q_{k-1}(x_k) = f(x_k)$$

hence the said identity holds

we take the coefficients of x^k on both sides, which yields the desired recursive formula

Invariance Theorem

the divided difference $f[x_0, x_1, \ldots, x_k]$ is invariant under all permutations of the arguments $x_0, x_1, \ldots, x_k$

this is because $f[x_0, x_1, \ldots, x_k]$ is the coefficient of x^k of the polynomial $p_k(x)$ of degree $\leq k$ that interpolates f at $x_0, x_1, \ldots, x_k$. $f[x_1, x_0, \ldots, x_k]$ is the coefficient of x^k of the polynomial $p_k(x)$ of degree $\leq k$ that interpolates f at $x_1, x_0, \ldots, x_k$. These two polynomials are the same

the generic recursive formula is

$$f[x_i, x_{i+1}, \ldots, x_{j-1}, x_j] =$$

$$\frac{f[x_{i+1}, x_{i+2}, \ldots, x_j] - f[x_i, x_{i+1}, \ldots, x_{j-1}]}{x_j - x_i}$$

Divided Difference Table

we can construct a divided difference table for f to facilitate computation of the coefficients of the interpolating polynomial

x	$f[\ ]$	$f[\ ,\ ]$	$f[\ ,\ ,\ ]$	$f[\ ,\ ,\ ,\ ]$
x_0	$f[x_0]$			
		$f[x_0, x_1]$		
x_1	$f[x_1]$		$f[x_0, x_1, x_2]$	
		$f[x_1, x_2]$		$f[x_0, x_1, x_2, x_3]$
x_2	$f[x_2]$		$f[x_1, x_2, x_3]$	
		$f[x_2, x_3]$		
x_3	$f[x_3]$			

the coefficients along the top diagonal are the ones needed to form the Newton form of the interpolating polynomial