CS321-002

Introduction to Numerical Methods

Lecture 3

Interpolation and Numerical Differentiation

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Polynomial Interpolation

given a set of discrete values, how can we estimate other values between these data

x	x_0	x_1	$\cdots$	x_n
y	y_0	y_1	$\cdots$	y_n

the method that we will use is called *polynomial interpolation*.

we assume the data we had are from the evaluation of a *smooth* function. We may be able to use a polynomial $p(x)$ to approximate this function, at least locally.

a condition: the polynomial $p(x)$ takes the given values at the given points (nodes), i.e., $p(x_i) = y_i$ with $0 \le i \le n$. The polynomial is said to *interpolate* the *table*, since we do not know the *function*.

Order of Interpolating Polynomial

a polynomial of degree 0, a constant function, interpolates one set of data

if we have two sets of data, we can have an interpolating polynomial of degree 1, a linear function

$$\begin{aligned} p(x) &= \left(\frac{x - x_1}{x_0 - x_1}\right) y_0 + \left(\frac{x - x_0}{x_1 - x_0}\right) y_1 \\ &= y_0 + \left(\frac{y_1 - y_0}{x_1 - x_0}\right) (x - x_0) \end{aligned}$$

review carefully if the *condition* is satisfied

interpolating polynomial can be written in several forms, the most well known ones are the Lagrange form and Newton form. Each has some advantages

Lagrange Form

for a set of fixed nodes $x_0, x_1, \ldots, x_n$, the *cardinal functions*, $l_0, l_1, \ldots, l_n$, are defined as

$$l_i(x_j) = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

we can interpolate any function $f(x)$ by the *Lagrange* form of the interpolating polynomial of degree $\leq n$

$$p_n(x) = \sum_{i=0}^{n} l_i(x)\, f(x_i)$$

note that $l_i(x)$ is of order n, but $p_n(x)$ is of order $\leq n$, and

$$p_n(x_j) = \sum_{i=1}^{n} l_i(x_j) f(x_i) = l_j(x_j) f(x_j) = f(x_j)$$

the (point exact) *condition* is satisfied

Cardinal Functions

the cardinal function is

$$l_i(x) = \prod_{j \neq i, j=0}^{n} \left(\frac{x - x_j}{x_i - x_j} \right) \qquad (0 \leq i \leq n)$$

what it looks like

$$\begin{aligned} l_i(x) \; = \; & \left(\frac{x - x_0}{x_i - x_0} \right) \left(\frac{x - x_1}{x_i - x_1} \right) \cdots \\ & \left(\frac{x - x_{i-1}}{x_i - x_{i-1}} \right) \left(\frac{x - x_{i+1}}{x_i - x_{i+1}} \right) \cdots \\ & \left(\frac{x - x_n}{x_i - x_n} \right) \end{aligned}$$

note that $l_i(x_j) = 0$, for $i \neq j$

and $l_i(x_i) = 1$

Step by Step Construction

for any table of data, we can construct a Lagrange interpolating polynomial. Its evaluation is a little bit costly, but we can always do that. The existence of the interpolating polynomial is guaranteed

can we construct the interpolating polynomial step by step, or if we discover some new data, can we add those data to make the interpolation more accurate?

we can use Newton form of the interpolating polynomial

let $p_k(x)$ be an interpolating polynomial for the date set $\{(x_i, y_i)\}$ with $0 \le i \le k$ such that $p_k(x_i) = y_i$

Newton Form - Cont.

We want to add another data (x_{k+1}, y_{k+1}) to have a new interpolating polynomial $p_{k+1}(x)$ such that $p_{k+1}(x_i) = y_i$ for $0 \leq i \leq (k+1)$. Let

$$p_{k+1}(x) = p_k(x) + c(x - x_0)(x - x_1)\cdots(x - x_k)$$

where c is an undetermined constant

since $p_{k+1}(x_{k+1}) = y_{k+1}$, we have

$$p_k(x_{k+1}) + c(x_{k+1} - x_0)(x_{k+1} - x_1) \cdots (x_{k+1} - x_k) = y_{k+1}$$

we can solve this equation for c, with the condition that $x_0, x_1, \ldots, x_{k+1}$ are all distinct

$$c = \frac{y_{k+1} - p_k(x_{k+1})}{(x_{k+1} - x_0)(x_{k+1} - x_1)\cdots(x_{k+1} - x_k)}$$

Uniqueness of Polynomial

is the interpolating polynomial unique?

if p and q are interpolating polynomials for the data set $\{(x_i, y_i)\}$ for $0 \leq i \leq n$ such that $p(x_i) = q(x_i) = y_i$

then the polynomial $r(x) = p(x) - q(x)$ of degree at most n is zero at $x_0, x_1, \ldots, x_n$. Note that a polynomial of degree n can have at most n roots, we must have $r(x) = 0$, or $p - q = 0$. Hence $p = q$

the interpolating polynomial is unique

it may be written in different forms

An Example

find the interpolating polynomial for this table

x	0	1	-1
y	-5	-3	-15

Lagrange form

$$
\begin{aligned}
l_0(x) &= \frac{(x-1)(x+1)}{(0-1)(0+1)} = -(x-1)(x+1) \\
l_1(x) &= \frac{(x-0)(x+1)}{(1-0)(1+1)} = \frac{1}{2}x(x+1) \\
l_2(x) &= \frac{(x-0)(x-1)}{(-1-0)(-1-1)} = \frac{1}{2}x(x-1)
\end{aligned}
$$

the interpolating polynomial is

$$p_2(x) = 5(x-1)(x+1) - \frac{3}{2}x(x+1) - \frac{15}{2}x(x-1)$$

Newton Form

the zeroth order polynomial is

$$p_0(x) = -5$$

let the 1st order interpolating polynomial be

$$p_1(x) = p_0 + c(x - x_0) = -5 + c(x - 0)$$

we want $p_1(x_1) = -3$, hence $-5 + c(1 - 0) = -3$, we have $c = 2$, it follows that

$$p_1(x) = -5 + 2x$$

let the 2nd order interpolating polynomial be

$$p_2(x) = p_1(x) + c(x - x_0)(x - x_1)$$

put $p_2(-1) = -15$, i.e., $-5 + 2(-1) + c(-1 - 0)(-1 - 1) = -15$. we have $c = -4$. The Newton form of the interpolating polynomial is

$$p_2(x) = -5 + 2x - 4x(x - 1)$$