Solutions of Homework 6: CS321-003, Spring 2006

Due Date: 1:50pm, April 26, 2006

Please show all steps in your work. Please be reminded that you should do your homework independently.

1. (10 points) Define $f(x) = 0$ if $x < 0$ and $f(x) = x^2$ if $x \geq 0$. Show that f and f' are continuous. Show that any quadratic spline with knots $t_0, t_1, \ldots, t_n$ is of the form

$$ax^2 + bx + c + \sum_{i=1}^{n-1} d_i f(x - t_i)$$

Solution. The function $f(x)$ is obviously continuous on $(-\infty, 0)$ and $[0, \infty)$. We only need to check its continuity at $x = 0$. For this, we verify that

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 0 = 0,$$
$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x^2 = 0.$$

Hence

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x),$$

and $f(x)$ is continuous at $x = 0$ and thus on $(-\infty, \infty)$.

The derivative of $f(x)$ on $(-\infty, \infty)$ is

$$f'(x) = \begin{cases} 0, & x < 0, \\ 2x, & x \geq 0. \end{cases}$$

It can be verified readily (similar to $f(x)$) that $f'(x)$ is continuous on $(-\infty, \infty)$.

By the definition of $f(x)$, we have, for $i = 0, 1, \ldots, n$,

$$f(x - t_i) = \begin{cases} 0, & x < t_i, \\ (x - t_i)^2, & x \geq t_i. \end{cases}$$

Thus, $f(x - t_i)$ is a quadratic function with respect to x, and $f(x - t_i)$ and $f'(x - t_i)$ are continuous on $(-\infty, \infty)$.

Define

$$q(x) = ax^2 + bx + c + \sum_{i=1}^{n-1} d_i f(x - t_i),$$

and

$$a = \min_i \{t_i\}, \qquad b = \max_i \{t_i\}.$$

It can be seen that $q(x)$ is a quadratic polynomial. Both $q(x)$ and $q'(x)$ are continuous on $[a, b]$.

For any quadratic spline, define $q(t_i) = y_i$, with just $(n+2)$ parameters $(a, b, c, d_1, d_2, \ldots, d_{n-1})$ to be fixed.

2. (10 points) Find a quadratic spline interpolant for these data

x	-1	0	$1/2$	1	2	$5/2$
y	2	1	0	1	2	3

Solution. Suppose the quadratic spline interpolant has the following form

$$Q(x) = \begin{cases} Q_0(x), & x \in [-1, 0] \\ Q_1(x), & x \in [0, 1/2] \\ Q_2(x), & x \in [1/2, 1] \\ Q_3(x), & x \in [1, 2] \\ Q_4(x), & x \in [2, 5/2] \end{cases}$$

Define $z_i = Q_i'(t_i)$, the following is the formula for Q_i

$$Q_i(x) = \frac{z_{i+1} - z_i}{2(t_{i+1} - t_i)}(x - t_i)^2 + z_i(x - t_i) + y_i.$$

It follows that

$$z_{i+1} = -z_i + 2\left(\frac{y_{i+1} - y_i}{t_{i+1} - t_i}\right), \qquad (0 \le i \le n-1).$$

By setting $z_0 = 0$, we can compute recursively,

$$\begin{aligned} z_1 &= -0 + 2\left(\frac{1-2}{0-(-1)}\right) = -2, \\ z_2 &= -(-2) + 2\left(\frac{0-1}{1/2-0}\right) = -2, \\ z_3 &= -(-2) + 2\left(\frac{1-0}{1-1/2}\right) = 6, \\ z_4 &= -6 + 2\left(\frac{2-1}{2-1}\right) = -4, \\ z_5 &= -(-4) + 2\left(\frac{3-2}{5/2-2}\right) = 8. \end{aligned}$$

Hence, the quadratic spline interpolant is

$$Q(x) = \begin{cases} Q_0(x) = -(x+1)^2 + 2, & x \in [-1, 0] \\ Q_1(x) = -2x + 1, & x \in [0, 1/2] \\ Q_2(x) = 8(x - 1/2)^2 - 2(x - 1/2), & x \in [1/2, 1] \\ Q_3(x) = -5(x-1)^2 + 6(x-1) + 1, & x \in [1, 2] \\ Q_4(x) = 12(x-2)^2 - 4(x-2) + 2, & x \in [2, 5/2] \end{cases}$$

3. (10 points) Determine if this function is a quadratic spline? Explain why or why not.

$$Q(x) = \begin{cases} x & -\infty < x \leq 1 \\ x^2 & 1 \leq x \leq 2 \\ 4 & 2 \leq x < \infty \end{cases}$$

Solution. $Q(x)$ is not a quadratic spline. The domain of definition is $(-\infty, \infty)$ which is not finite. Furthermore,

$$\lim_{x \to 1^-} Q'(x) = \lim_{x \to 1^-} 1 = 1,$$
$$\lim_{x \to 1+} Q'(x) = \lim_{x \to 1+} 2x = 2.$$

It follows that $Q'(x)$ is not continuous at $x = 1$, which violates the definition of a quadratic spline.

4. (10 points) Determine the parameters a, b, c, d and e so that S is a natural cubic spline

$$S(x) = \begin{cases} a + b(x-1) + c(x-1)^2 + d(x-1)^3 & x \in [0, 1] \\ (x-1)^3 + ex^2 - 1 & x \in [1, 2] \end{cases}$$

Solution. We first compute the derivatives of the individual functions

$$\begin{aligned} S_0'(x) &= b + 2c(x-1) + 3d(x-1)^2 \\ S_0''(x) &= 2c + 6d(x-1) \\ S_1'(x) &= 3(x-1)^2 + 2ex \\ S_1''(x) &= 6(x-1) + 2e \end{aligned}$$

We will make use of the continuity condition and the definition of the natural cubic spline.

From $S_0(1) = S_1(1)$, we have $a = c - 1$.

From $S_0'(1) = S_1'(1)$, we have $b = 2e$.

From $S_0''(1) = S_1''(1)$, we have $2c = 2e$, with $c = e$.

We also have

$$S_0''(0) = S''(2) = 0$$

$$2c - 6d = 0, \qquad 2e + 6 = 0,$$

we have $e = -3$, $c = -3$. Then $a = e - 1 = -4$, $b = 2e = -6$, and $d = c/3 = -1$.

5. (10 points) Determine the coefficients so that the function

$$S(x) = \begin{cases} x^2 + x^3 & 0 \leq x \leq 1 \\ a + bx + cx^2 + dx^3 & 1 \leq x \leq 2 \end{cases}$$

is a cubic spline and has the property $S_1'''(x) = 12$.

Solution. The derivatives are

$$\begin{aligned} S_0'(x) &= 2x + 3x^2 \\ S_0''(x) &= 2 + 6x \\ S_1'(x) &= b + 2cx + 3dx^2 \\ S_1''(x) &= 2c + 6dx \\ S_1'''(x) &= 6d \end{aligned}$$

Given the condition $S_1'''(x) = 12$, we have $d = 2$.

By continuity, we have $S_0'(1) = S_1'(1)$, which yields $5 = b + 2c + 3d$.

From $S_0''(1) = S_1''(1)$, we have $8 = 2c + 6d$.

We can solve the above two equations and get $c = -2$ and $b = 3$.

From $S_0(1) = S_1(1)$, we have $2 = a + b + c + d$, which gives us $a = -1$.