Homework 4: CS321-003, Spring 2006

Answer Sheet

1. Using Taylor series

$$f(x+h) = f(x) + h\,f'(x) + \frac{1}{2!}h^2 f''(x) + \frac{1}{3!}h^3 f'''(x) + \cdots \quad (1)$$

$$f(x+2h) = f(x) + 2h\,f'(x) + \frac{1}{2!}(2h)^2 f''(x) + \frac{1}{3!}(2h)^3 f'''(x) + \cdots \quad (2)$$

What we need here is to keep the $f'(x)$ term, so we need to get rid of the $f''(x)$ term. To that end, we multiply Eq. (1) by 4 and subtract Eq. (2) from it, we have

$$4f(x+h) - f(x+2h) = 3f(x) + 2h\,f'(x) + \frac{4}{3!}h^3 f'''(x) - \frac{1}{3!}(2h)^3 f'''(x) + \cdots.$$

Hence, the approximate formula is

$$f'(x) \approx \frac{1}{2h}[4f(x+h) - f(x+2h) - 3f(x)]$$

and the truncation error is $\frac{1}{3}h^2 f'''(\xi)$ for some ξ between x and $x+h$.

2. Using Taylor's theorem

$$f(x+2h) = f(x) + 2h\,f'(x) + \frac{1}{2!}(2h)^2 f''(x) + \frac{1}{3!}(2h)^3 f'''(\xi_1) \quad (3)$$

$$f(x-2h) = f(x) - 2h\,f'(x) + \frac{1}{2!}(2h)^2 f''(x) - \frac{1}{3!}(2h)^3 f'''(\xi_2) \quad (4)$$

for some number ξ_1 between x and $x+2h$, and ξ_2 between x and $x-2h$.

Subtracting Equation (4) from Equation (3), we have

$$f(x+2h) - f(x-2h) = 4h\,f'(x) + \frac{2}{3!}(2h)^3[f'''(\xi_1) + f'''(\xi_2)].$$

Hence, after dropping the f''' terms, we obtained our approximation formula as

$$f'(x) \approx \frac{f(x+2h) - f(x-2h)}{4h}.$$

The truncation error of this formula is

$$\frac{1}{3!}(2h)^2[f'''(x_1) + f'''(\xi_2)] = \frac{4h^2}{3}\,f'''(\xi_3)$$

where ξ_3 is a number between x_1 and x_2. It follows that the approximation formula is accurate of order $O(h^2)$.

3. Let the order of $O(h)$ approximation be

$$\phi(h) = L - c_1 h - c_2 h^2 - c_3 h^3 - \cdots \tag{5}$$

If we replace h by $h/2$ in (5), we have

$$\phi(h/2) = L - \frac{c_1}{2} h - \frac{c_2}{4} h^2 - \frac{c_3}{8} h^3 - \cdots \tag{6}$$

Now, multiplying (6) by 2 and subtracting it from (5), we have

$$\phi(h) - 2\phi(h/2) = -L - \frac{c_2}{2} h^2 - \frac{3c_3}{4} h^3 - \cdots$$

With an order of $O(h^2)$ approximation, we have

$$L \approx 2\phi(h/2) - \phi(h).$$

4. Let the partition be $[x_0, x_1, \ldots, x_i, \ldots, x_n]$ with $x_0 = a$ and $x_n = b$. Note that $h = (b-a)/n$. Since the function is decreasing, in each trapezoid $[x_i, x_{i+1}]$, we have $m = f(x_{i+1})$ and $M = f(x_i)$. Hence, the lower sums and the upper sums are, respectively,

$$\begin{aligned} L(f, P) &= \sum_{i=0}^{n-1} m_i (x_{i+1} - x_i) = \sum_{i=0}^{n-1} f(x_{i+1}) h \\ U(f, P) &= \sum_{i=0}^{n-1} M_i (x_{i+1} - x_i) = \sum_{i=0}^{n-1} f(x_i) h \end{aligned}$$

It follows that

$$U(f, P) - L(f, P) = \sum_{i=0}^{n-1} h(f(x_i) - f(x_{i+1})) = \frac{(b-a)}{n}[f(a) - f(b)]$$

5. Here $f(x) = \frac{1}{x^2}$ and $h = 1/2$. Note that the function is decreasing on $[1, 2]$.

For the lower sums

$$U(f) = h[f(1.5) + f(2.0)] = 0.5(\frac{1}{1.5^2} + \frac{1}{2^2}) \approx 0.3472$$

For the trapezoid rule, we have

$$T = h[f(1.5) + (f(1.0) + f(2.0))/2] = 0.5[\frac{1}{1.5^2} + (1 + \frac{1}{2^2})/2] \approx 0.5347$$