CS 633 3D Computer Animation

Solution Set - HW 3 (40 points)

Due: 2/8/08

1. If we want the user to have an impression that the object is moving in constant speed, we need to parametrize the path curve by arc length. For a B-spline curve, how should the re-parametrization process be done? Note that a B-spline curve usually has several segments and each segment is defined differently.

Sol.

Let $C(t)$ be a cubic B-spline curve with n segments $C_1(t)$, $C_2(t)$, ..., $C_n(t)$. To re-parametrize $C(t)$, we need to build an arc length table first. The arc length table is built as follows.

For $0 \le t \le 1$, the arc length is computed using the expression $C(t) = C_1(t)$.

If L_1 is the length of $C_1(t)$, then for $1 \le t \le 2$ the arc length is computed by first computing the arc length of $C(t) = C_2(t-1)$ and then adding this arc length to L_1.

In general, if L_1, L_2, ..., L_{k-1} are the lengths of the segments $C_1(t)$, $C_2(t)$, ... $C_{k-1}(t)$, then for $k-1 \le t \le k$ the arc length is computed by first computing the arc length of $C(t) = C_k(t-k+1)$ and then adding this arc length to $L_1 + L_2 + \cdots + L_{k-1}$.

2. Prove the third equation on page 104 of the textbook, i.e.,

$$s_3(t) = \frac{v_0 t_1}{2} + v_0(t_2 - t_1) + [v_0 - \frac{v_0(t-t_2)}{2(1-t_2)}](t - t_2)$$

when $t_2 < t < 1$. To prove the above equation, you need to show the following equation first.

$$v_3(t) = -\frac{v_0}{1-t_2}(t-t_2) + v_0$$

(10 points)

Sol.

First note that

$$s_3(t) = \int_{t_2}^{t} v_3(t)\, dt + s_2(t_2)\,, \qquad t_2 \le t \le 1$$

Since

$$s_2(t_2) = \frac{v_0 t_1}{2} + v_0(t_2 - t_1)$$

we have

$$s_3(t) = \int_{t_2}^{t} v_3(t)\ dt\ + \frac{v_0 t_1}{2} + v_0(t_2 - t_1).$$

The acceleration between t_2 and 1 is a constant $-b$, so the velocity between t_2 and 1 can be expressed as

$$v_3(t) = \int_{t_2}^{t} -b dt\ + v_2(t_2) = -\,b(t - t_2) + v_2(t_2)$$

The value of $v_2(t_2)$ is v_0 and the value of $v_3(t)$ at $t = 1$ equals 0. Therefore,

$$b = \frac{v_0}{1 - t_2}$$

and so

$$s_3(t) = \int_{t_2}^{t} [\,-\frac{v_0}{1 - t_2}(t - t_2) + v_0\,]\ dt + \frac{v_0 t_1}{2} + v_0(t_2 - t_1)$$

$$= -\frac{v_0(t - t_2)^2}{2(1 - t_2)} + v_0(t - t_2) + \frac{v_0 t_1}{2} + v_0(t_2 - t_1)$$

3. There are two approaches to define a *spherical linear interpolation* (see page 111 for the definition of this term) between two unit quaternions. In the second approach, $slerp(q_1,\ q_2;\ u)$ is defined as follows:

$$slerp(q_1,\ q_2;\ u) = \frac{\sin((1-u)\theta)}{\sin\theta}\ q_1 + \frac{\sin(u\theta)}{\sin\theta}\ q_2$$

Prove that when $u = 1/2$ the above definition indeed gives the mid-point of the arc between q_1 and q_2. (10 points)

Sol.

To show that when $u = 1/2$ the above definition indeed gives the mid-point of the arc between q_1 and q_2, we have to prove that

$$slerp(q_1,\ q_2;\ 1/2) = \frac{q_1 + q_2}{|q_1 + q_2|}\ . \qquad (*)$$

When $u = 1/2$, we have

$$slerp(q_1, q_2; 1/2) = \frac{\sin(\theta/2)}{\sin\theta}\, q_1 + \frac{\sin(\theta/2)}{\sin\theta}\, q_2$$

$$= \frac{\sin(\theta/2)}{\sin\theta}\,(q_1 + q_2)$$

$$= \frac{1}{2\cos(\theta/2)}\,(q_1 + q_2)$$

Let $q_1 = [w_1, (x_1, y_1, z_1)]$ and $q_2 = [w_2, (x_2, y_2, z_2)]$. Since q_1 and q_2 are unit quaternions and $q_1 \cdot q_2 = \cos\theta$, we have

$$|q_1 + q_2| = \sqrt{(w_1 + w_2)^2 + (x_1 + x_2)^2 + (y_1 + y_2)^2 + (z_1 + z_2)^2}$$

$$= \sqrt{w_1^2 + w_2^2 + x_1^2 + x_2^2 + y_1^2 + y_2^2 + z_1^2 + z_2^2 + 2w_1w_2 + 2x_1x_2 + 2y_1y_2 + 2z_1z_2}$$

$$= \sqrt{2\cos\theta + 2}$$

$$= \sqrt{2(2\cos^2(\theta/2) - 1) + 2}$$

$$= 2\cos(\theta/2)$$

Hence, (*) is proved.

4. Prove the second approach generates the same curve as the first approach. (10 points)

Sol.

We will show that, for each $0 \le u \le 1$, we have

$$q_1(q_1^{-1}q_2)^u = \frac{\sin((1-u)\theta)}{\sin\theta}\, q_1 + \frac{\sin(u\theta)}{\sin\theta}\, q_2 \qquad (**)$$

where $\cos\theta = q_1 \cdot q_2$.

Let $q_1 = [\cos\alpha, v\sin\alpha]$, $q_2 = [\cos\beta, v\sin\beta]$, where v is a unit vector in the direction of the rotation axis.

$$q_1(q_1^{-1}q_2)^u = q_1([\cos\alpha, -v\sin\alpha][\cos\beta, v\sin\beta])^u$$

$$= q_1([\cos\alpha\,\cos\beta + \sin\alpha\,\sin\beta,\ v\cos\alpha\,\sin\beta - v\sin\alpha\,\cos\beta])^u$$

$$= q_1([\cos(\beta - \alpha),\ v\,\sin(\beta - \alpha)]^u$$

By setting $\theta \equiv \beta - \alpha$, we have

$$q_1(q_1^{-1}q_2)^u = [\cos\alpha, v\sin\alpha][cos\theta, v\sin\theta]^u$$

$$= [\cos\alpha,\ v\sin\alpha][cos(u\theta),\ v\sin(u\theta)]$$

$$= [\cos\alpha\ \cos(u\theta) - \sin\alpha\ \sin(u\theta),\ v\cos\alpha\ \sin(u\theta) + v\sin\alpha\ \cos(u\theta)]$$

$$= [cos(\alpha + u\theta),\ v\sin(\alpha + u\theta)]$$

We claim that

$$\cos(\alpha + u\theta) = \frac{\cos\alpha\ \sin((1-u)\theta) + \cos\beta\ \sin(u\theta)}{\sin\theta} \qquad (1)$$

$$\sin(\alpha + u\theta) = \frac{\sin((1-u)\theta)\ \sin\alpha + \sin(u\theta)\ \sin\beta}{\sin\theta} \qquad (2)$$

To prove (1), note that

$$\sin\alpha\ \sin\beta = \cos\theta - \cos\alpha\ \cos\beta.$$

Hence,

$$\cos(\alpha + u\theta) = \cos\alpha\ \cos(u\theta) - \sin\alpha\ \sin(u\theta)$$

$$= \cos\alpha\ \cos(u\theta) - \sin\alpha\ \sin(u\theta)\ \frac{\sin(\beta-\alpha)}{\sin\theta}$$

$$= \cos\alpha\ \cos(u\theta) - \sin\alpha\ \sin(u\theta)\left(\frac{\cos\alpha\ \sin\beta - \sin\alpha\ \cos\beta}{\sin\theta}\right)$$

$$= \cos\alpha\ \cos(u\theta) - \frac{\sin\alpha\ \sin\beta\ \cos\alpha\ \sin(u\theta) - \sin^2\alpha\ \cos\beta\ \sin(u\theta)}{\sin\theta}$$

$$= \cos\alpha\ \cos(u\theta) - \frac{(cos\theta - \cos\alpha\ \cos\beta)\cos\alpha\ \sin(u\theta) - \sin^2\alpha\ \cos\beta\ \sin(u\theta)}{\sin\theta}$$

$$= \frac{\cos\alpha\ \cos(u\theta)\ \sin\theta - \cos\theta\ \cos\alpha\ \sin(u\theta) + \cos^2\alpha\ \cos\beta\ \sin(u\theta) + \sin^2\alpha\ \cos\beta\ \sin(u\theta)}{\sin\theta}$$

$$= \frac{\cos\alpha\ (\sin\theta\ \cos(u\theta) - \cos\theta\ \sin(u\theta)) + \cos\beta\ \sin(u\theta)}{\sin\theta}$$

$$= \frac{\cos\alpha\ \sin((1-u)\theta)) + \cos\beta\ \sin(u\theta)}{\sin\theta}.$$

To prove (2), recall that $\theta = \beta - \alpha$ and $\cos\theta = \cos\alpha\ \cos\beta + \sin\alpha\ \sin\beta$. Hence

$$\sin(\alpha + u\theta) = \frac{\sin\theta}{\sin\theta}\,(\sin\alpha\ \cos(u\theta) + \cos\alpha\ \sin(u\theta))$$

$$= \frac{1}{\sin\theta}\,(\sin\theta\ \sin\alpha\cos(u\theta) + \sin(\beta-\alpha)\ \cos\alpha\ \sin(u\theta))$$

$$= \frac{1}{\sin\theta}\left(\sin\theta\,\sin\alpha\cos(u\theta) + \sin\beta\,\cos^2\alpha\,\sin(u\theta) - \sin\alpha\,\cos\alpha\,\cos\beta\,\sin(u\theta)\right)$$

$$= \frac{1}{\sin\theta}\,(\sin\beta\,\sin(u\theta) + \sin\theta\,\sin\alpha\cos(u\theta) - \sin\beta\,\sin(u\theta) + \sin\beta\,\cos^2\alpha\,\sin(u\theta)$$

$$- \sin\alpha\,\cos\alpha\,\cos\beta\,\sin(u\theta))$$

$$= \frac{1}{\sin\theta}\,(\sin\beta\,\sin(u\theta) + \sin\theta\,\sin\alpha\cos(u\theta) - \sin^2\alpha\,\sin\beta\,\sin(u\theta)$$

$$- \sin\alpha\,\cos\alpha\,\cos\beta\,\sin(u\theta))$$

$$= \frac{1}{\sin\theta}\,(\sin\beta\,\sin(u\theta) + \sin\theta\,\sin\alpha\cos(u\theta) - \sin\alpha\,\cos\theta\,\sin(u\theta))$$

$$= \frac{1}{\sin\theta}\,(\sin\beta\,\sin(u\theta) + \sin\alpha\,(\sin\theta\,\cos(u\theta) - \cos\theta\,\sin(u\theta)))$$

$$= \frac{1}{\sin\theta}\,(\sin\beta\,\sin(u\theta) + \sin\alpha\,\sin((1-u)\theta))$$

Therefore, (**) is proved.