5.3.3 Numerical Solution to IK (conti)

Def: A is an $m \times n$ matrix. A^+ is called the *pseudo inverse* of A if A^+ satisfies the following four conditions:

1. $A\, A^+\, A = A$
2. $A^+\, A\, A^+ = A^+$
3. $(A\, A^+)^T = A\, A^+$
4. $(A^+\, A)^T = A^+\, A$

The *pseudo inverse* always exists and is unique (Ben-Israel and Greville, *Generalized Inverses: Theory and Applications*, Wiley-Interscience, New York, 1974)

5.3.3 Numerical Solution to IK (conti)

Theorem: A is an $m \times n$ matrix. A^+ is the pseudo inverse of A. If A is of full rank then

$$A^+ = \begin{cases} A^T (A A^T)^{-1}, & m < n \\ A^{-1}, & m = n \\ (A^T A)^{-1} A^T, & m > n \end{cases}$$

Proof Show A^+ defined the above way satisfies the four properties of the above definition.

5.3.3 Numerical Solution to IK (conti)

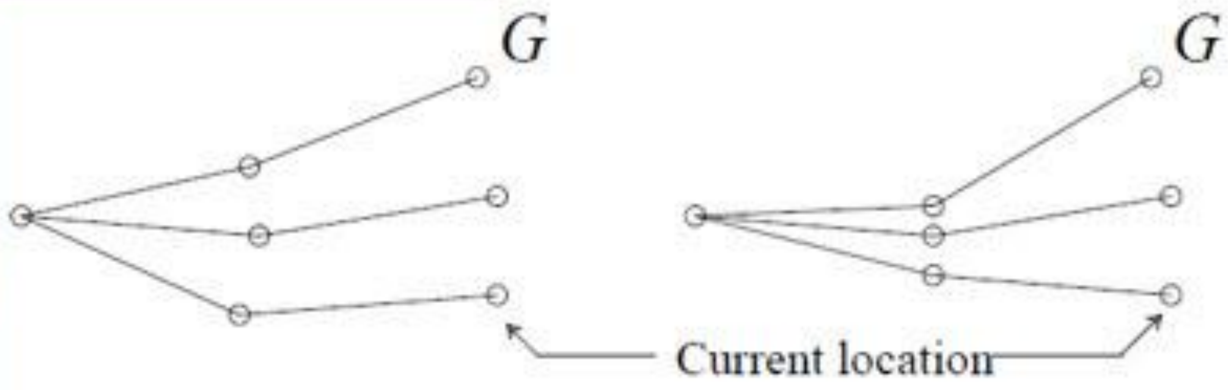
Example:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 0 \end{bmatrix}$$

$$A^+ = A^T\,(A\,A^T)^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix}^{-1}$$

$$= \frac{1}{9} \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 5 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 1 & 4 \\ 1 & -5 \\ 4 & -2 \end{bmatrix}$$

Now, consider the following two cases:



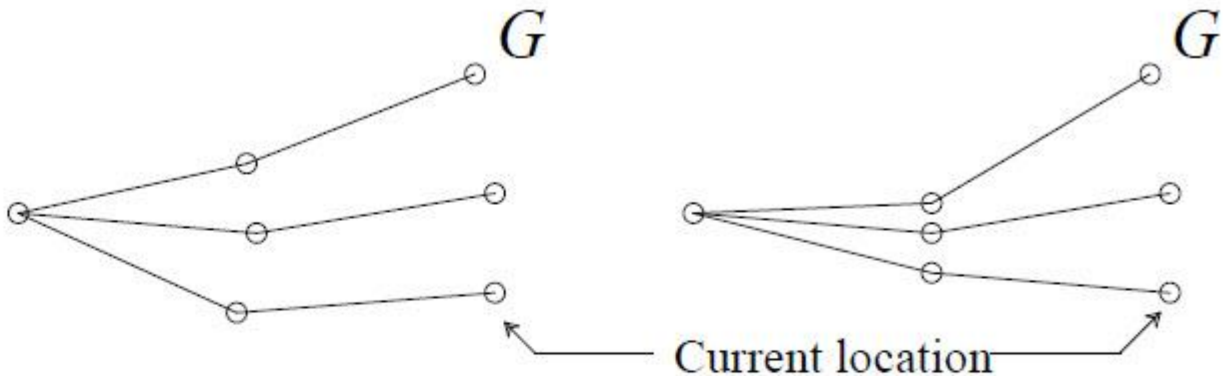
Same current position, same desired position.
But the one on the right is more natural

Add a control expression to the pseudo inverse Jacobian solution to better control the kinematic model:

$$\dot{\theta} = J^{+} V \qquad \textit{original solution}$$

$$\dot{\theta} = J^{+}V + (J^{+}J - I) \cdot Z$$

Now, consider the following two cases:



$$\dot{\theta} = J^{+} V \qquad \textit{original solution}$$

$$\dot{\theta} = J^{+}V + (J^{+}J - I) \cdot Z$$

How?

First, adding the control expression does not change the velocities of the end effector

$$\dot{\theta} = J^{+} V \qquad \textit{original solution}$$

$$\dot{\theta} = J^{+}V + (J^{+}J - I) \cdot Z$$

How?

First, adding the control expression does not change the velocities of the end effector

Let

$$\dot{\theta} = (J^{+} J - I) Z$$

for some Z, then consider

$$
\begin{aligned}
V &= J \cdot \dot{\theta} \\
&= J(J^{+}J - I) Z \\
&= J J^{+} J Z - J I Z \\
&= J Z - J Z \\
&= 0
\end{aligned}
$$

Therefore, if $\dot{\theta} \equiv J^{+}V + (J^{+}J - I)Z$, we would still get the same V.

How to bias the solution toward specific joint angles?

Define

$$H = \sum_{i=1}^{n} \alpha_i \, (\theta_i - \theta_{ci})^k$$

Why?

θ_i : *current joint angles*

θ_{ci} : *desired joint angles*

α_i : *desired angle gains*

indicates the relative importance of the associated desired angle

The higher the gain, the stiffer the joint

k : *kth norm(keven)*

usually k=2

$$Z = \nabla_\theta H = \frac{dH}{d\theta} = (k\alpha_1(\theta_1 - \theta_{c1})^{k-1}, \ldots, k\alpha_n(\theta_n - \theta_{cn})^{k-1})$$

How to bias the solution toward specific joint angles?

Now consider

$$\dot{\theta} = J^{+} V + (J^{+}J - I)\nabla_{\theta}H$$

How to find $\dot{\theta}$? Note

$$\dot{\theta} = J^{+}(V + J\nabla_{\theta}H) - \nabla_{\theta}H$$

$$= J^{T}(JJ^{T})^{-1}(V + J\nabla_{\theta}H) - \nabla_{\theta}H$$

$$= J^{T} \beta - \nabla_{\theta}H$$

$\dot{\theta} = J^T \beta - \nabla_\theta H$

$$\beta = \left(JJ^T\right)^{-1}\left(V + J\,\nabla_\theta\, H\right)$$

First, solve

$$V + J \cdot \nabla_\theta H = (J\,J^T)\,\beta$$

for β.

Then compute

$$\dot{\theta} = J^T\,\beta - \nabla_\theta H$$

End of Kinematic III