5.3.2 The Jacobian

- Iterative numeric solution
- Construct the motion incrementally

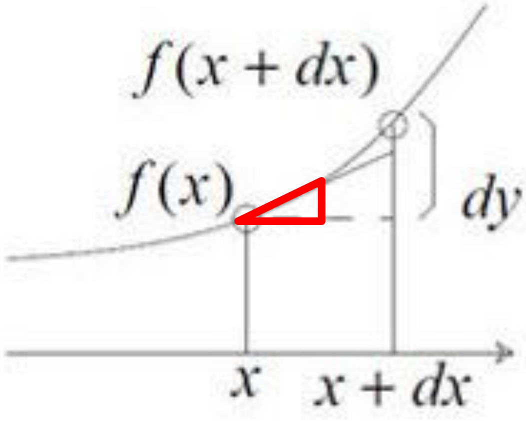
Solving a system by analysis is difficult

To use incremental method, we need to use Jacobian. Why?

What is a Jacobian?

Consider $y = f(x)$

$$f' \sim \frac{dy}{dx} \quad \text{or} \quad dy \sim f'dx$$



Hence, *f (x + dx) can be approximated by*

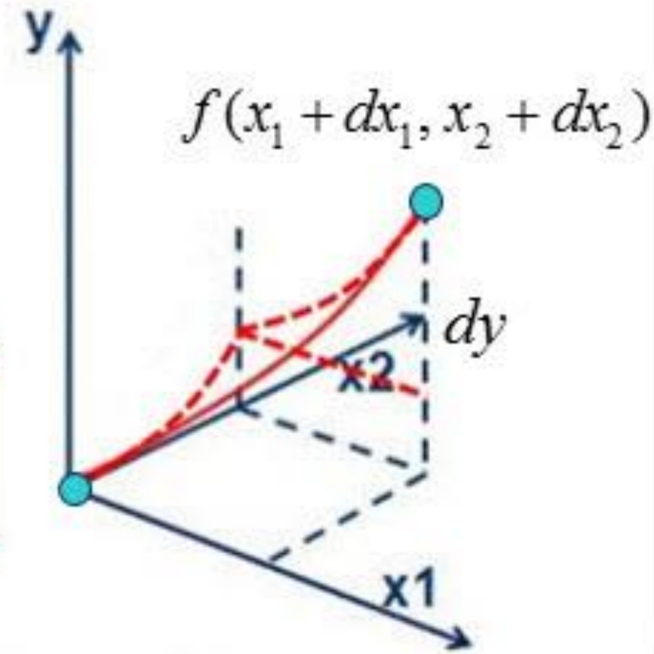
$$f(x + dx) = f(x) + dy$$

$$\sim f(x) + f'dx$$

Works only when dx is small

In the 2-variable case,

$$y = f(x_1, x_2),$$



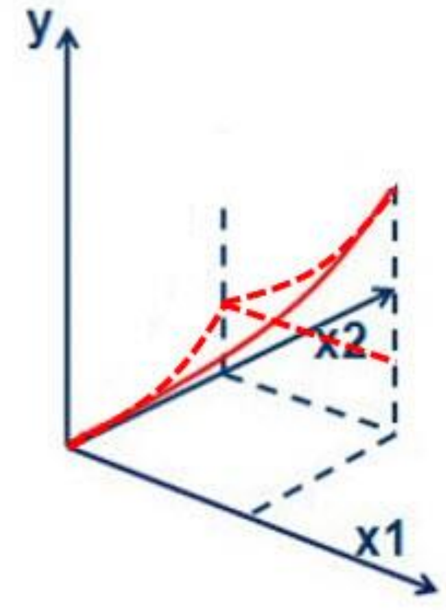
$$dy \sim \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 \qquad (*)$$

$f(x_1 + dx_1, x_2 + dx_2)$ can be approximated by

$$f(x_1 + dx_1, x_2 + dx_2) = f(x_1, x_2) + dy$$

$$\sim f(x_1, x_2) + [\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}] \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix}$$

Why is (*) true?



Let $x_2 + dx_2$ be fixed, then

$g(x) \equiv f(x, x_2 + dx_2)$ can be considered

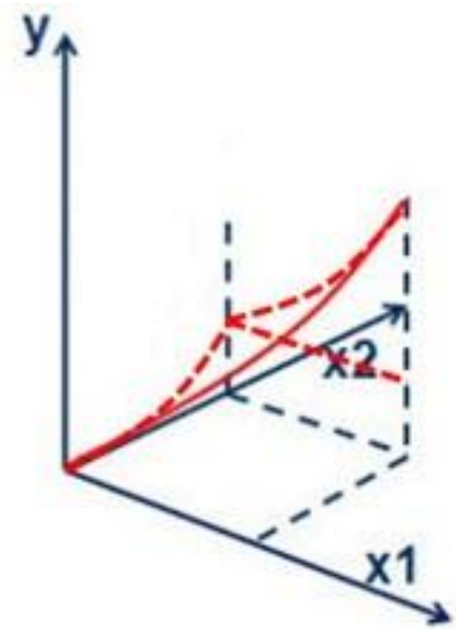
as a function of one variable x, therefore

$$g(x_1 + dx_1) \approx g(x_1) + g'(x_1)\, dx_1$$

$$= f(x_1, x_2 + dx_2) + \frac{\partial f(x_1, x_2 + dx_2)}{\partial x_1} dx_1$$

$$\approx f(x_1, x_2) + \frac{\partial f(x_1, x_2)}{\partial x_2} dx_2 + \frac{\partial f(x_1, x_2)}{\partial x_1} dx_1$$

If $f(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2))$ then



$$dy = (dy_1, dy_2) \sim \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix}$$

$f(x_1 + dx_1, x_2 + dx_2)$ can be approximated by

$$f(x_1 + dx_1, x_2 + dx_2) = f(x_1, x_2) + dy$$

$$\sim f(x_1, x_2) + \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix}$$

Now Consider:

$$F : (x_1, x_2, x_3, x_4) \rightarrow (y_1, y_2, y_3, y_4)$$

$$\begin{cases} y_1 = f_1(x_1, x_2, x_3, x_4) \\ y_2 = f_2(x_1, x_2, x_3, x_4) \\ y_3 = f_3(x_1, x_2, x_3, x_4) \\ y_4 = f_4(x_1, x_2, x_3, x_4) \end{cases}$$

$$dy_i \sim [\frac{\partial f_i}{\partial x_1}, \frac{\partial f_i}{\partial x_2}, \frac{\partial f_i}{\partial x_3}, \frac{\partial f_i}{\partial x_4}] \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \\ dx_4 \end{bmatrix}$$

$$dy = \begin{bmatrix} dy_1 \\ dy_2 \\ dy_3 \\ dy_4 \end{bmatrix}^T \sim \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_4} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_2}{\partial x_4} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} & \frac{\partial f_3}{\partial x_4} \\ \frac{\partial f_4}{\partial x_1} & \frac{\partial f_4}{\partial x_2} & \frac{\partial f_4}{\partial x_3} & \frac{\partial f_4}{\partial x_4} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \\ dx_4 \end{bmatrix}$$

$$dY = \frac{\partial F}{\partial X} \cdot dX$$

$$\frac{\partial F}{\partial X} \text{ is called a } \textit{Jacobian}$$

The Jacobian maps the velocities of *X* to the velocities of *Y*

$$\dot{Y} = J(X) \cdot \dot{X}$$

So a Jacobian is a *linear transformation* (like a rotation)

In our case, x_i are the joint angles and y_i are position and orientation of the end effector:

$$V = J(\theta) \cdot \dot{\theta}$$

with

$$V = [v_x, v_y, v_z, \omega_x, \omega_y, \omega_z]$$

$$\dot{\theta} = [\dot{\theta}_1, \dot{\theta}_2, \cdots, \dot{\theta}_n]$$

$$V = J(\theta) \cdot \dot{\theta}$$

$$J(\theta) = \begin{bmatrix} \frac{\partial V_x}{\partial \theta_1} & \frac{\partial V_x}{\partial \theta_2} & \cdot & \cdot & \frac{\partial V_x}{\partial \theta_n} \\ \frac{\partial V_y}{\partial \theta_1} & \frac{\partial V_y}{\partial \theta_2} & \cdot & \cdot & \frac{\partial V_y}{\partial \theta_n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial \Omega_z}{\partial \theta_1} & \frac{\partial \Omega_z}{\partial \theta_2} & \cdot & \cdot & \frac{\partial \Omega_z}{\partial \theta_n} \end{bmatrix}$$

$$V = J(\theta) \cdot \dot{\theta}$$

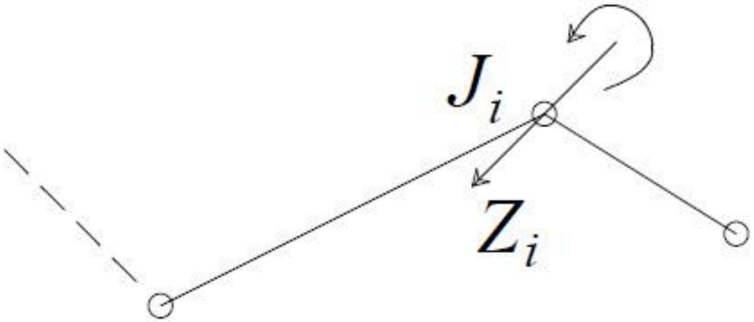
V : vector of linear and rotational velocities
represents the desired change in the end effector
e.g. $v_x = (G - E)_x$

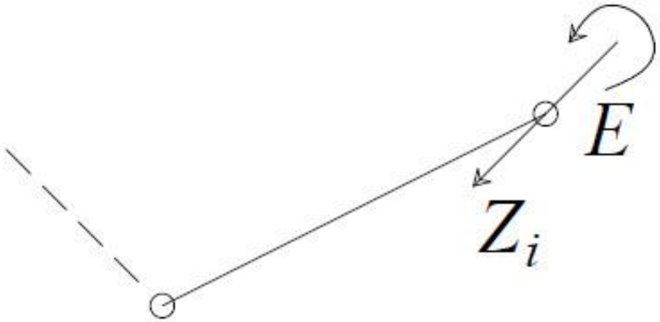
$\dot{\theta}$: vector of joint angle velocities
unknowns of the equation

$J(\theta)$: each term of $J(\theta)$ relates the change of a specific joint to a specific change in the end effector

Rotational change:

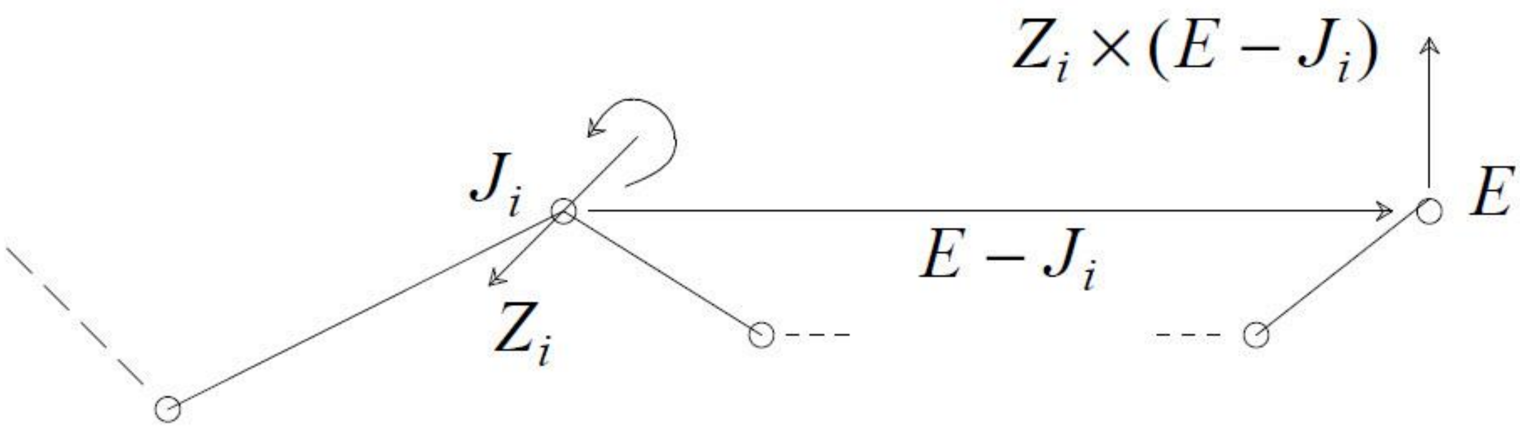
- merely the velocity of the joint angle about the axis of revolution at the ioint under consideration



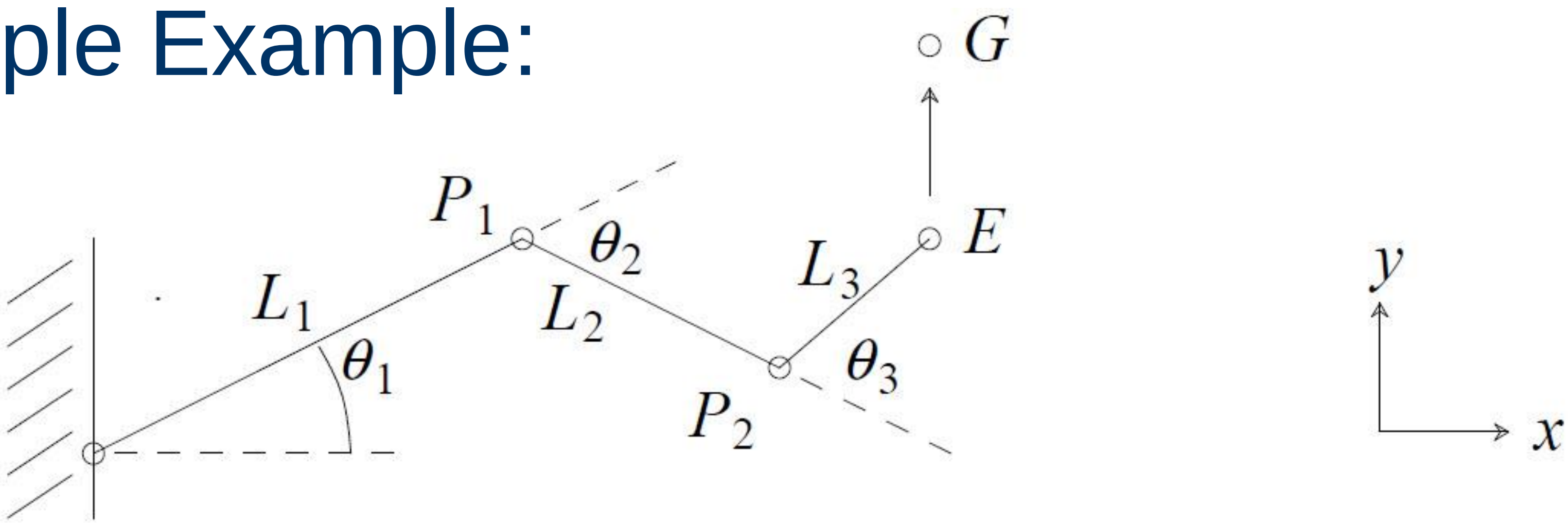


Linear change (in the end effector) :

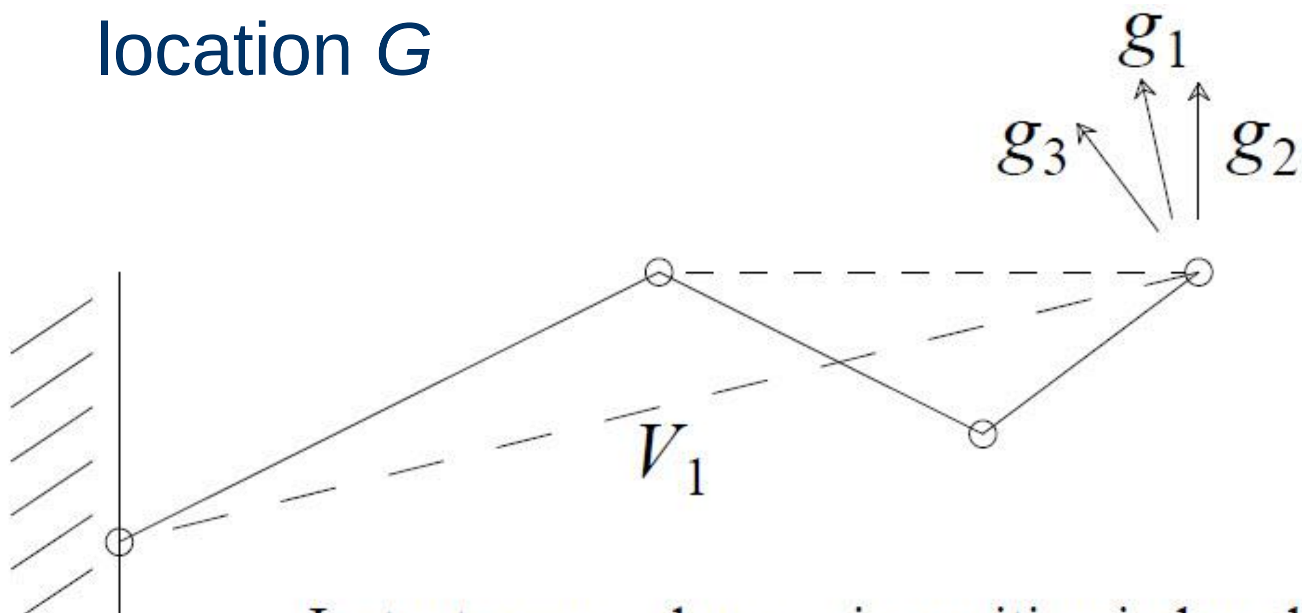
- the cross product of the axis of revolution and the vector from the joint to the end effector



A Simple Example:



Objective: move the end effector *E* to the global location *G*



Orientation of *E* is of no concern

Instantaneous changes in position induced by joint angle rotations

Therefore, the corresponding equation is:

$$\begin{bmatrix} (G-E)_x \\ (G-E)_y \\ (G-E)_x \end{bmatrix} =$$

$$\begin{bmatrix} (Z \times E)_x & (Z \times (E-P_1))_x & (Z \times (E-P_2))_x \\ (Z \times E)_y & (Z \times (E-P_1))_y & (Z \times (E-P_2))_y \\ (Z \times E)_z & (Z \times (E-P_1))_z & (Z \times (E-P_2))_z \end{bmatrix} \cdot \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

where Z=(0, 0, 1)

Or, simply.

$$V = J \cdot \dot{\theta}$$

5.3.3 Numerical Solutions to IK

How to solve $V = J \cdot \dot{\theta}$ for $\dot{\theta}$?

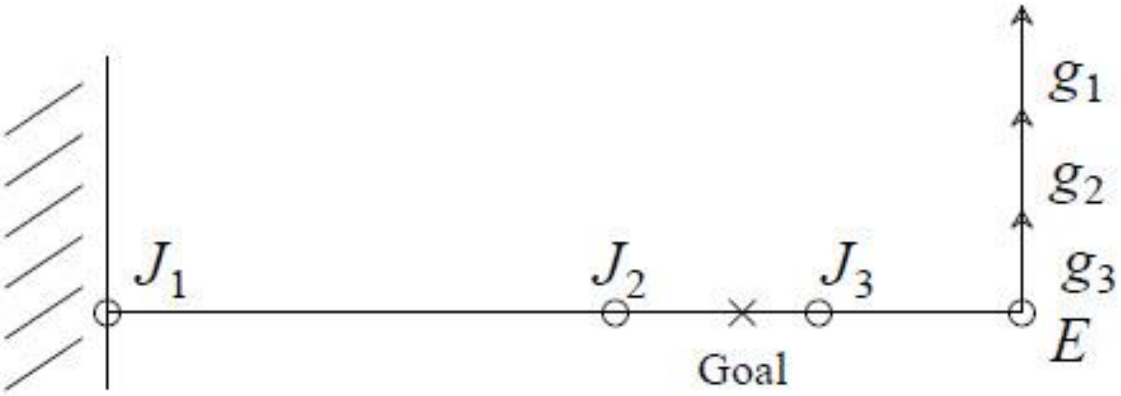
(1)
Non-singular (J is a square matrix)

$$\dot{\theta} = J^{-1} \cdot V$$

seldom to happen!!!

(2)
Singular (J^{-1} does not exist)

.



(3)

If the manipulator is *redundant*

($dim(J(\theta)) = m \times n, \quad m < n$

$J(\theta)$ is of full rank)

Since $rank(J) = m$ and the $m \times m$ matrix (JJ^T) is invertible, we can define

$$\beta \equiv (JJ^T)^{-1} V$$

So,

$$(JJ^T)\ \beta = V$$

$$\rightarrow \quad JJ^T \cdot \beta = J \cdot \dot{\theta}$$

$$\rightarrow \quad J(J^T \cdot \beta - \dot{\theta}\ ) = 0$$

$$\rightarrow \quad J^T\ \beta = \dot{\theta}$$

$$\rightarrow \quad J^T(JJ^T)^{-1}\ V = \dot{\theta}$$

.

$$J^+ \equiv J^T(JJ^T)^{-1}$$

is called the *pseudo inverse* of J

In real-life implementation, $\dot{\theta}$ should be computed as follows:

First, use an efficient method such as LU decomposition to solve the following equation for β:

$$(J^T J)\beta = V$$

Then substitute β into the following equation to solve for $\dot{\theta}$:

$$J^T \beta = \dot{\theta}$$

Then we use Euler method to update the joint angles. The Jacobian has changed at the next time step, so the computation must be performed again and another step taken. This process repeats until the end effector reaches the goal configuration within some acceptable tolerance.

$\dot{\theta}$ can also be computed as

$$\dot{\theta} = \left(J^T J\right)^{-1} J^T \mathrm{V} \qquad \text{Why?}$$

One possible way to find a solution to an under-determined system like the following one

$$\mathrm{M}\,\mathbf{X} = \mathbf{Y} \qquad (*)$$

(M is an $m \times n$ matrix with $n > m$, **X** is an unknown vector of dimension n and **Y** is a constant vector of dimension m) is to solve the following system for **X**.

$$(\mathrm{M}^{\mathrm{T}}\mathrm{M})\,\mathbf{X} = \mathrm{M}^{\mathrm{T}}\mathbf{Y} \qquad (**)$$

Note that if we define $F(\mathbf{X})$ as follows:

$$F(\mathbf{X}) \equiv (M\mathbf{X} - \mathbf{Y})^T (M\mathbf{X} - \mathbf{Y}),$$

we get a non-negative function whose minimum occurs at a point **X** where Eq. (*) is satisfied (why?). Hence, to find a solution for (*), we simply compute the derivative of $F(\mathbf{X})$ with respect to **X**, set it to zero, and solve for **X**. Note that

$$F(\mathbf{X}) = (\mathbf{X}^T M^T - \mathbf{Y}^T)(M\mathbf{X} - \mathbf{Y})$$

$$= \mathbf{X}^T M^T M\mathbf{X} - \mathbf{X}^T M^T \mathbf{Y} - \mathbf{Y}^T M\mathbf{X} + \mathbf{Y}^T \mathbf{Y}.$$

Since $X^T M^T Y = Y^T MX$, we have

$$F(\mathbf{X}) = \mathbf{X}^T M^T M\mathbf{X} - 2\mathbf{X}^T M^T \mathbf{Y} + \mathbf{Y}^T \mathbf{Y}$$

Since $X^T M^T Y = Y^T M X$, we have

$$F(\mathbf{X}) = \mathbf{X}^T M^T M\mathbf{X} - 2\mathbf{X}^T M^T \mathbf{Y} + \mathbf{Y}^T \mathbf{Y}$$

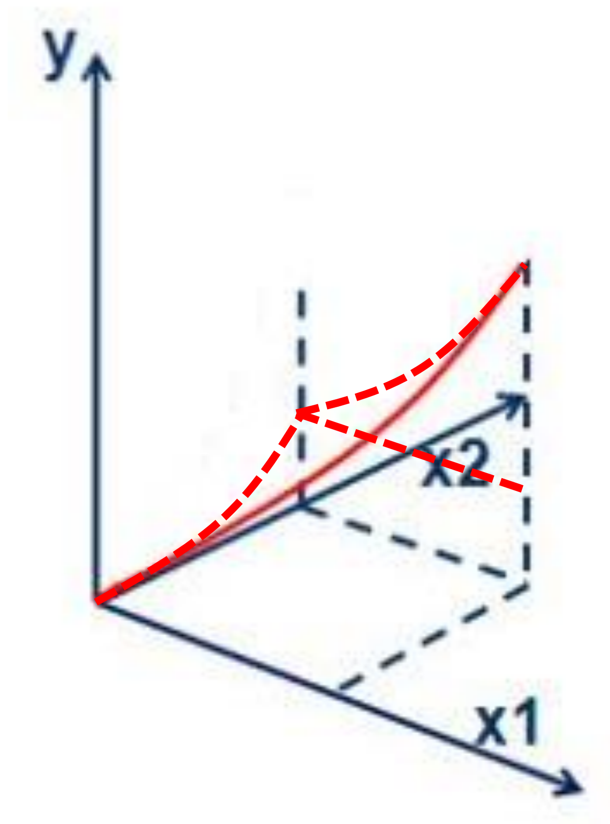
By differentiating *F*(**X**) with respect to **X** and setting it to zero,

$$\frac{dF(\mathbf{X})}{d\mathbf{X}} = 2M^T M\mathbf{X} - 2M^T \mathbf{Y} = 0$$

we get (**). Hence, solving (*) is equivalent to solving (**) for **X**.

From this one can see now why $\dot{\theta}$ can also be computed as $\dot{\theta} = \left(J^T J\right)^{-1} J^T V$

End of Kinematic II


y
x2
x1