CS 633 Computer Animation

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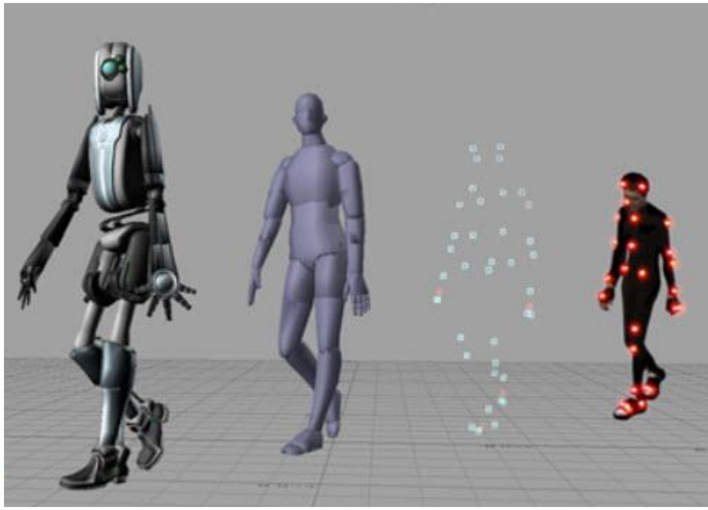
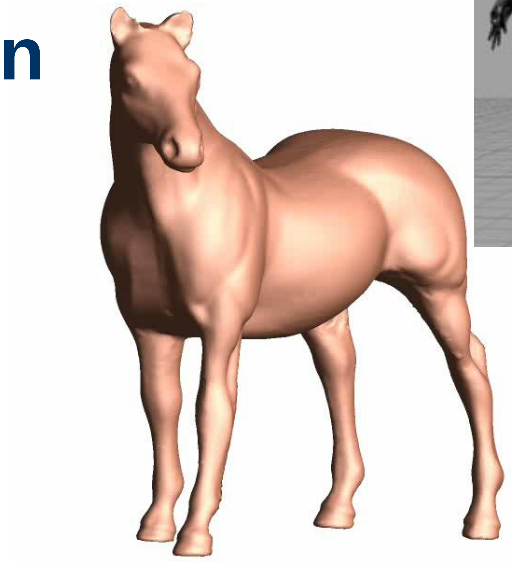
1. Introduction

1.1 What is Computer Animation

- **the process of using "continuous image" to convey information**

- **deals with motion**





horse_sph_morph_loop.avi

1. Introduction

1.2 Applications

- **motion films**

  (https://www.youtube.com/watch?v

- **television**

- **advertising**

  (read Chapter 1)

2. Technical Background

- **The display pipeline:**

  Object space -

  World space -

  Eye space –

  viewing parameters

  field of view

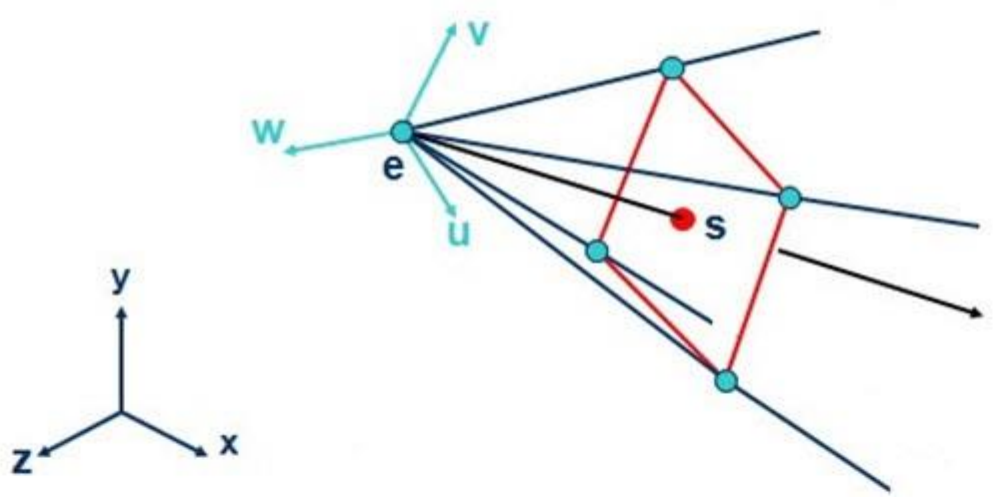
  Image space -

  Screen space -

2. Technical Background

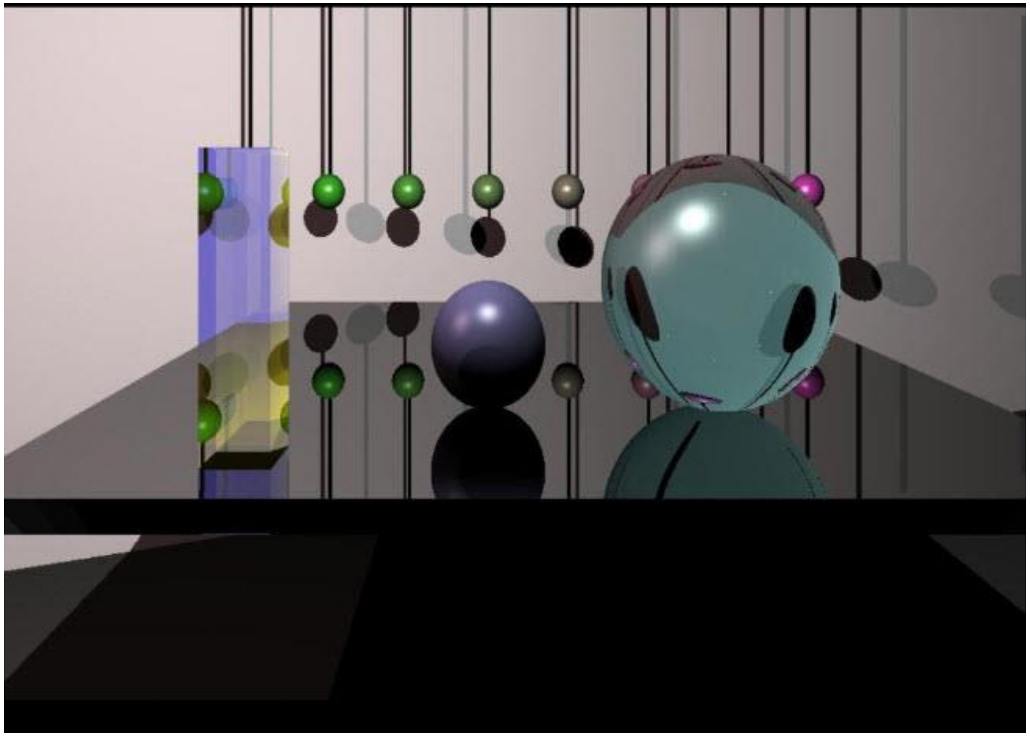
- **Ray Casting:**

  - act of tracing rays through world space



2. Technical Background

- **Ray Casting:**
  - implicitly accomplishes the perspective transformation (HOW ?)



2. Technical Background

- **Homogeneous Coordinates:**

$$\left(\frac{x}{w}, \frac{y}{w}, \frac{z}{w} \right) = \left[x, y, z, w \right]$$

- why do we want to use homogeneous coordinates ?

2. Technical Background

- **Transformation Matrices:**

$$4 \times 4$$

$$\begin{bmatrix} & M & \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

2. Technical Background

- **Compounding Transformations:**

after

before

$$P' = M_1 * M_2 * M_3 * P$$

$$M = M_1 * M_2 * M_3$$

$$P' = M * P$$

2. Technical Background

- **Basic Transformations:**

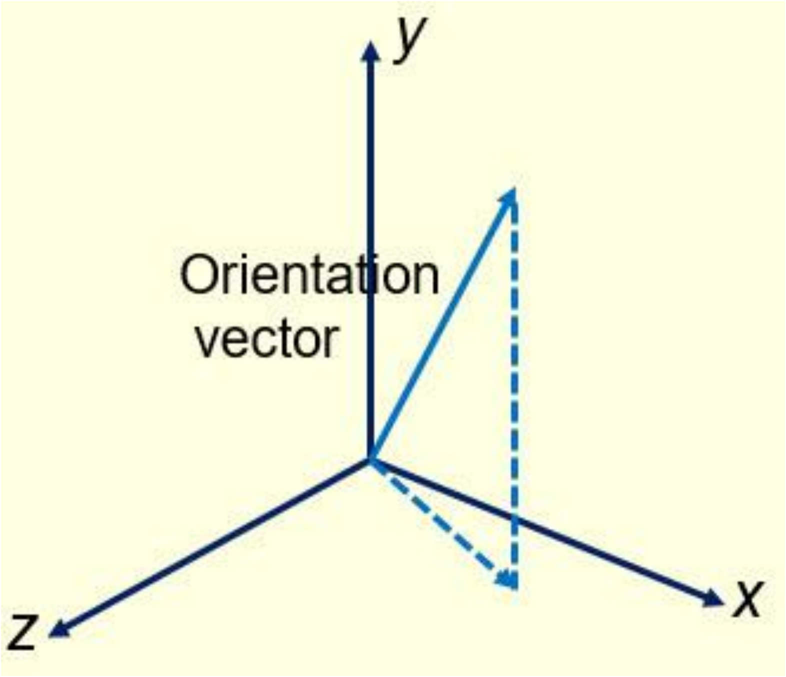
  Translation

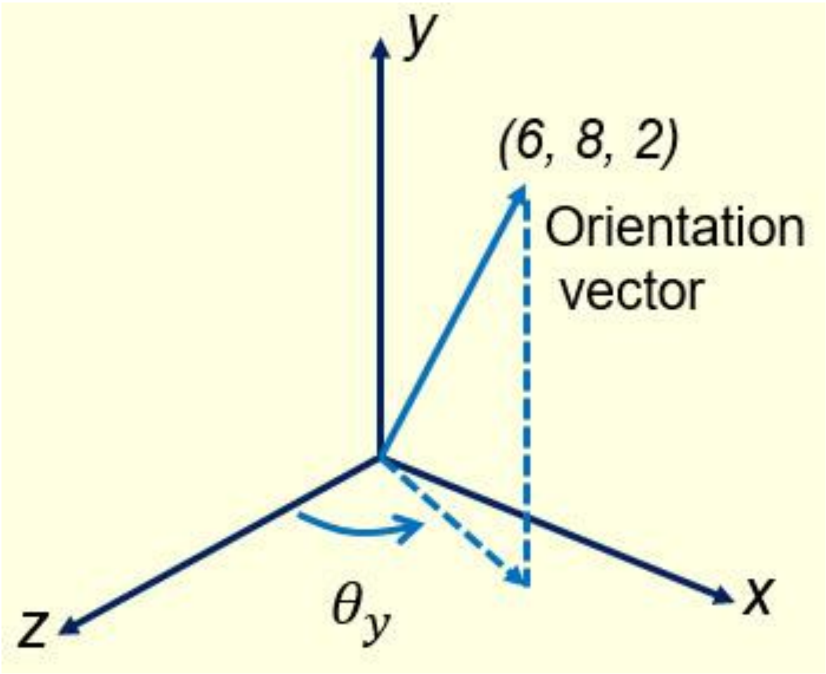
  Rotation

  Scaling (reflection)

Representing an arbitrary orientation:

- Fixed angle representation $(\theta_x, \theta_y, \theta_z)$





Representing an arbitrary orientation:

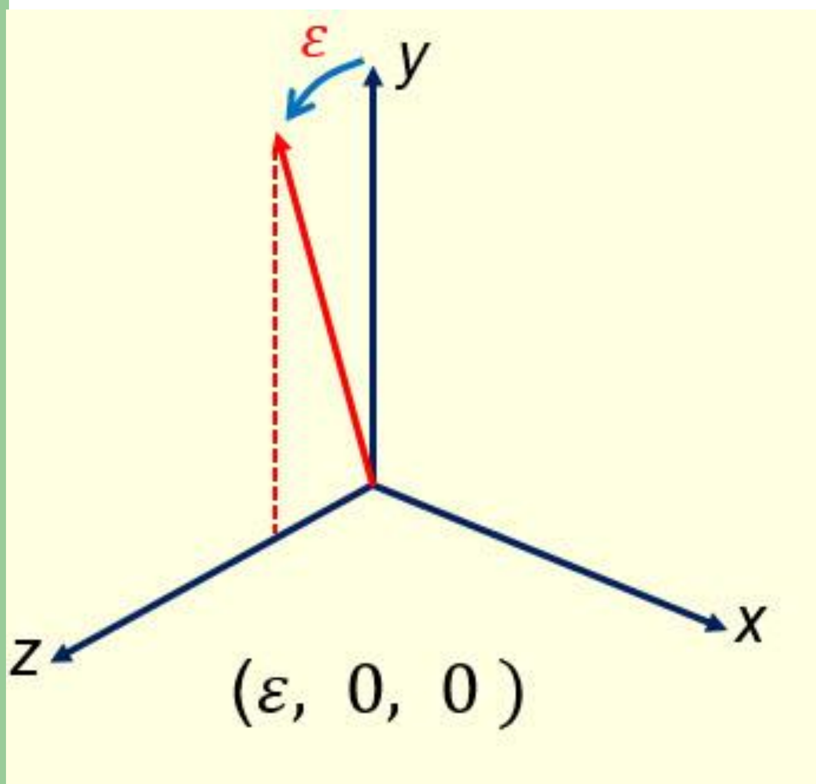
- **Problems**

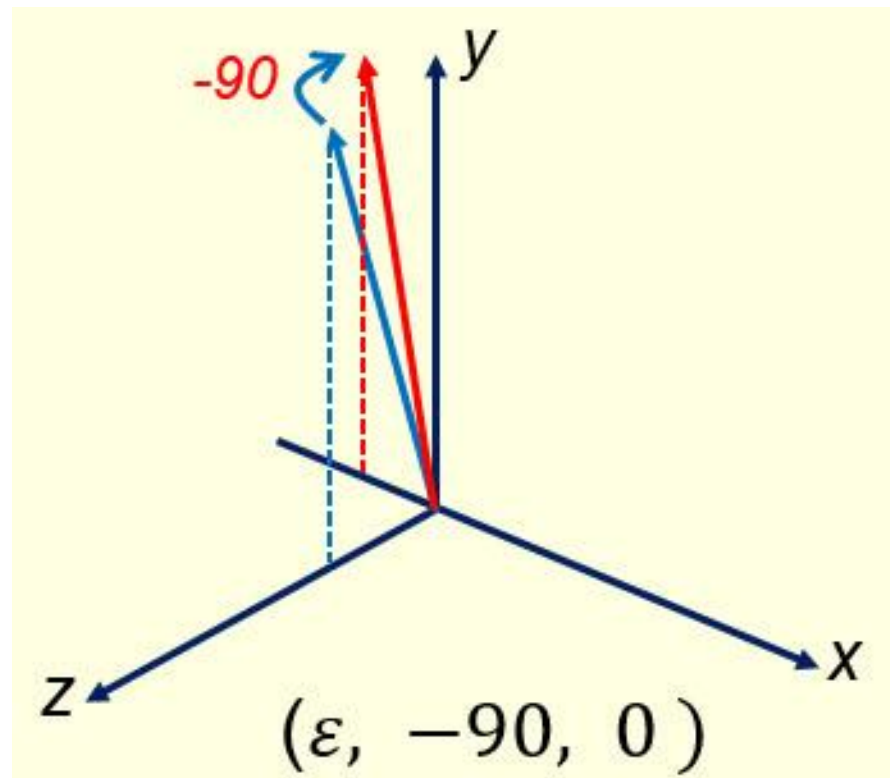
For (0, -90, 0)

$(\varepsilon, -90, 0)$ and $(0, -90, \varepsilon)$

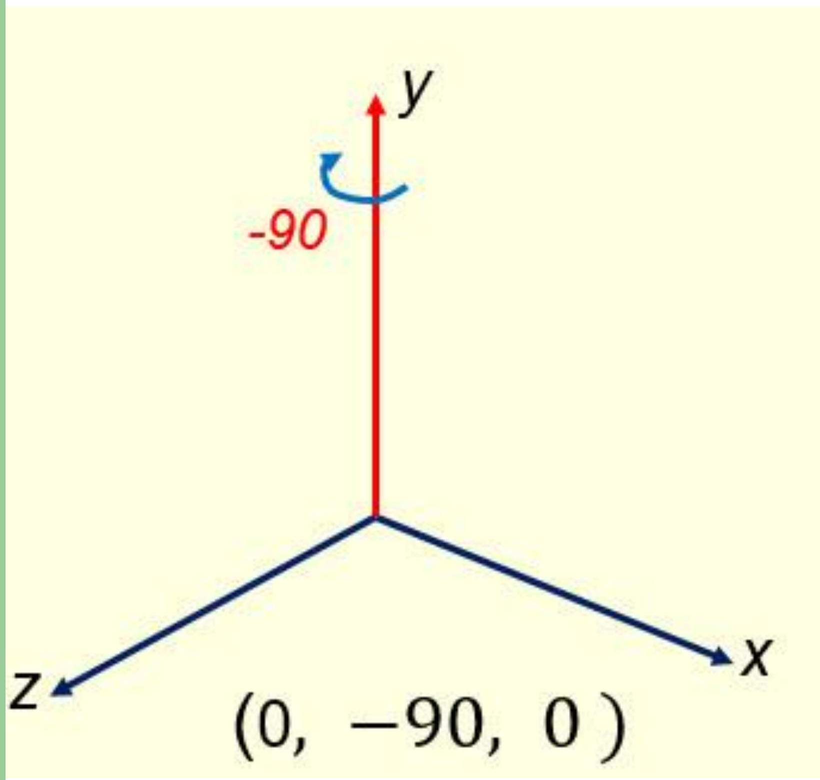
are the same. Why?

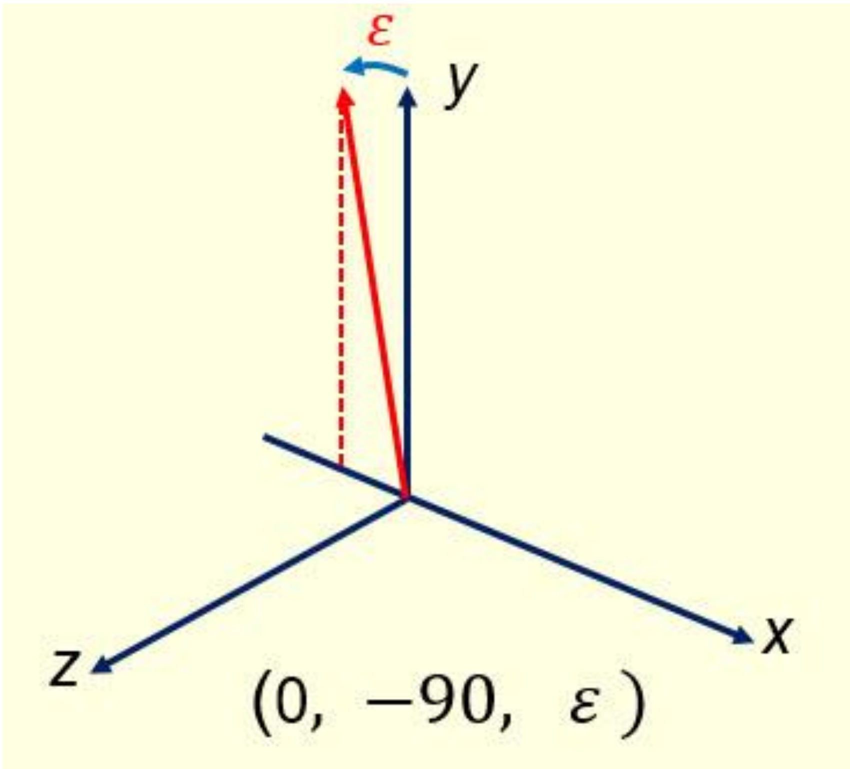
Representing an arbitrary orientation:





Representing an arbitrary orientation:





Representing an arbitrary orientation:

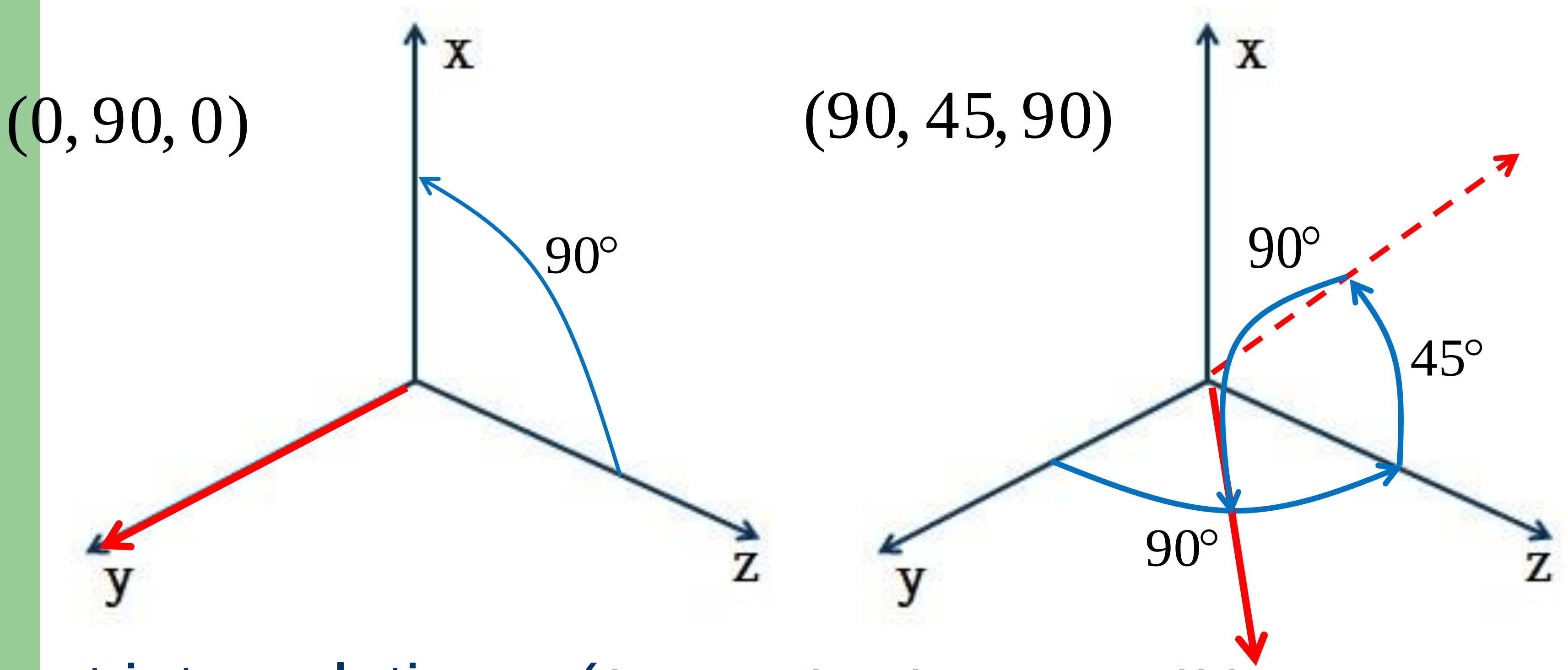
- Interpolation is difficult

Consider (0 , 90 , 0) and (90 , 45 , 90)

Direct interpolation: (45 , 67.5 , 45)

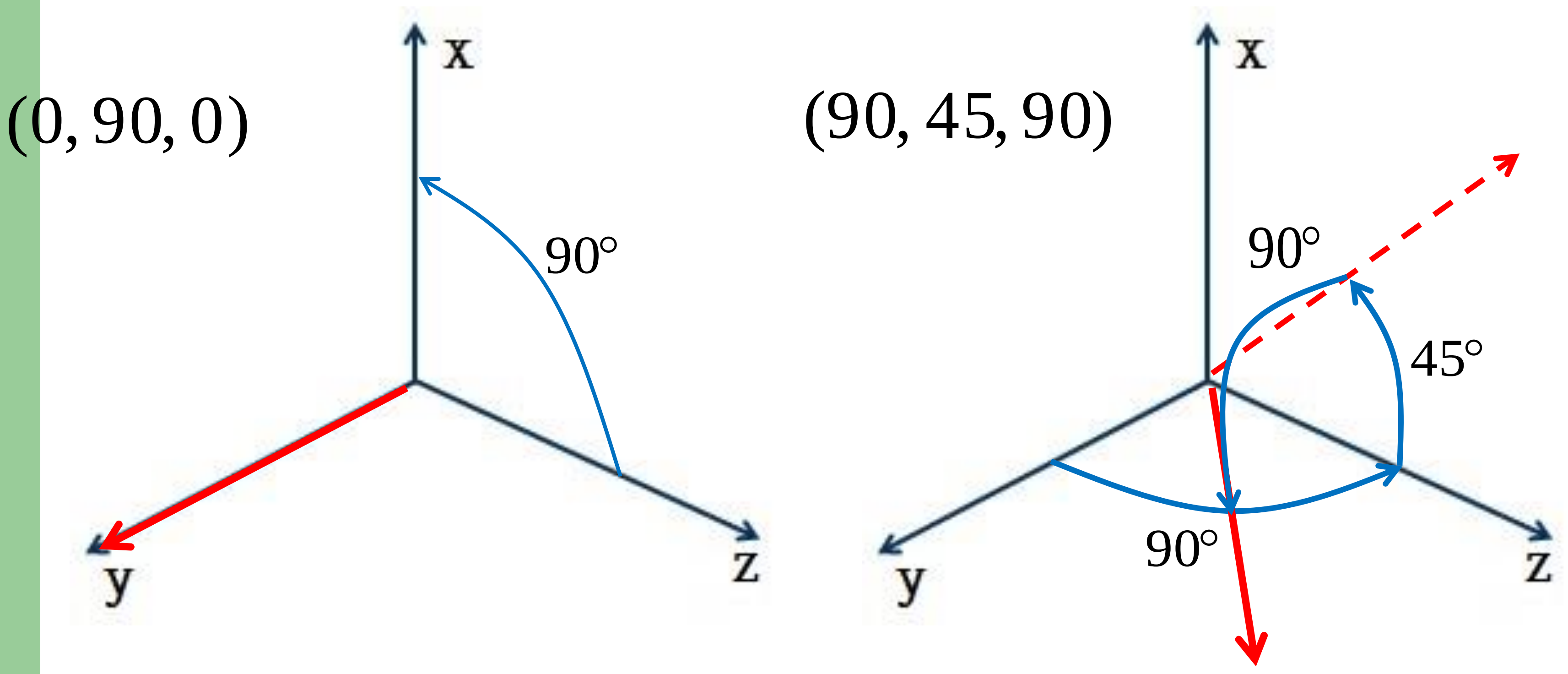
Desired interpolation: (90 , 67.5 , 90)

Representing an arbitrary orientation:



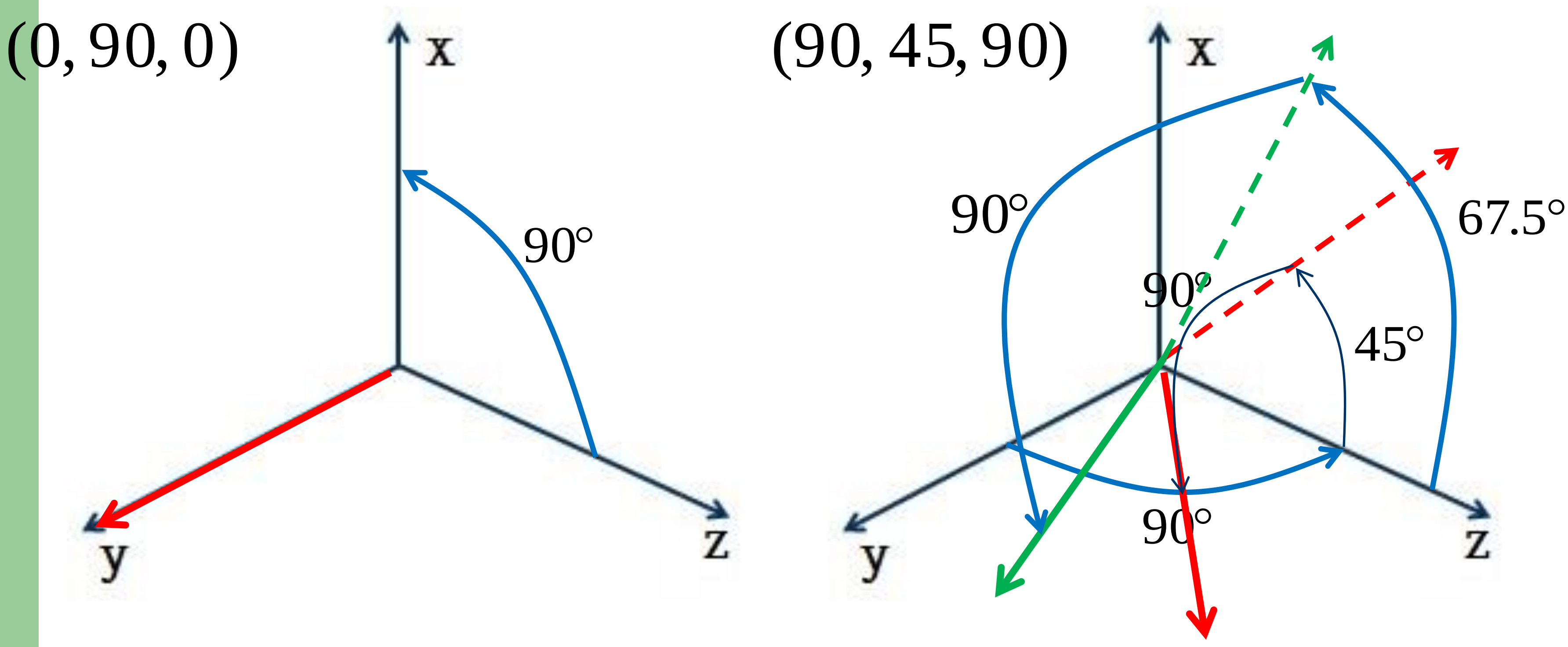
Direct interpolation = ((0, 90, 0) + (90, 45, 90))/2
= (45, 67.5, 45)

Representing an arbitrary orientation:



Question: Is (45, 67.5, 45) coplanar with (0, 90, 0) and (90, 45, 90) ?

Representing an arbitrary orientation:



Desired interpolation = (90, 67.5, 90)

Quaternions

Motivation:

- multiplying complex numbers can be interpreted as a rotation in two dimensions.

$$r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

- *Can hyper-complex numbers be defined so their multiplication can be viewed as a rotation in three dimensions?*

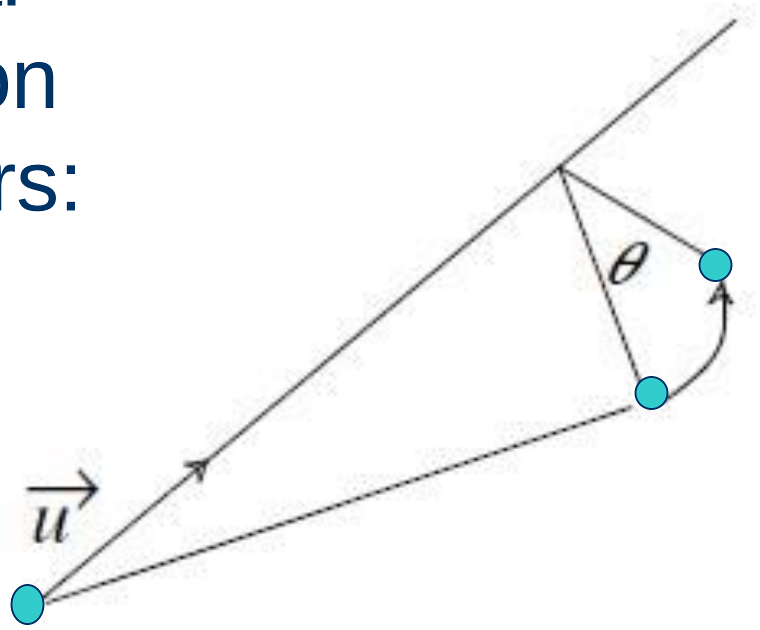
Quaternions

Remember that a general rotation in three dimension is defined by four numbers:

one for rotation angle

and three

for rotation axis



Quaternions: several approaches

1. 2×2 **matrices of complex numbers**

$$q = \begin{pmatrix} z & w \\ -w^* & z^* \end{pmatrix} = \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}$$

$$q = a\,\mathbf{U} + b\,\mathbf{I} + c\,\mathbf{J} + d\,\mathbf{K}$$

where

$$\mathbf{U} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{I} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \mathbf{K} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

Quaternions: several approaches

2. **Four dimensional vector space**

one of the bases:

$$\mathbf{i} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \qquad \mathbf{j} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{k} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \qquad \mathbf{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

4x4 Identity matrix

where

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathrm{I} \qquad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}$$

$$\mathbf{jk} = -\mathbf{kj} = \mathbf{i} \qquad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}$$

Quaternions: several approaches

3. **Combination of a *scaler* and a *vector***

$$q \equiv [\ s\ ,\ \mathbf{v}\ ]$$

$$s = w$$

$$\mathbf{v} = (\ x\ ,\ y\ ,\ z\ )$$

$$q \equiv w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

Conjugate quaternion: $q^* = w - x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$

Sum/difference:

$$q_2 \pm q_2 \equiv (w_1 \pm w_2) + (x_1 \pm x_2)\mathbf{i} + (y_1 \pm y_2)\mathbf{j} + (z_1 \pm z_2)\mathbf{k}$$

$$= \left[s_1 \pm s_2\ ,\ (\ \mathbf{v}_1 \pm \mathbf{v}_2\ ) \right]$$

Quaternions: several approaches

Product:

Inner product — Cross product

$$q_1 \cdot q_2 = \left[s_1 s_2 - \mathbf{v}_1 \cdot \mathbf{v}_2,\ \ s_1 \mathbf{v}_2 + s_2 \mathbf{v}_1 + \mathbf{v}_1 \otimes \mathbf{v}_2 \right]$$

Norm:

$$|q| = \sqrt{q \cdot q^*} = \sqrt{q^* \cdot q} = \sqrt{w^2 + x^2 + y^2 + z^2}$$

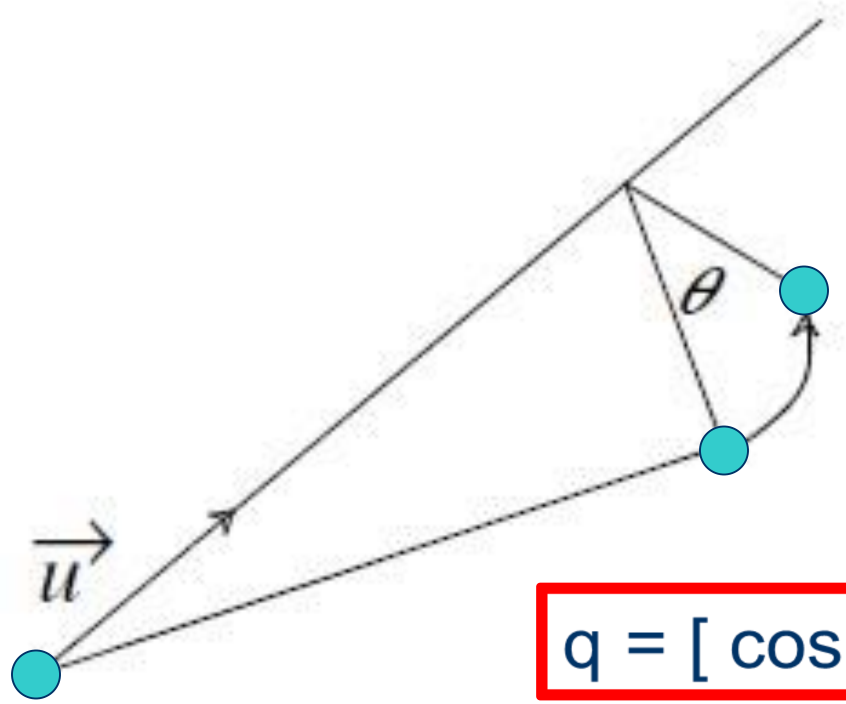
Inverse:

$$q^{-1} = \frac{q^*}{qq^*} = \frac{q^*}{|q|^2} = \frac{q^*}{w^2 + x^2 + y^2 + z^2}$$

Division:

$$q_1 / q_2 = q_1 \cdot q_2^{-1}$$

Representing rotations using quaternion:

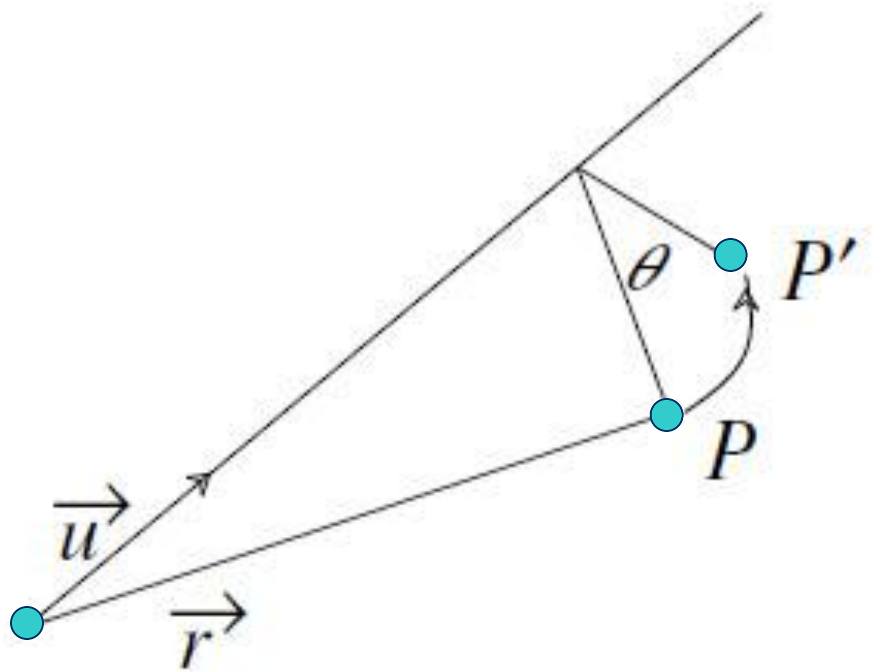


u ($= \vec{u}$): unit vector

$q = [\ \cos(\theta/2)\ ,\ \sin(\theta/2)\ \mathbf{u}\ ]$

Representing rotations using quaternion:

Rotate a point **P** by an angle θ about a unit axis **u:**



u ($= \vec{u}$):
unit vector

Representing rotations using quaternion:

1. Represent the point **P** as

   [0, **r**]   (or, [0, **P**])

2. represent the rotation by a quaternion

$$q = [\ \cos(\frac{\theta}{2})\ ,\ \sin(\frac{\theta}{2})\ \mathbf{u}\ ]$$

3. perform the rotation

$$q \cdot [\ 0\ ,\ \mathbf{r}\ ] \cdot q^{-1}$$

$$(\text{or,}\ \ \mathbf{P}' = q \cdot [\ 0\ ,\ \mathbf{P}\ ] \cdot q^{-1}\ )$$

Representing rotations using quaternion:

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3. perform the rotation

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$$(\text{or, }\ \mathbf{P}' = q \cdot [\ 0\ ,\mathbf{P}\ ] \cdot q^{-1}\ )$$

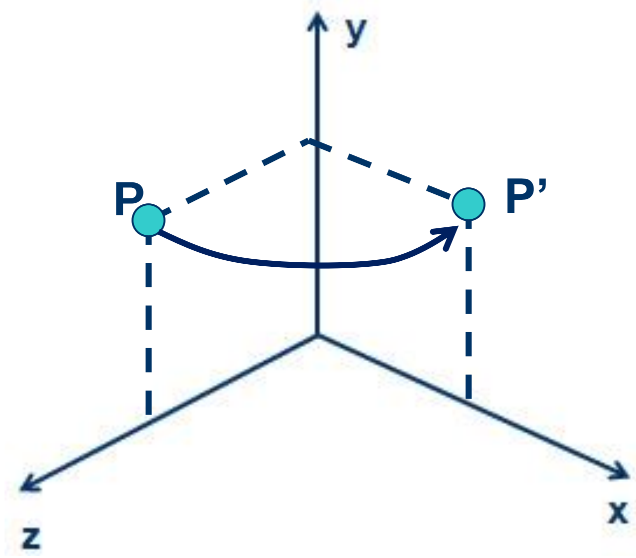
Representing rotations using quaternion:

Example: Consider a 90° rotation of point **P** = (0, 1, 1) about the *y-axis.*

After rotation, we should get

P' = (1, 1, 0)

Would we?



Representing rotations using quaternion:

$$[\ 0\ ,\ \mathbf{P}\ ] = [0\ ,\ \ (0,\ 1,\ 1\ )\ ]$$

$$q = [\ \frac{\sqrt{2}}{2}\ ,\ \ \frac{\sqrt{2}}{2}\,(0,\ 1,\ 0)\ ]$$

$$q^{-1} = [\ \frac{\sqrt{2}}{2}\ ,\ -\frac{\sqrt{2}}{2}\,(0,\ 1,\ 0)\ ]$$

Representing rotations using quaternion:

Hence,

$$q \cdot [0, \mathbf{P}] \cdot q^{-1}$$

$$= \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}(0,1,0)\right] \cdot [0,(0,1,1)] \cdot \left[\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}(0,1,0)\right]$$

$$= [\frac{\sqrt{2}}{2} * 0 - \frac{\sqrt{2}}{2}(0,1,0) \cdot (0,1,1), \frac{\sqrt{2}}{2}(0,1,1) +$$

$$0 * \frac{\sqrt{2}}{2}(0,1,0) + \frac{\sqrt{2}}{2}(0,1,0) \otimes (0,1,1)] \cdot$$

$$[\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}(0,1,0)]$$

Representing rotations using quaternion:

$$=[-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}(1,1,1)]\cdot\left[\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}(0,1,0)\right]$$

$$=[-\frac{\sqrt{2}}{2}*\frac{\sqrt{2}}{2}+\frac{1}{2}(1,1,1)\cdot(0,1,0),\ \frac{1}{2}(0,1,0)+$$

$$\frac{1}{2}(1,1,1)-\frac{1}{2}(1,1,1)\otimes(0,1,0)]$$

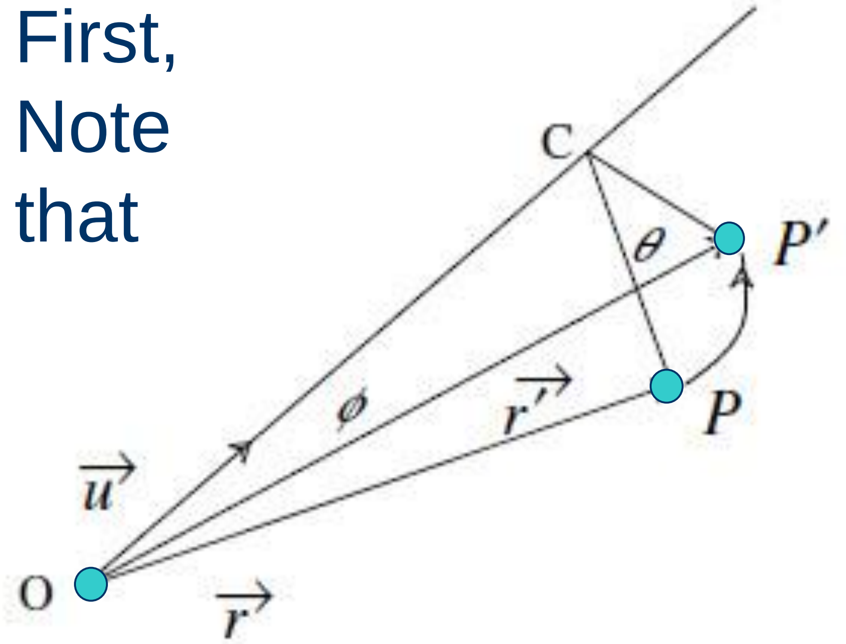
$$=[0,(1,1,0)]$$

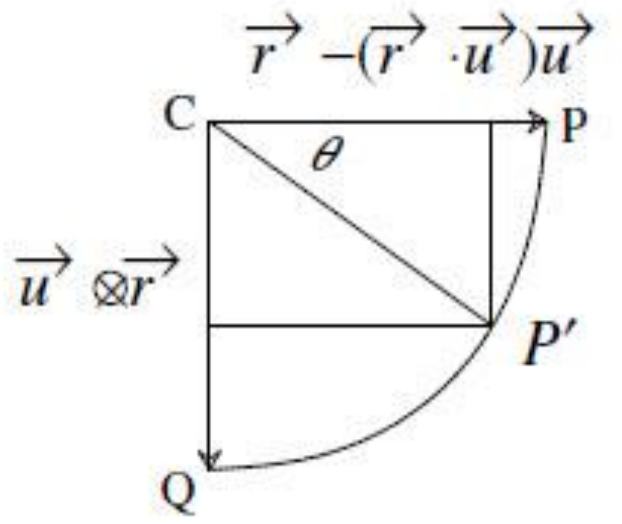
$$=[0,\mathbf{P}']$$

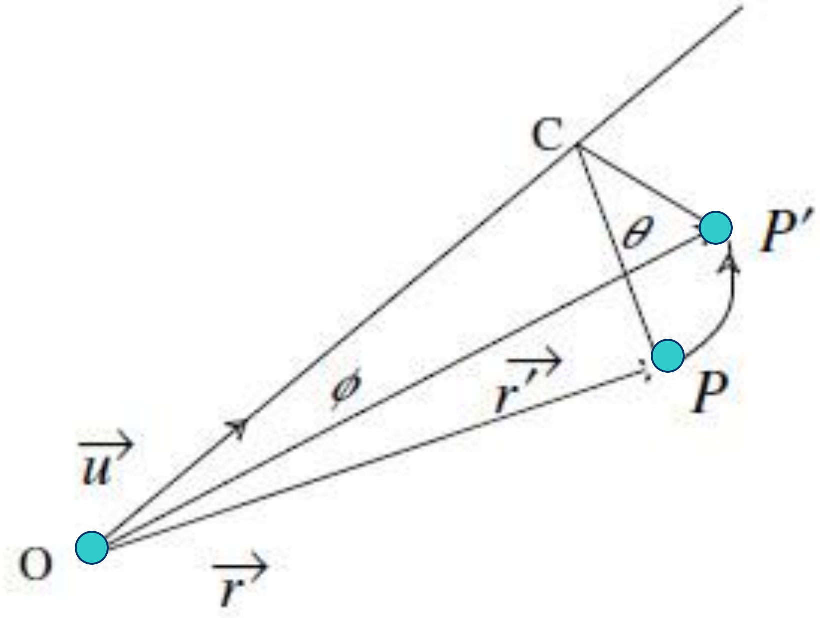
Representing rotations using quaternion:

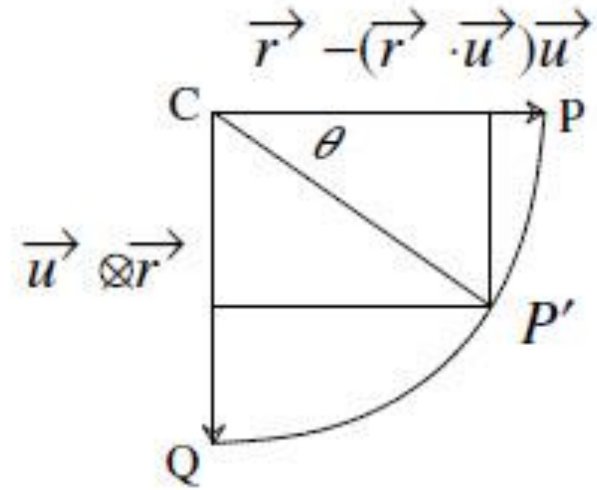
Prove that the triple product $q \cdot [\,0\,,\mathbf{P}\,] \cdot q^{-1}$ indeed performs a rotation of **P** about **u.**

First,
Note
that









$$\overrightarrow{CP'} = \cos\theta\,[\,\vec{r} - (\vec{r} \cdot \vec{u})\,\vec{u}\,] + \sin\theta(\vec{u} \otimes \vec{r})$$

$$\vec{r'} = (\vec{r} \cdot \vec{u})\vec{u} + \cos\theta\,[\,\vec{r} - (\vec{r} \cdot \vec{u})\vec{u}\,] + \sin\theta(\vec{u} \otimes \vec{r})$$

$$q = [\cos\left(\frac{\theta}{2}\right),\ \sin\left(\frac{\theta}{2}\right)\vec{u}\ ]$$

Now prove that $q \cdot [\,0\,,\mathbf{P}\,] \cdot q^{-1}$ would give us $[0\,,\ \mathbf{P}']$

$$q \cdot [0, \vec{r}] \cdot q^{-1}$$

$$= [\cos\left(\frac{\theta}{2}\right), \sin\left(\frac{\theta}{2}\right)\vec{u}] \cdot [0, \vec{r}] \cdot [\cos\left(\frac{\theta}{2}\right), -\sin\left(\frac{\theta}{2}\right)\vec{u}]$$

$$= [\cos\left(\frac{\theta}{2}\right) * 0 - \sin\left(\frac{\theta}{2}\right)\vec{u} \cdot \vec{r}, \cos\left(\frac{\theta}{2}\right)\vec{r} + 0 * \sin\left(\frac{\theta}{2}\right)\vec{u} + \sin\left(\frac{\theta}{2}\right)\vec{u} \otimes \vec{r}]$$

$$\cdot [\cos\left(\frac{\theta}{2}\right), -\sin\left(\frac{\theta}{2}\right)\vec{u}]$$

$$= [-\sin\left(\frac{\theta}{2}\right)\vec{u} \cdot \vec{r}, \cos\left(\frac{\theta}{2}\right)\vec{r} + \sin\left(\frac{\theta}{2}\right)\vec{u} \otimes \vec{r}] \cdot [\cos\left(\frac{\theta}{2}\right), -\sin\left(\frac{\theta}{2}\right)\vec{u}]$$

Now prove that $q \cdot [\,0\,,\mathbf{P}\,] \cdot q^{-1}$ would give us $[0\,,\ \mathbf{P}']$

$$= [-\sin\left(\frac{\theta}{2}\right)\vec{u}\cdot\vec{r},\ \cos\left(\frac{\theta}{2}\right)\vec{r} + \sin\left(\frac{\theta}{2}\right)\vec{u}\otimes\vec{r}]\cdot[\cos\left(\frac{\theta}{2}\right),\ -\sin\left(\frac{\theta}{2}\right)\vec{u}]$$

$$= [-\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\vec{u}\cdot\vec{r} + \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\vec{r}\cdot\vec{u} + \sin^2\left(\frac{\theta}{2}\right)(\vec{u}\otimes\vec{r})\cdot\vec{u},$$

$$\sin^2\left(\frac{\theta}{2}\right)(\vec{u}\cdot\vec{r})\vec{u} + \cos^2\left(\frac{\theta}{2}\right)\vec{r} + \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\vec{u}\otimes\vec{r}$$

$$-\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\vec{r}\otimes\vec{u} - \sin^2\left(\frac{\theta}{2}\right)(\vec{u}\otimes\vec{r})\otimes\vec{u}]$$

$$\left(\vec{u}\otimes\vec{r}\right)\otimes\vec{u}=\vec{r}-(\vec{u}\cdot\vec{r})\vec{u}$$

Now prove that $q\cdot[\,0\,,\mathbf{P}\,]\cdot q^{-1}$ would give us $[0\,,\;\mathbf{P}']$

$$=[-\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\vec{u}\cdot\vec{r}+\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\vec{r}\cdot\vec{u}+\sin^2\left(\frac{\theta}{2}\right)\left(\vec{u}\otimes\vec{r}\right)\cdot\vec{u},$$

0

$$\sin^2\left(\frac{\theta}{2}\right)(\vec{u}\cdot\vec{r})\vec{u}+\cos^2\left(\frac{\theta}{2}\right)\vec{r}+\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\vec{u}\otimes\vec{r}$$

$$-\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\vec{r}\otimes\vec{u}-\sin^2\left(\frac{\theta}{2}\right)\left(\vec{u}\otimes\vec{r}\right)\otimes\vec{u}]$$

$$=[0,\sin\theta(\vec{u}\otimes\vec{r})+\sin^2\left(\frac{\theta}{2}\right)(\vec{u}\cdot\vec{r})\vec{u}+\cos^2\left(\frac{\theta}{2}\right)\vec{r}-\sin^2\left(\frac{\theta}{2}\right)\vec{r}$$

$$+\sin^2\left(\frac{\theta}{2}\right)(\vec{r}\cdot\vec{u})\vec{u}\;]$$

Now prove that $q \cdot [\,0\,, \mathbf{P}\,] \cdot q^{-1}$ would give us $[0\,,\ \mathbf{P}']$

$$= [0, \sin\theta(\vec{u} \otimes \vec{r}) + \cos\theta\,\vec{r} + 2\sin^2\left(\frac{\theta}{2}\right)(\vec{u}\cdot\vec{r})\,\vec{u}]$$

$$= [0, \sin\theta(\vec{u} \otimes \vec{r}) + \cos\theta\,\vec{r} +$$

$$(\cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right) - \cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right))((\vec{r}\cdot\vec{u})\vec{u})]$$

$$= [0, (\vec{r}\cdot\vec{u})\vec{u} + \cos\theta(\vec{r} - (\vec{r}\cdot\vec{u})\vec{u}) + \sin\theta(\vec{u} \otimes \vec{r})]$$

$$= [0, \vec{r}\,']$$

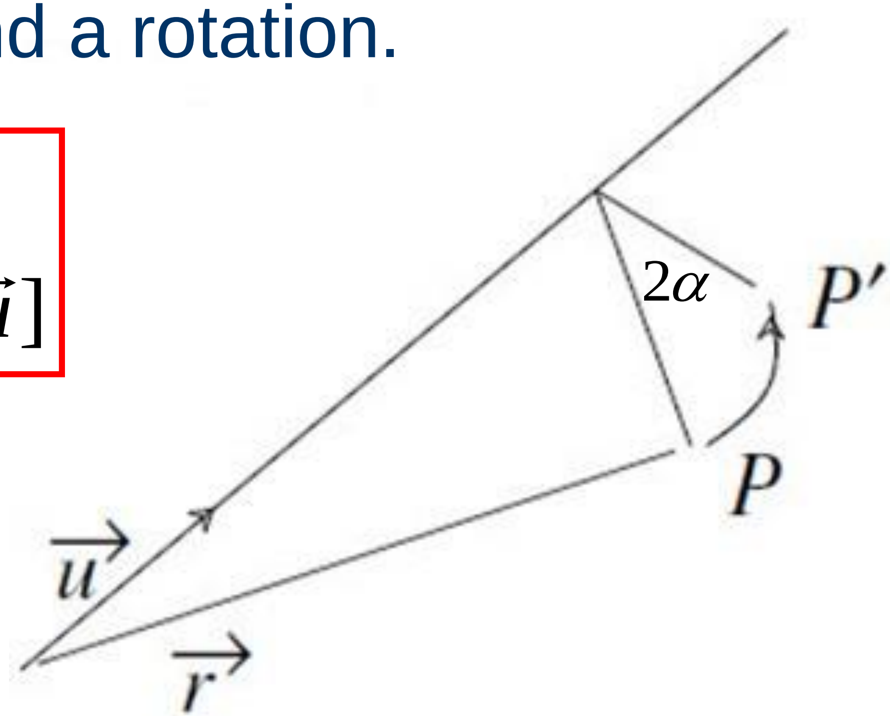
$$= [0, \mathrm{P}']$$

Geometric Meaning of Quaternions

Quaternions provide a clear difference between a vector (point) and a rotation.

$$\text{Point/vector}: [0, P] = [0, \vec{r}]$$

$$\text{Rotation}: q = [\cos\alpha, \sin\alpha\, \vec{u}]$$



All the operations for vectors such as vector addition/subtraction, scalar multiplication hold.

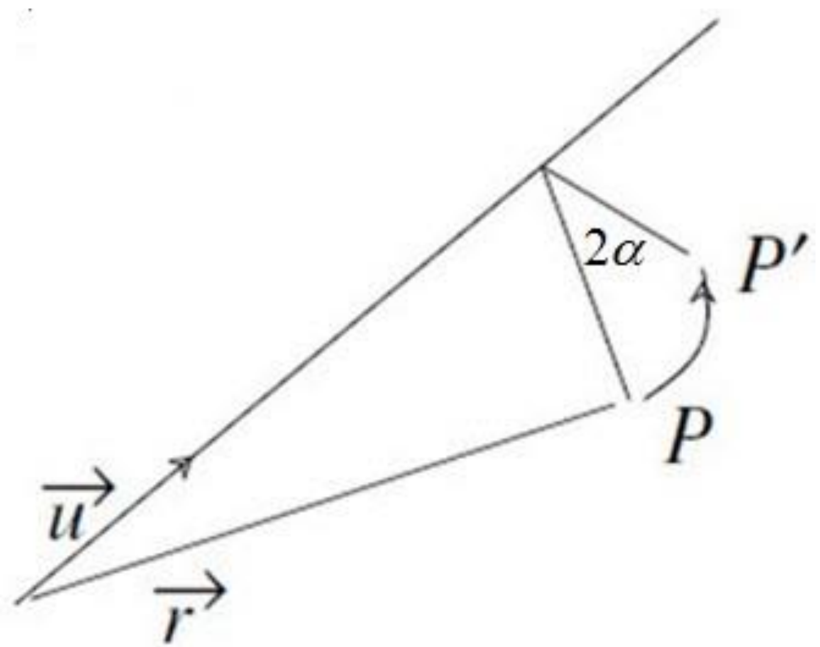
Geometric Meaning of Quaternions

Rotation is performed as: $q \cdot [0, P] \cdot q^{-1}$

Why?

Let $R(P) \equiv q[0, P]q^{-1}$

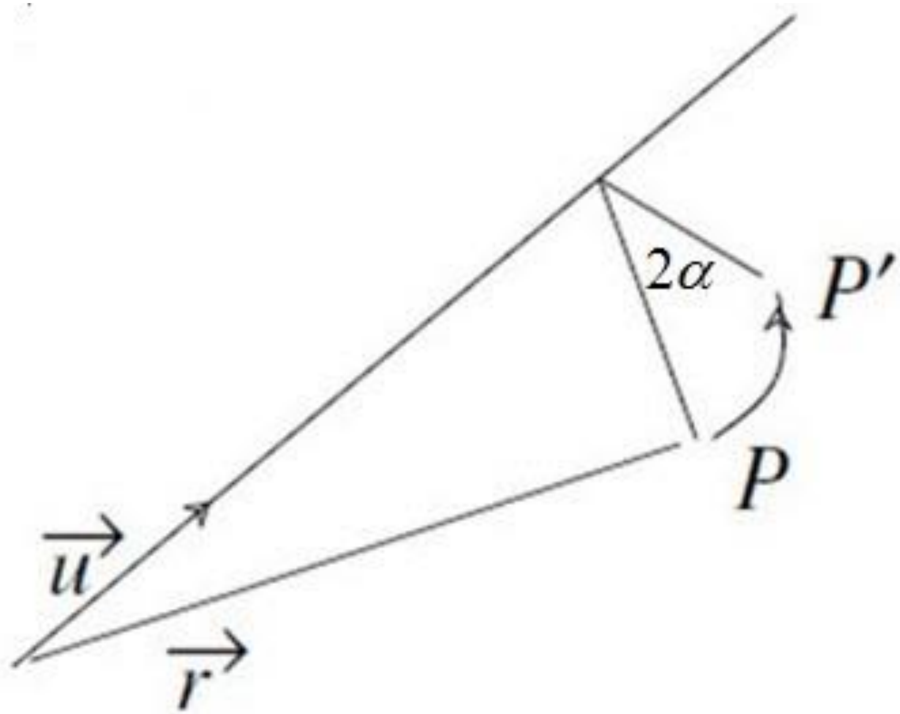
For *R(P)* to be a rotation,
It must satisfy four properties:



Geometric Meaning of Quaternions

For *R(P)* to be a rotation, it must satisfy four properties:

1. *R(P)* is a 3D vector
2. *R(P)* preserves length
3. *R(P)* is a linear transformation
4. *R(P)* does not have a reflection component



R(P) is a 3D vector

Let *W(p)* represent the real part of the quaternion *p*. Then we need to show that *W(R(P))=0*

First, note that quaternion multiplication is *distributive* and *associative*. Besides, we have

$$W(p) = (p + p^*)/2$$

and

$$(pq)^* = q^* p^*$$

R(P) is a 3D vector

Hence,

$$W(R(P)) = W(q[0, P]q^*)$$

$$= \{q[0, P]q^* + (q[0, P]q^*)^*\} / 2$$

$$= \{q[0, P]q^* + q[0, -P]q^*\} / 2$$

$$= q\{\frac{[0, P] + [0, -P]}{2}\}q^*$$

$$= q[0, \vec{0}]q^*$$

$$= 0$$

R(P) preserves length

Let *N(p)* represent the length of the quaternion *p*,

i.e., if $p = w + x\vec{i} + y\vec{j} + z\vec{k}$

then $N(p) = w^2 + x^2 + y^2 + z^2$

Note that $N(pq) = N(p)N(q)$

Hence,

$$N(R(P)) = N(q[0, P]q^*)$$

$$= N(q)N([0, P])N(q^*)$$

$$= N([0, P])$$

R(*P*) is a linear transformation

Let *a* be a scalar and *P*, *Q* be 3D vectors, then

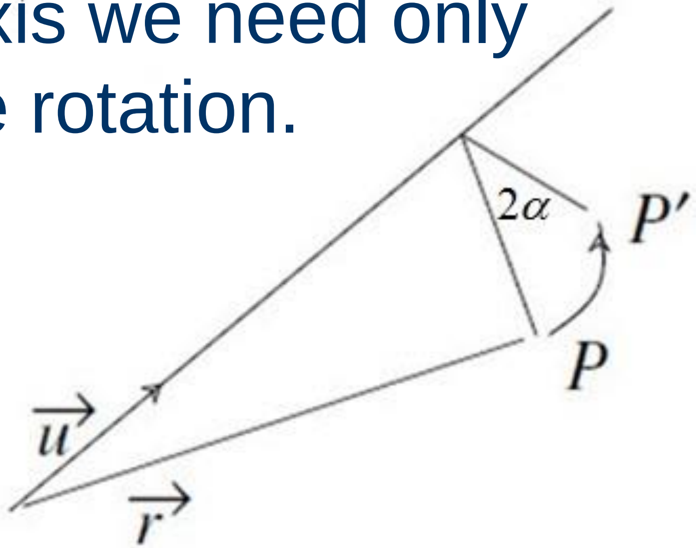
$$R(aP+Q) = q[0, aP+Q]q^*$$
$$= q\{[0, aP]+[0, Q]\}q^*$$
$$= q[0, aP]q^* + q[0, Q]q^*$$
$$= q(a[0, P])q^* + q[0, Q]q^*$$
$$= a(q[0, P]q^*) + q[0, Q]q^*$$
$$= aR(P) + R(Q)$$

R(P) does not have a reflection component

Consider R as a function of q for a fixed vector $\vec{r}$.
That is, R(q) = $q[0, \vec{r}\,]q^*$.
This function is a continuous function of q. For each q it is a linear transformation with determinant D(q), so the determinant itself is a continuous function of q.
Thus, $\lim_{q\to 1} R(q) = R(1) = I$, the identity function (the limit is taken along any path of quaternions which approach the quaternion 1) and $\lim_{q\to 1} D(q) = D(1) = 1$.
By continuity, D(q) is identically 1 and R(q) does not have a reflection component.

$\vec{u}$ is the unit rotation axis

To see that $\vec{u}$ is a unit rotation axis we need only show that $\vec{u}$ is unchanged by the rotation. Indeed,



$$
\begin{aligned}
R(\vec{u}) &= q[0,\vec{u}]q^* \\
&= [\cos\alpha, \sin\alpha\,\vec{u}][0,\vec{u}][\cos\alpha, -\sin\alpha\,\vec{u}] \\
&= [-\sin\alpha, \cos\alpha\,\vec{u}][\cos\alpha, -\sin\alpha\,\vec{u}] \\
&= [-\sin\alpha\cos\alpha + \cos\alpha\sin\alpha, \sin^2\alpha\,\vec{u} + \cos^2\alpha\,\vec{u}] \\
&= [0,\vec{u}]
\end{aligned}
$$

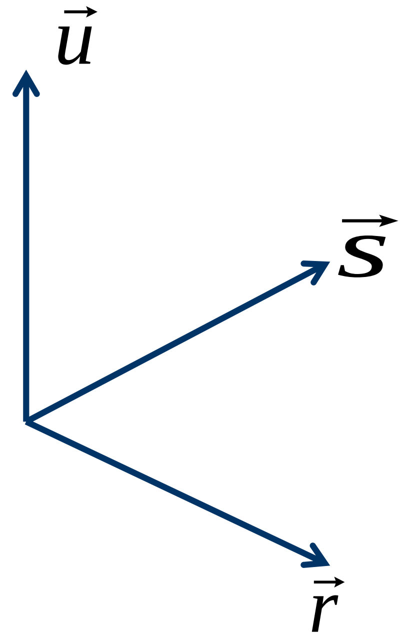
The rotation angle is 2α

To see that the rotation angle is 2α, let $\vec{u}, \vec{r}$ and $\vec{s}$ be a right-handed set of orthonormal vectors, i.e.,
The vectors are all of unit length;

$$\vec{u}\cdot\vec{r}=\vec{u}\cdot\vec{s}=\vec{r}\cdot\vec{s}=0$$; and

$$\vec{u}\otimes\vec{r}=\vec{s},\ \ \vec{r}\otimes\vec{s}=\vec{u}\ \ \text{and}$$

$$\vec{s}\otimes\vec{u}=\vec{r}.$$

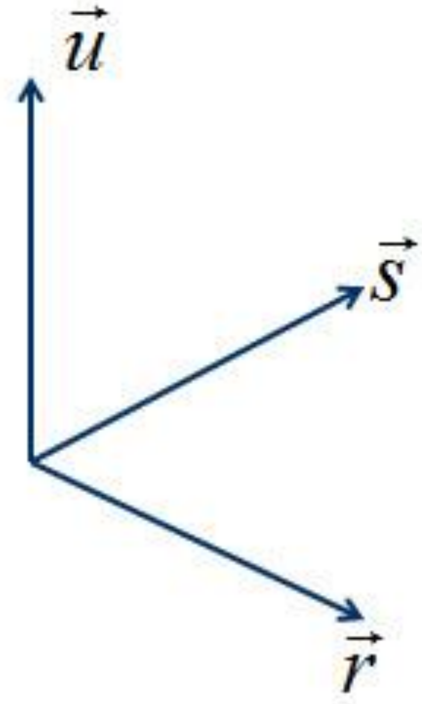


The vector $\vec{r}$ is rotated by an angle ϕ to the vector $q[0,\vec{r}]q^*$, so $\vec{r}\cdot\left(q[0,\vec{r}]q^*\right)=\cos(\phi)$.

The rotation angle is 2α

A quaternion q may also be viewed as a 4D vector (w; x; y; z). The dot product of two quaternions is

$$q_0 \cdot q_1 = w_0 w_1 + x_0 x_1 + y_0 y_1 + z_0 z_1 = W(q_0 q_1^*)$$



Also,

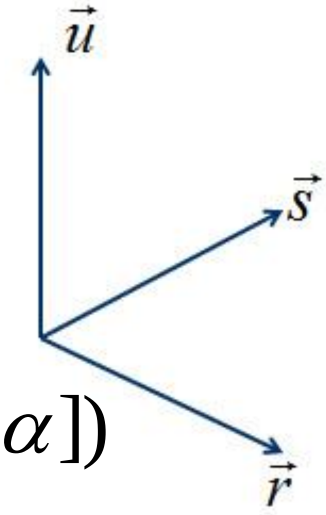
$$[0, \vec{r}]^* = -[0, \vec{r}]$$

and

$$[0, p]^2 = -1$$

for any unit quaternion $[0, p]$ with zero real part.

The rotation angle is 2α



$$\cos(\phi) = \vec{r} \cdot \left(q[0, \vec{r}\,]q^* \right)$$

$$= W([0, \vec{r}\,]^* q[0, \vec{r}\,]q^*)$$

$$= W(-[0, \vec{r}\,][\cos\alpha, \vec{u}\sin\alpha][0, \vec{r}\,][\cos\alpha, -\vec{u}\sin\alpha])$$

$$= W([0, -\vec{r}\cos\alpha + \vec{s}\sin\alpha][0, \vec{r}\cos\alpha + \vec{s}\sin\alpha])$$

$$= W([\cos^2\alpha - \sin^2\alpha, -2\vec{u}\cos\alpha\sin\alpha])$$

$$= \cos^2\alpha - \sin^2\alpha$$

$$= \cos(2\alpha)$$

Hence, the rotation angle is $\phi = 2\alpha$

End of Introduction & Technical Background