3. Interpolating Values

- Methods for precisely specifying the motion of objects

- Animator uses these techniques to directly control how the objects will move
3. Interpolating Values

Interpolation

- Foundation of most animation
  - path curve (space curve) construction
  - motion control
3.1 Space Curve Construction

**Space curve**: path to be followed by object

- **Given**: position-time pairs constraining the motion

- **Want**: a curve interpolating these position-time pairs
3.1 Space Curve Construction

How to find the appropriate function:

- Interpolation vs approximation
- Complexity (computational complexity)
  - piecewise, polynomial
- Continuity
  - $C1$, $C2$
- Global vs local control

Independent of shape complexity

Affect local shape only
3.1 Space Curve Construction

Typical space curve generation technique:
- uniform cubic B-spline curve interpolation

$$C(t) = \frac{(1-t)^3}{6} P_0 + \frac{4-6t^2+3t^3}{6} P_1 + \frac{1+3t+3t^2-3t^3}{6} P_2 + \frac{t^3}{6} P_3$$

0 ≤ t ≤ 1

Uniform cubic B-spline Curve segment
3.1 Space Curve Construction

A uniform cubic B-spline curve:

$P_0, P_1, ..., P_7$ are given
3.1 Space Curve Construction

A uniform cubic B-spline curve:

\[ P_0, P_1, \ldots, P_7 \] are given
3.1 Space Curve Construction

Reverse process:

Find \( P_0, P_1, ..., P_7 \) so that the corresponding uniform cubic B-spline curve would interpolate \( D_0, D_1, ..., D_5 \).

Given:
\( D_0, D_1, ..., D_5 \)
3.1 Space Curve Construction

Reverse process:

Find $P_0, P_1, ..., P_7$ so that the corresponding uniform cubic B-spline curve would interpolate $D_0, D_1, ..., D_5$.

Given:

$D_0, D_1, ..., D_5$
Closed Curve Fitting using Uniform Cubic B-spline Curves

Given data points $D_i = (x_i, y_i)$, $i = 0, 1, ..., n$, ($n \geq 3$), $D_0 = D_n$, $D_1$, $D_2$, $D_0 = D_9$, $D_8$
Closed Curve Fitting using Uniform Cubic B-spline Curves

Given data points \( D_i = (x_i, y_i), \; i = 0, 1, ..., n, (n \geq 3), \; D_0 = D_n \), how can a cubic B-spline curve \( S(u) \) that interpolates these points be constructed?
Closed Curve Fitting using Uniform Cubic B-spline Curves

Given data points \( D_i = (x_i, y_i), \ i = 0, 1, \ldots, n, \ (n \geq 3), \ D_0 = D_n, \)
how can a cubic B-spline curve \( S(u) \) that interpolates these points be constructed?

The cubic B-spline curve has \( n \) segments with \( D_{i-1} \) and \( D_i \) being the start and end points of \( C_i(t) \).
Closed Curve Fitting using Uniform Cubic B-spline Curves

The curve must have $n+3$ control points $P_i, i = 0, 1, ..., n+2$, with

- $P_0 = P_n$
- $P_1 = P_{n+1}$
- $P_2 = P_{n+2}$

to ensure $C^2$-continuity at $D_0 = D_n$. 

$C_9(t)$

$D_8$

$C_1(t)$

$D_1$

$D_2$

$D_0 = D_9$

$P_0 = P_n$

$P_n = P_0$

$P_{n+2} = P_2$
Closed Curve Fitting using Uniform Cubic B-spline Curves

Parameter space: \([0, n]\)

For \(i \leq u \leq i + 1\),
\[
S(u) = C_{i+1}(u - i)
\]

In particular,
\[
S(i) = D_i , \quad i = 0, 1, 2, ..., n
\]
Closed Curve Fitting using Uniform Cubic B-spline Curves

We have

\[ S(0) = D_0 \]
\[ S(1) = D_1 \]
\[ \vdots \]
\[ S(n-1) = D_{n-1} \]
\[ S(n) = D_n \]

\[ P_0 + 4P_1 + P_2 = 6D_0 \]
\[ P_1 + 4P_2 + P_3 = 6D_1 \]
\[ \vdots \]
\[ P_{n-1} + 4P_n + P_{n+1} = 6D_{n-1} \]
\[ P_n + 4P_{n+1} + P_{n+2} = 6D_n \]
Closed Curve Fitting using Uniform Cubic B-spline Curves

In matrix form

\[
\begin{bmatrix}
4 & 1 & 1 & 1 \\
1 & 4 & 1 & 1 \\
1 & 4 & 1 & 1 \\
\vdots & \vdots & \vdots & \vdots \\
1 & 4 & 1 & 1 \\
1 & 4 & 1 & 1 \\
1 & 4 & 1 & 1 \\
1 & 1 & 4 & 1 \\
\end{bmatrix}
\begin{bmatrix}
P_1 \\
P_2 \\
P_3 \\
\vdots \\
P_{n-2} \\
P_{n-1} \\
P_n \\
\end{bmatrix}
\begin{bmatrix}
6D_0 \\
6D_1 \\
6D_2 \\
\vdots \\
6D_{n-3} \\
6D_{n-2} \\
6D_{n-1} \\
\end{bmatrix}
\]
Given data points $D_i = (x_i, y_i)$, $i = 0, 1, \ldots, n$, $(n \geq 3)$, how can an open cubic B-spline curve $S(u)$ that interpolates these points be built?

The cubic B-spline curve has $n$ segments $C_i(t)$, $C_2(t)$, $\ldots$, $C_n(t)$ with $D_{i-1}$ and $D_i$ being the start and end points of $C_i(t)$. 

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**Open Curve Fitting using Uniform Cubic B-spline Curves**
To get the curve constructed, how many control points are needed? Consider the following case:

\[ n + 3 \text{ control points: } P_i, i = 0, 1, 2, \ldots, n + 2 \]
Open Curve Fitting using Uniform Cubic B-spline Curves

Parameter space: [0, n]

For \( i \leq u \leq i + 1 \),
\[ S(u) = C_{i+1}(u - i) \]

In particular,
\[ S(i) = D_i, \]
\[ i = 0, 1, 2, \ldots, n \]
Open Curve Fitting using Uniform Cubic B-spline Curves

We have

\[ S(0) = D_0 \quad P_0 + 4P_1 + P_2 = 6D_0 \]
\[ S(1) = D_1 \quad P_1 + 4P_2 + P_3 = 6D_1 \]
\[ \vdots \]
\[ S(n-1) = D_{n-1} \quad P_{n-1} + 4P_n + P_{n+1} = 6D_{n-1} \]
\[ S(n) = D_n \quad P_n + 4P_{n+1} + P_{n+2} = 6D_n \]
Open Curve Fitting using Uniform Cubic B-spline Curves

Need two more conditions. We set \( S''(0) = 0, \quad S''(n) = 0 \)

we get

\[ P_0 - 2P_1 + P_2 = 0 \quad \text{and} \quad P_1 = D_0 \]

\[ P_n - 2P_{n+1} + P_{n+2} = 0 \quad \text{and} \quad P_{n+1} = D_n \]
Open Curve Fitting using Uniform Cubic B-spline Curves

Relationship between data points and control points:

\[
\begin{align*}
P_1 &= D_0 \\
P_0 &= 2P_1 - P_2 \\
P_{n+1} &= D_n \\
P_{n+2} &= 2P_{n+1} - P_n
\end{align*}
\]

\(n=5\)
We have

\[ \begin{align*}
\sum_{n=0}^{u} &= 6D^{u-1} - D^u \\
\sum_{n=1}^{u-1} &= 4p + \sum_{n=1}^{u-2} \\
\sum_{n=1}^{u-2} &= 6D \\
\sum_{n=2}^{u} &= 4p + \sum_{n=3}^{u-1} \\
\sum_{n=3}^{u-1} &= 6D \\
\sum_{n=4}^{u} &= 4p + \sum_{n=1}^{u-2} \\
\sum_{n=2}^{u-2} &= 6D \\
\sum_{n=3}^{u-3} &= 4p + \sum_{n=4}^{u-1} \\
\sum_{n=4}^{u-1} &= 6D \\
\sum_{n=5}^{u} &= 4p + \sum_{n=1}^{u-2} \\
\sum_{n=2}^{u-2} &= 6D \\
\sum_{n=3}^{u-3} &= 4p + \sum_{n=4}^{u-1} \\
\sum_{n=4}^{u-2} &= 6D \\
\sum_{n=5}^{u} &= 4p + \sum_{n=1}^{u-2} \\
\sum_{n=2}^{u-2} &= 6D \\
\sum_{n=3}^{u-3} &= 4p + \sum_{n=4}^{u-1} \\
\sum_{n=4}^{u-3} &= 6D \\
\sum_{n=5}^{u} &= 4p + \sum_{n=1}^{u-2} \\
\sum_{n=2}^{u-2} &= 6D \\
\sum_{n=3}^{u-3} &= 4p + \sum_{n=4}^{u-1} \\
\sum_{n=4}^{u-4} &= 6D \\
\sum_{n=5}^{u} &= 4p + \sum_{n=1}^{u-2} \\
\sum_{n=2}^{u-2} &= 6D \\
\end{align*} \]

Cubic B-spline Curves
Open Curve Fitting using Uniform
The matrix form of this system is:

\[
\begin{bmatrix}
4 & 1 & & & & \\
1 & 4 & 1 & & & \\
1 & 4 & 1 & & & \\
& & & \ddots & & \\
1 & 4 & 1 & & & \\
1 & 4 & 1 & & & \\
\end{bmatrix}
\begin{bmatrix}
P_2 \\
P_3 \\
P_4 \\
\vdots \\
P_{n-2} \\
P_{n-1} \\
P_n \\
\end{bmatrix}
= 
\begin{bmatrix}
6D_1 - D_0 \\
6D_2 \\
6D_3 \\
\vdots \\
6D_{n-3} \\
6D_{n-2} \\
6D_{n-1} - D_n \\
\end{bmatrix}
\]

This system of equations can be solved using Gaussian elimination without pivoting.
End of Interpolation I