CS633 3D Computer Animation
Solution Set - HW 6 (40 points)
Due: 4/19/2018

1. Given the following two primitive implicit surfaces

$$f_1(p) = (x-2)^2 + (y-0)^2 + (z-0)^2 - 1$$

$$f_2(p) = (x+2)^2 + (y-0)^2 + (z-0)^2 - 1$$

what is the shape of the implicit surface

$$F(p) = \sum_{i=1}^{2} w_i f_i(p) - T$$

when (a) $w_1 = w_2 = 1$ and $T = 4$; (b) $w_1 = w_2 = 1$ and $T = 2$; (c) $w_1 = -w_2 = 1$ and $T = 6$; (d) $w_1 = -w_2 = 1$ and $T = 4$.

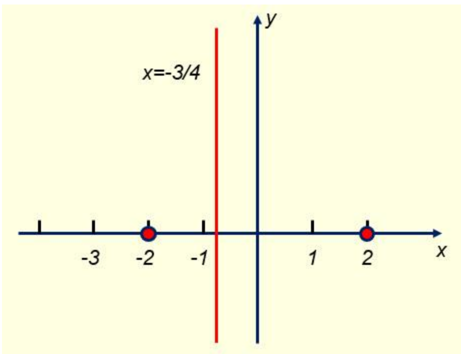
Use 2D examples to illustrate your results. (10 points)

Sol:

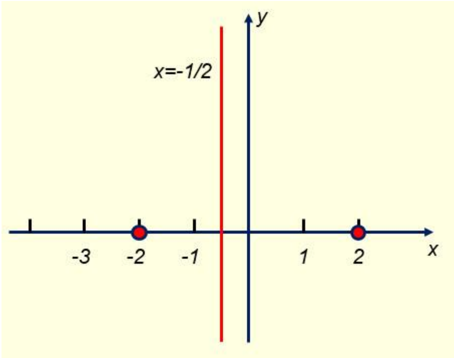
(a) In this case, $F(p) = 2x^2 + 2y^2 + 2z^2 + 2 \geq 2$ for any 3D point, no points would satisfy the implicit function, hence the implicit surface is an empty set.

(b) In this case, $F(p) = 2x^2 + 2y^2 + 2z^2 + 4 \geq 4$ for any 3D point, no points would satisfy the implicit function, hence the implicit surface is also an empty set.

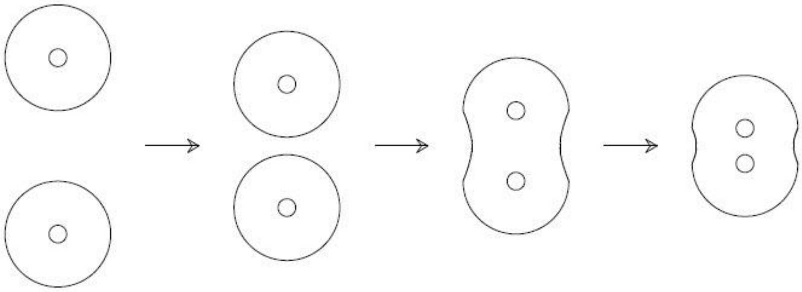
(c) In this case, $F(p) = -8x - 6$. The implicit function is satisfied when $x = -3/4$. Hence the implicit surface is a plane that is perpendicular to the x-axis at $x = -3/4$ (see the following figure).



(d) In this case, $F(p) = -8x - 4$. The implicit function is satisfied when $x = -1/2$. Hence the implicit surface is a plane that is perpendicular to the x-axis at $x = -1/2$ (see the following figure).



2. One technique to model the merging of two water drops



is to use the following implicit function

$$F(x,y,z) = \sum_{i=1}^{2} \frac{1}{(x-x_i)^2+(y-y_i)^2+(z-z_i)^2} - 1 \qquad (1)$$

where (x_1, y_1, z_1) is the center point of the first water drop and $(x_2, y_{2,} z_2)$ is the center point of the second water drop. Can this technique be extended to cover the merging of three water drops or even four and five water drops? Why or why not. Justify your answer. (10 points)

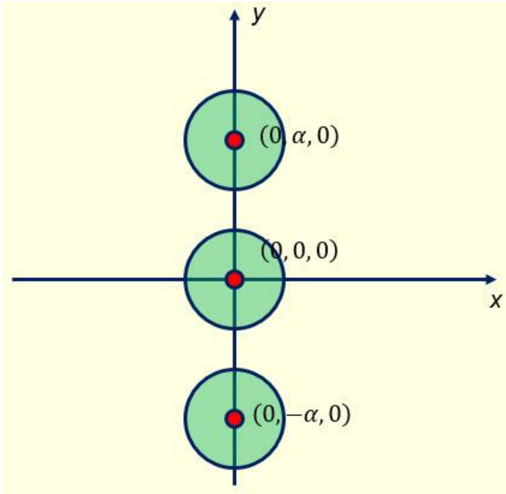
Sol.

This technique can be extended to any number of water drops. If n water drops are considered, simply change equation (1) to the following form

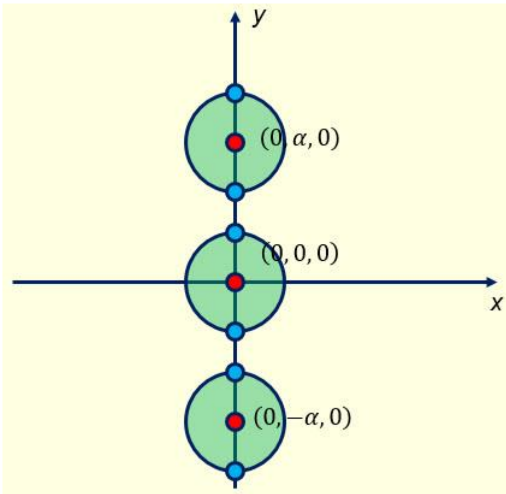
$$F(x,y,z) = \sum_{i=1}^{n} \frac{\beta_i}{(x-x_i)^2+(y-y_i)^2+(z-z_i)^2} - T \qquad (2)$$

where β_i, $0 \le i \le n$, and T are weights to be set by the user and (x_i, y_i, z_i) is the center point of drop i.

To show that this is indeed possible, consider the case $n = 3$ for equation (2) with $\beta_i = 1$ for all i, $T = 1$, $(x_1, y_1, z_1) = (0, \alpha, 0)$, $(x_2, y_2, z_2) = (0, 0, 0)$, and $(x_3, y_3, z_3) = (0, -\alpha, 0)$ with α being a positive number. We will show that when α is large enough, we have three separate components such as the case show in the following figure, and when (x_1, y_1, z_1) and (x_3, y_3, z_3) are moved toward (x_2, y_2, z_2), these three components will merge into a single component.



To show that we would have three separate component when α is large enough, we will show that the implicit function would have six intersection points with the y-axis when α is large enough. These intersection points are shown as blue dots in the following figure.



Note that the implicit function in this case is of the following form:

$$F(x,y,z) = \frac{1}{x^2+(y-\alpha)^2+z^2} + \frac{1}{x^2+y^2+z^2} + \frac{1}{x^2+(y+\alpha)^2+z^2} - 1 \tag{3}$$

Intersection points of (3) with the y-axis satisfy the following equation

$$\frac{1}{(y-\alpha)^2} + \frac{1}{y^2} + \frac{1}{(y+\alpha)^2} = 1 \tag{4}$$

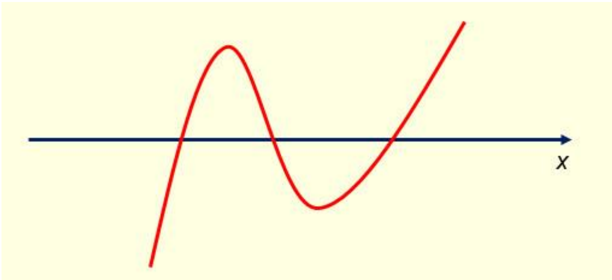
Equation (4) is equivalent to

$$y^6 - (4\alpha^2+3)y^4 + \alpha^4 y^2 - \alpha^4 = 0 \tag{5}$$

Set $x = y^2$, equation (5) can be written as

$$f(x) = x^3 - (4\alpha^2 + 3)x^2 + \alpha^4 x - \alpha^4 = 0 \quad (6)$$

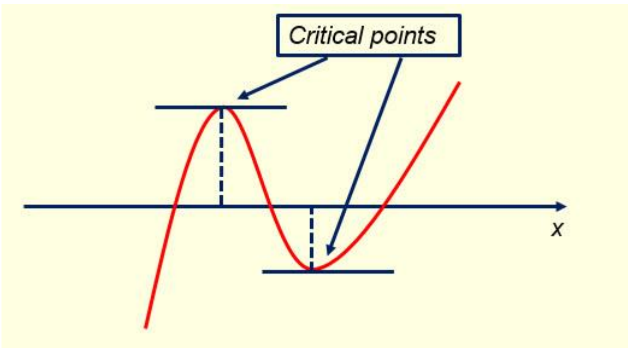
Solutions of equation (6) are the intersection points with the x-axis



However, we need to show that these intersection points have positive x coordinates when α is large enough. By taking the derivative of equation (6) and set it to zero, we get the critical points of (6)

$$f'(x) = 3x^2 - (8\alpha^2 + 6)x + \alpha^4 = 0$$

$$x = \frac{8\alpha^2 + 6 \pm \sqrt{52\alpha^2 + 96\alpha + 36}}{6}$$



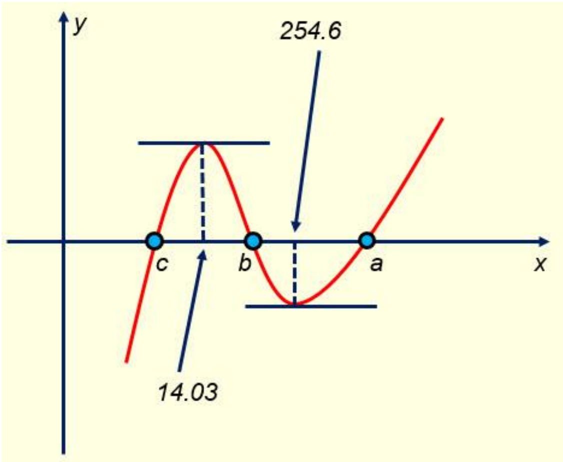
Now consider the case $\alpha = 10$. In this case, the critical points are $x = 14.03$ and $x = 254.6$ (see the following figure). We have

$$f(255) = -7083700 < 0$$
$$f(14) = 62645 > 0$$
$$f(0) = -10000 < 0$$

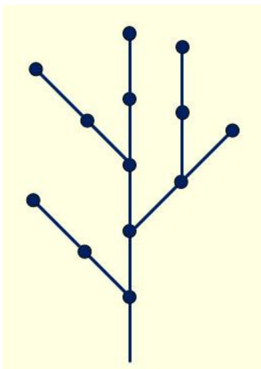
Hence all the roots of $f(x)$ are positive. Let a, b, c be the roots of $f(x)$ (see the following figure). Then we have six solutions for y:

$$y = \pm\sqrt{a} \qquad y = \pm\sqrt{b} \qquad y = \pm\sqrt{c}$$

Hence, the implicit function has six intersection points with the y-axis. It is obvious the three separate components of the implicit surface will merge into one component when the center points of these components move toward each other.



3. Find the terminal string whose corresponding graphics produced by the turtle interpretation is as follows. See slides 15 and 17 of the notes: "Special Models for Animation II" for the definitions of F, +, -, [and]. (10 points).



Sol.

F[+FF]F[−F[+FF]F]F[+FF]FF

4. For the example (Fractal plant) given in slide 32 of the notes: "Special Modesl for Animation II", show the corresponding graphics produced by the turtle interpretation when n=3 . (10 points)

Sol.

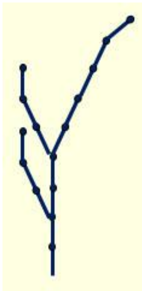
When n=3, the terminal string is of the following form:

FFFF-[[Z]+Z]+FFFF[+FFFFZ]-Z ……………… (1)

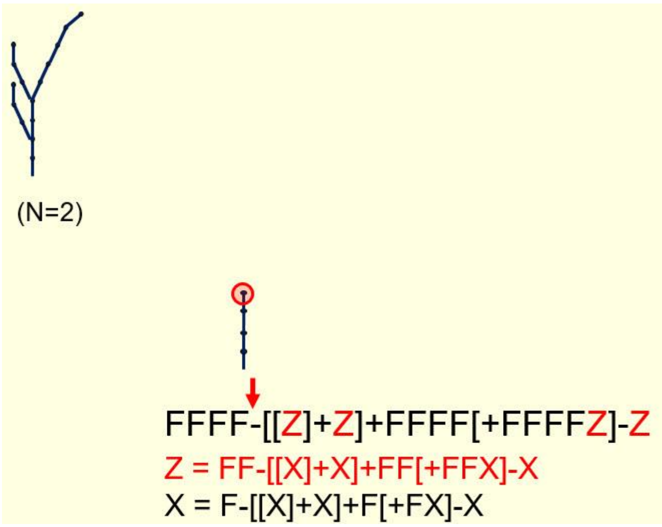
with Z = FF-[[Y]+Y]+FF[+FFY]-Y

and Y=F-[[X]+X]+F[+FX]-X

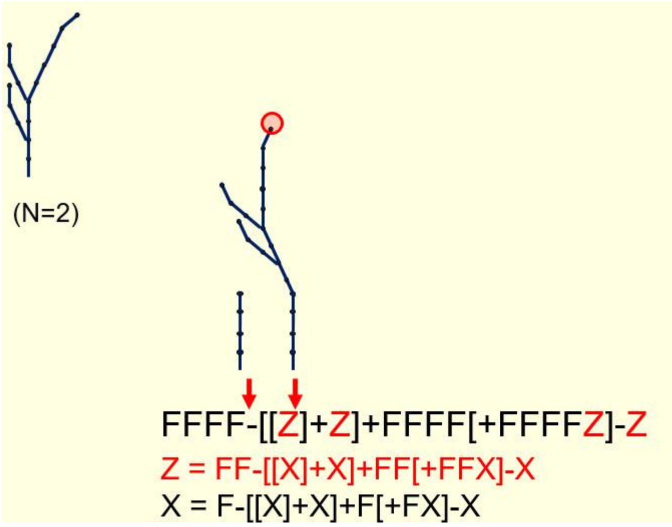
Each occurrence of Z is a branch of the following form (why):



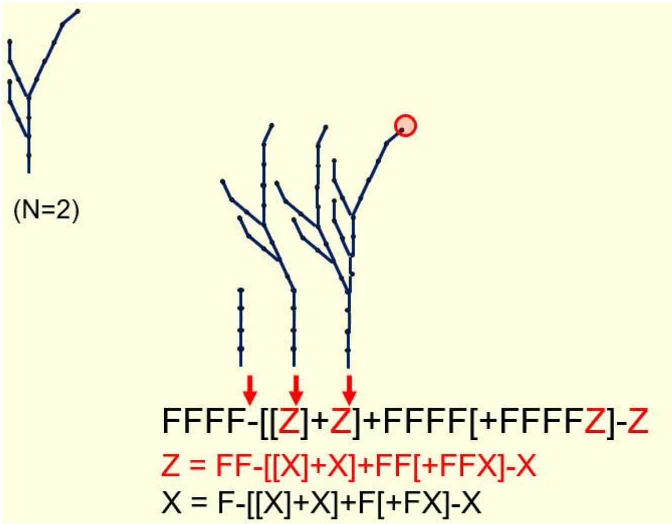
When we are at the point marked by the first red arrow, we have



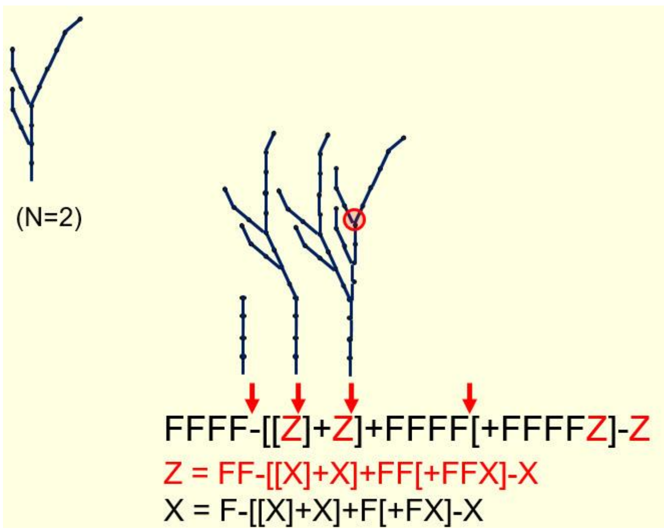
When we are at the point marked by the second red arrow, we have



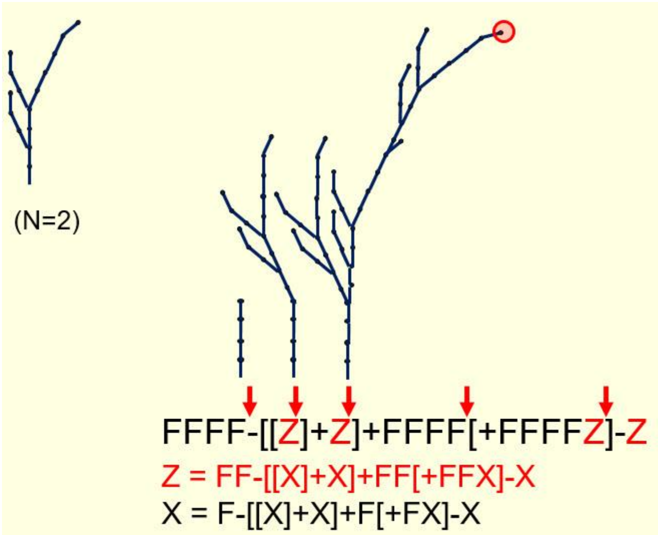
When we are at the point marked by the third red arrow, we have



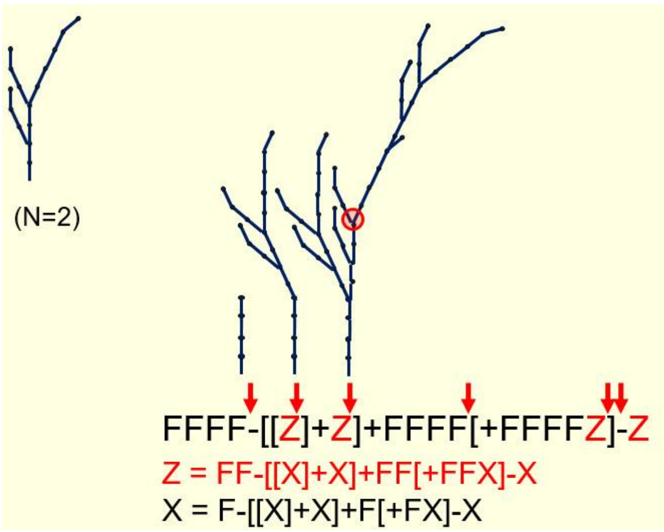
When we are at the point marked by the fourth red arrow, we have



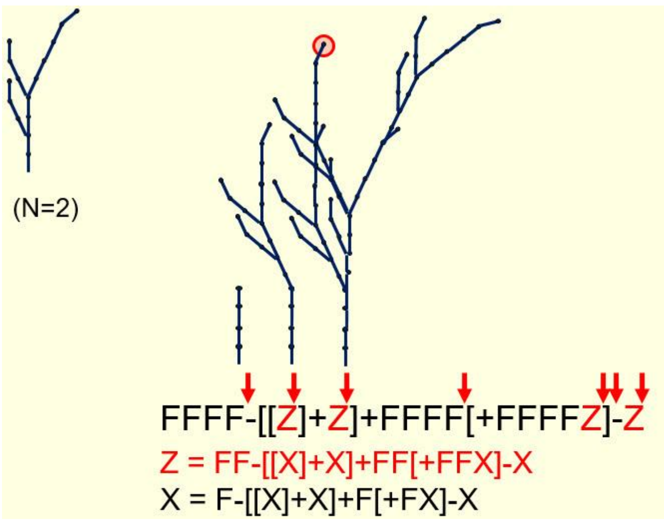
When we are at the point marked by the fifth red arrow, we have



When we are at the point marked by the sixth red arrow, we have



When we are at the point marked by the seventh red arrow, we have



Hence, when n=3, we have