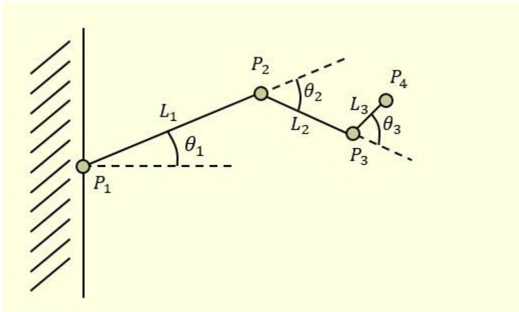
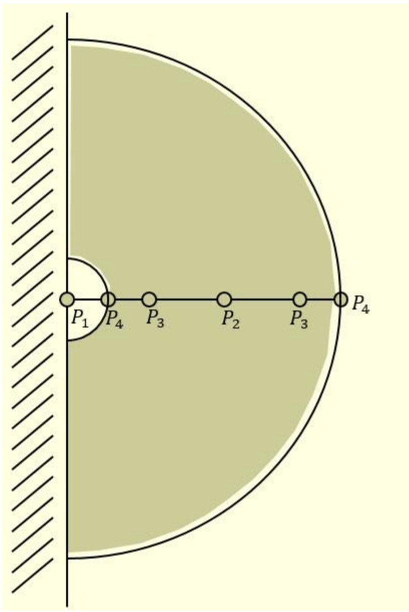
CS633 3D Computer Animation
Solution Set - HW 5 (40 points)
Due: 4/5/2018

1. Use a drawing to show the "reachable workspace" of the following robot arm. Here we assume $|L_2| = 2|L_3| = |L_1|/2$, θ_2 and θ_3 can be any value, and $-\pi/2 \le \theta_1 \le \pi/2$. (5 points)



Sol.



2. When solving a *kinematic modeling problem* (such as moving the end effector of a robotic manipulator from one point to another point), we prefer *iterative numeric method* to *analytic method*. Why? If necessary, use an example to justify your answer. (5 points)

Sol.

Because analytic solutions are not tractable in many cases. Sometime it is not even possible to find analytic solution for a three-link arm. On the other hand, iterative numeric method can always provide us with a solution, no matter how complicated the system is, as long as the desired new location is reachable.

3. One possible way to find a solution to an underdetermined system like the following one

$$M\,\mathbf{X} = \mathbf{Y}$$

(M is an $m \times n$ matrix with $n > m$, **X** is an unknown vector of dimension n and **Y** is a constant vector of dimension m) is to solve the following system for **X**. Why?

$$(M^T M)\,\mathbf{X} = M^T \mathbf{Y}$$

Note that here $M^T M$ is a square matrix of dimension $n \times n$ and $M^T\mathbf{Y}$ is a constant vector of dimension n. This is a very important technique in solving an underdetermined system (of course, important for us as well). (10 points)

Sol.

Note that if we define $F(\mathbf{X})$ as follows:

$$F(\mathbf{X}) \equiv (M\mathbf{X} - \mathbf{Y})^T (M\mathbf{X} - \mathbf{Y}),$$

we get a non-negative function whose minimum occurs at a point **X** where Eq. (*) is satisfied (why?).

Hence, to find a solution for (*), we simply compute the derivative of $F(\mathbf{X})$ with respect to **X**, set it to zero, and solve for **X**. Note that

$$F(\mathbf{X}) = (\mathbf{X}^T M^T - \mathbf{Y}^T)(M\mathbf{X} - \mathbf{Y})$$

$$= \mathbf{X}^T M^T M\mathbf{X} - \mathbf{X}^T M^T \mathbf{Y} - \mathbf{Y}^T M\mathbf{X} + \mathbf{Y}^T \mathbf{Y}.$$

Since $X^T M^T Y = Y^T M X$, we have

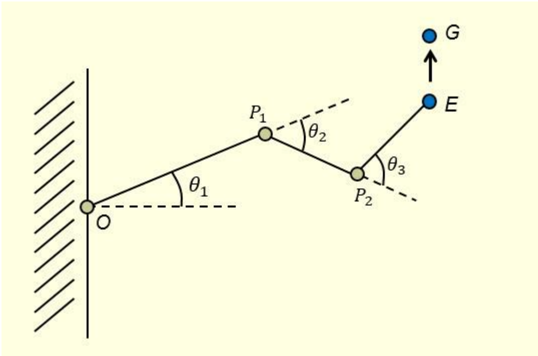
$$F(\mathbf{X}) = \mathbf{X}^T M^T M\mathbf{X} - 2\mathbf{X}^T M^T \mathbf{Y} + \mathbf{Y}^T \mathbf{Y}$$

By differentiating $F(\mathbf{X})$ with respect to **X** and setting it to zero,

$$\frac{dF(\mathbf{X})}{d\mathbf{X}} = 2M^T M\mathbf{X} - 2M^T \mathbf{Y} = 0$$

we get (**). Hence, solving (*) is equivalent to solving (**) for **X**.

4. For a robotic manipulator with four joints (see the following figure), what is the corresponding $V = J \cdot \dot{\theta}$ if we want to move the end effector **E** to the global location **G**. The origin of the coordinate system is at **O** and orientation of the end effector is of no concern. (5 points)



Sol.

$$\begin{bmatrix} (\mathbf{G}-\mathbf{E})_x \\ (\mathbf{G}-\mathbf{E})_y \\ (\mathbf{G}-\mathbf{E})_z \end{bmatrix} = \begin{bmatrix} (\mathbf{Z}\times\mathbf{E})_x & (\mathbf{Z}\times(\mathbf{E}-\mathbf{P}_1))_x & (\mathbf{Z}\times(\mathbf{E}-\mathbf{P}_2))_x & (\mathbf{Z}\times(\mathbf{E}-\mathbf{P}_3))_x \\ (\mathbf{Z}\times\mathbf{E})_y & (\mathbf{Z}\times(\mathbf{E}-\mathbf{P}_1))_y & (\mathbf{Z}\times(\mathbf{E}-\mathbf{P}_2))_y & (\mathbf{Z}\times(\mathbf{E}-\mathbf{P}_3))_y \\ (\mathbf{Z}\times\mathbf{E})_z & (\mathbf{Z}\times(\mathbf{E}-\mathbf{P}_1))_z & (\mathbf{Z}\times(\mathbf{E}-\mathbf{P}_2))_z & (\mathbf{Z}\times(\mathbf{E}-\mathbf{P}_3))_z \end{bmatrix} \cdot \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

where $\mathbf{Z} = (0, 0, 1)$.

5. The purpose of adding a "control expression" to a pseudo-inverse Jacobian solution is to better control the kinematic model. In the above example, if we want to move the end effector (**E**) to a new location **G**, and if we would like the rotation to be performed mostly on the third joint $\mathbf{P_2}$, then how should the "control expression" be defined in this case? (5 points)

Sol.

Choose relatively small α_3 and relatively large α_1 and α_2 in the following control expression:

$$H = \alpha_1(\theta_1 - \theta_{c1})^2 + \alpha_2(\theta_2 - \theta_{c2})^2 + \alpha_3(\theta_3 - \theta_{c3})^2$$

6. In the paper "Surface Simplification Using Quadric Metrics", the squared distance (error) of a point **v** = (x, y, z) to a plane can be defined as $\Delta(\mathrm{v}) = \mathrm{vQv^T}$ for a symmetric matrix **Q** (slide 13 of notes: Special Models for Animation I). Why? (10 points)

Sol.

The distance between a point **v** = (x, y, z) and a plane (a, b, c, d) is:

$$\frac{|\, ax + by + cz + d \,|}{\sqrt{a^2 + b^2 + c^2}}$$

If the normal of the plane is normalized, i.e., $\sqrt{a^2 + b^2 + c^2} = 1$, then this distance can be expressed as

$$\frac{|\, ax + by + cz + d \,|}{\sqrt{a^2 + b^2 + c^2}} = |\, ax + by + cz + d \,|$$

$$= \left| [a, b, c, d] \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \right|$$

$$= |\, \mathbf{p}^T \cdot \mathbf{v} \,|$$

$\mathbf{p}^T \cdot \mathbf{v}$ is a number. We have $(\mathbf{p}^T \cdot \mathbf{v})^T = \mathbf{p}^T \cdot \mathbf{v}$. Hence, the squared distance can be expressed as

$$\Delta \equiv |\, \mathbf{p}^T \cdot \mathbf{v} \,|^2 = (\mathbf{p}^T \cdot \mathbf{v}) \cdot (\mathbf{p}^T \cdot \mathbf{v}) = (\mathbf{p}^T \cdot \mathbf{v})^T \cdot (\mathbf{p}^T \cdot \mathbf{v}) = \mathbf{v}^T \mathbf{p}\mathbf{p}^T \mathbf{v}$$

$$= \mathbf{v}^T \, \mathbf{Q} \, \mathbf{v}$$

where $\mathbf{Q} \equiv \mathbf{p}\mathbf{p}^T$ is a 4×4 matrix. $\mathbf{Q}$ is a symmetric matrix because $\mathbf{Q}^T = (\mathbf{p}\mathbf{p}^T)^T = \mathbf{p}\mathbf{p}^T = \mathbf{Q}$.