CS 633 3D Computer Animation

Solution Set - HW 2 (40 points)

1. Let q be a rotation quaternion, i.e., q is a quaternion of the following form

$$q = [\cos(\theta), \sin(\theta)\,(x, y, z)]$$

with (x, y, z) being a unit vector. The question you need to answer here is: Does $-q$ represent the same rotation? Justify your answer. (5 points)

Sol:

Yes. Note that $(-q)^* = -(q^*)$. Therefore,

$$(-q)[0, \vec{r}](-q)^* = (-q)[0, \vec{r}]\big(-(q^*)\big) = q[0, \vec{r}]q^*$$

for any $[0, \vec{r}]$.

2. Each rotation quaternion is a unit quaternion (a quaternion with unit length). Is a unit quaternion also a rotation quaternion? Justify your answer. (5 points)

Sol:

Yes. If $q = [w, (x, y, z)]$ is a unit quaternion, i.e., $|q| = \sqrt{w^2 + x^2 + y^2 + z^2} = 1$, then by defining

$$cos\theta = w$$
$$sin\theta = \sqrt{x^2 + y^2 + z^2}$$
$$\vec{u} = \frac{(x,y,z)}{sin\theta}$$

q can be expressed as $[cos\theta,\ sin\theta\,\vec{u}]$ with $\vec{u}$ being an unit vector. Hence, q is also a rotation quaternion.

3. I have shown you in class how to find the control points of a uniform cubic B-spline curve that interpolates six given data points $\mathbf{D}_0$, $\mathbf{D}_1$, $\mathbf{D}_2$, $\mathbf{D}_3$, $\mathbf{D}_4$ and $\mathbf{D}_5$ (pages 17-24 of Interpolating Values I). This curve has five segments. So, it needs eight control points $\mathbf{P}_0$, $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, $\mathbf{P}_4$, $\mathbf{P}_5$, $\mathbf{P}_6$ and $\mathbf{P}_7$. To find these control points, you need to solve a system of four equations to find $\mathbf{P}_2$, $\mathbf{P}_3$, $\mathbf{P}_4$ and $\mathbf{P}_5$ first, and then compute $\mathbf{P}_0$ and , $\mathbf{P}_7$. You don't have to compute $\mathbf{P}_1$ and $\mathbf{P}_6$ because they are equal to $\mathbf{D}_0$ and $\mathbf{D}_5$, respectively. Now, here is the question: to build a uniform cubic B-spline curve to interpolate 11 data points: $\mathbf{D}_0$, $\mathbf{D}_1$, ..., $\mathbf{D}_8$ and $\mathbf{D}_9$, how many segments should it have? and how many control points should this curve have? (2.5 points)

Sol.

9 segments, 12 control points.

4. To build a uniform cubic B-spline curve to interpolate 10 data points: $\mathbf{D}_0$, $\mathbf{D}_1$, ..., $\mathbf{D}_8$ and $\mathbf{D}_9$, we also need to solve a system of equations for some of the control points. How big is this system (i.e., how many equations are there in this system)? Which control points (using the notations of Question 1) will be computed from this system? (2.5 points)

Sol.

A system of 8 equations. The control points to be solved from these equations are $\mathbf{P}_2$, $\mathbf{P}_3$, ..., $\mathbf{P}_8$ and $\mathbf{P}_9$.

The remaining control points are computed as follows:

$$\mathbf{P}_1 = \mathbf{D}_0,\ \mathbf{P}_{10} = \mathbf{D}_9$$

$$\mathbf{P}_0 = 2\mathbf{P}_1 - \mathbf{P}_2,\ \mathbf{P}_{11} = 2\mathbf{P}_{10} - \mathbf{P}_9$$

5. Here is a more specific question. What is the exact form of the system of equations to be solved in Question 4? Give your answer in matrix form. (5 points)

Sol.

The matrix form is as follows:

$$\begin{bmatrix} 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \\ P_8 \\ P_9 \end{bmatrix} = \begin{bmatrix} 6D_1 - D_0 \\ 6D_2 \\ 6D_3 \\ 6D_4 \\ 6D_5 \\ 6D_6 \\ 6D_7 \\ 6D_8 - D_9 \end{bmatrix}$$

Once we have $\mathbf{P}_2$, $\mathbf{P}_3$, ... $\mathbf{P}_8$, and $\mathbf{P}_9$, we then use the following equations to compute the remaining control points.

$$\mathbf{P}_1 = \mathbf{D}_0,\ \mathbf{P}_{10} = \mathbf{D}_9,\ \mathbf{P}_0 = 2\mathbf{P}_1 - \mathbf{P}_2,\ \mathbf{P}_{11} = 2\mathbf{P}_{10} - \mathbf{P}_9.$$

6. If we use the approach covered in slide 10-16 to form a closed B-spline curve to interpolate these 10 data points but with $P_0 = P_9$, then what system of equations should we solve to get the control points, in matrix form? (5 points)

Sol.

The following system:

$$\begin{bmatrix} 4 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & 0 & 0 & 0 \\ & & & \vdots & & & & \\ 0 & 0 & 0 & 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ \vdots \\ P_7 \\ P_8 \\ P_9 \end{bmatrix} = \begin{bmatrix} 6D_0 \\ 6D_1 \\ 6D_2 \\ \vdots \\ 6D_6 \\ 6D_7 \\ 6D_8 \end{bmatrix}$$

The remaining points P_0, P_{10}, P_{11} can then be found from the following equations:

$$P_0 = P_9,\ P_{10} = P_1,\ P_{11} = P_2.$$

7. If we want the user to have an impression that the object is moving in constant speed, we need to parametrize the path curve by arc length. For a B-spline curve, how should the re-parametrization process be done? Note that a B-spline curve usually has several segments and each segment is defined differently. (5 points)

Sol:

Let $C(t)$ be a cubic B-spline curve with n segments $C_1(t)$, $C_2(t)$, ..., $C_n(t)$. To reparametrize $C(t)$, we need to build an arc length table first. The arc length table is built as follows.

For $0 \le t \le 1$, the arc length is computed using the expression $C(t) = C_1(t)$.

If L_1 is the length of $C_1(t)$, then for $1 \le t \le 2$ the arc length is computed by first computing the arc length of $C(t) = C_2(t-1)$ and then adding this arc length to L_1.

In general, if L_1, L_2, ..., L_{k-1} are the lengths of the segments $C_1(t)$, $C_2(t)$, ... $C_{k-1}(t)$, then for $k-1 \le t \le k$ the arc length is computed by first computing the arc length of $C(t) = C_k(t - k + 1)$ and then adding this arc length to $L_1 + L_2 + \ldots + L_{k-1}$.

8. Prove the third equation on slide 32 of notes: Interpolating Values II, i.e.,

$$s_3(t) = \frac{v_0 t_1}{2} + v_0(t_2 - t_1) + [v_0 - \frac{v_0(t - t_2)}{2(1 - t_2)}](t - t_2)$$

when $t_2 < t < 1$. To prove the above equation, you need to show the following equation first. (5 points)

$$v_3(t) = -\frac{v_0}{1 - t_2}(t - t_2) + v_0$$

Sol.

First note that

$$s_3(t) = \int_{t_2}^{t} v_3(t)\ dt\ + s_2(t_2)\,, \qquad t_2 \le t \le 1$$

Since

$$s_2(t_2) = \frac{v_0 t_1}{2} + v_0(t_2 - t_1)$$

we have

$$s_3(t) = \int_{t_2}^{t} v_3(t)\ dt\ + \frac{v_0 t_1}{2} + v_0(t_2 - t_1).$$

The acceleration between t_2 and 1 is a constant $-b$, so the velocity between t_2 and 1 can be expressed as

$$v_3(t) = \int_{t_2}^{t} -bdt\ + v_2(t_2) = -\,b(t - t_2) + v_2(t_2)$$

The value of $v_2(t_2)$ is v_0 and the value of $v_3(t)$ at $t = 1$ equals 0. Therefore,

$$b = \frac{v_0}{1 - t_2}$$

and so

$$s_3(t) = \int_{t_2}^{t} [-\frac{v_0}{1 - t_2}(t - t_2) + v_0\ ]\ dt + \frac{v_0 t_1}{2} + v_0(t_2 - t_1)$$

$$= -\frac{v_0(t - t_2)^2}{2(1 - t_2)} + v_0(t - t_2) + \frac{v_0 t_1}{2} + v_0(t_2 - t_1)$$

9. There are two approaches to define a *spherical linear interpolation* (see slides 5 and 8 of notes: Interpolating Values III for the definitions of this term) between two unit quaternions. In the second approach, *slerp*($q1$, $q2$; u) is defined as follows:

$$slerp(q_1, q_2; u) = \frac{\sin((1 - u)\theta)}{\sin\theta} q_1 + \frac{\sin(u\theta)}{\sin\theta} q_2$$

Prove the second approach generates the same curve as the first approach. (5 points)

Sol.

We will show that for each $0 \le u \le 1$, we have

$$q_1(q_1^{-1}q_2)^u = \frac{sin((1-u)\theta)}{sin\theta} q_1 + \frac{sin(u\theta)}{sin\theta} q_2 \qquad (**)$$

where $cos\theta = q_1 \cdot q_2$.

Let $q_1 = [\cos\alpha, v\sin\alpha]$, $q_2 = [\cos\beta, v\sin\beta]$, where v is a unit vector in the direction of the rotation axis.

$$\begin{aligned} q_1(q_1^{-1}q_2)^u &= q_1([\cos\alpha, -v\sin\alpha][\cos\beta, v\sin\beta])^u \\ &= q_1([\cos\alpha\ \cos\beta + \sin\alpha\ \sin\beta, v\cos\alpha\ \sin\beta - v\sin\alpha\ \cos\beta])^u \\ &= q_1([\cos(\beta-\alpha),\ v\ \sin(\beta-\alpha)]^u \end{aligned}$$

By setting $\theta \equiv \beta - \alpha$, we have

$$\begin{aligned} q_1(q_1^{-1}q_2)^u &= [\cos\alpha, v\sin\alpha][cos\theta, v\sin\theta]^u \\ &= [\cos\alpha, v\sin\alpha][cos(u\theta), v\sin(u\theta)] \\ &= [\cos\alpha\ \cos(u\theta) - \sin\alpha\ \sin(u\theta), v\cos\alpha\ \sin(u\theta) + v\sin\alpha\ \cos(u\theta)] \\ &= [cos(\alpha+u\theta), v\sin(\alpha+u\theta)] \end{aligned}$$

We claim that

$$\cos(\alpha+u\theta) = \frac{\cos\alpha\ \sin((1-u)\theta) + \cos\beta\ \sin(u\theta)}{\sin\theta} \tag{1}$$

$$\sin(\alpha+u\theta) = \frac{\sin((1-u)\theta)\ \sin\alpha + \sin(u\theta)\ \sin\beta}{\sin\theta} \tag{2}$$

To prove (1), note that

$$\sin\alpha\ \sin\beta = \cos\theta - \cos\alpha\ \cos\beta.$$

Hence,

$$\begin{aligned} \cos(\alpha+u\theta) &= \cos\alpha\ \cos(u\theta) - \sin\alpha\ \sin(u\theta) \\ &= \cos\alpha\ \cos(u\theta) - \sin\alpha\ \sin(u\theta)\,\frac{\sin(\beta-\alpha)}{\sin\theta} \\ &= \cos\alpha\ \cos(u\theta) - \sin\alpha\ \sin(u\theta)\left(\frac{\cos\alpha\ \sin\beta - \sin\alpha\ \cos\beta}{\sin\theta}\right) \\ &= \cos\alpha\ \cos(u\theta) - \frac{\sin\alpha\ \sin\beta\ \cos\alpha\ \sin(u\theta) - \sin^2\alpha\ \cos\beta\ \sin(u\theta)}{\sin\theta} \\ &= \cos\alpha\ \cos(u\theta) - \frac{(cos\theta - \cos\alpha\ \cos\beta)\cos\alpha\ \sin(u\theta) - \sin^2\alpha\ \cos\beta\ \sin(u\theta)}{\sin\theta} \\ &= \frac{\cos\alpha\ \cos(u\theta)\ \sin\theta - \cos\theta\ \cos\alpha\ \sin(u\theta) + \cos^2\alpha\ \cos\beta\ \sin(u\theta) + \sin^2\alpha\ \cos\beta\ \sin(u\theta)}{\sin\theta} \\ &= \frac{\cos\alpha\ (\sin\theta\ \cos(u\theta) - \cos\theta\ \sin(u\theta)) + \cos\beta\ \sin(u\theta)}{\sin\theta} \\ &= \frac{\cos\alpha\ \sin((1-u)\theta)) + \cos\beta\ \sin(u\theta)}{\sin\theta}. \end{aligned}$$

To prove (2), recall that $\theta = \beta - \alpha$ and $\cos\theta = \cos\alpha\ \cos\beta + \sin\alpha\ \sin\beta$. Hence

$$\begin{aligned}\sin(\alpha + u\theta) &= \frac{\sin\theta}{\sin\theta}\left(\sin\alpha\ \cos(u\theta) + \cos\alpha\ \sin(u\theta)\right)\\ &= \frac{1}{\sin\theta}\left(\sin\theta\ \sin\alpha\cos(u\theta) + \sin(\beta-\alpha)\cos\alpha\ \sin(u\theta)\right)\end{aligned}$$