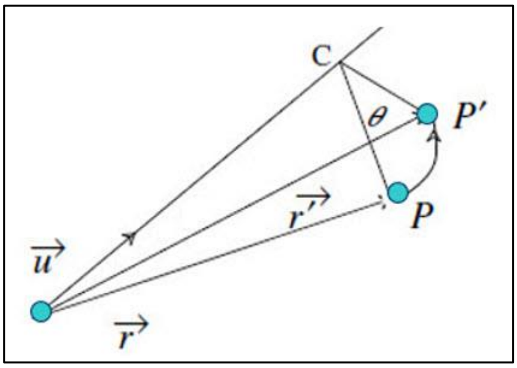
CS633 Computer Animation
Homework Assignment 1 (40 points)
Due: 2/6/2018

1. Your assignment for the first question is to download a copy of the "animated 2D bouncing sample program", compile it, test run it, fix a problem of this sample program, and turn a copy of your revised version of the sample program in on the due date.

   Solution:
   A better version of the sample program (with the problem of the sample program fixed) is attached underneath this solution set.

2. Fixed angle representation $(\theta_x, \ \theta_y, \theta_z)$ has problems when representing an arbitrary orientation and performing orientation interpolation. Using quaternions to represent points and rotations can avoid those problems, but it seems that rotation implemented using quaternions cannot be integrated with transformations implemented using homogeneous coordinates. Is this so? In the following figure, let

$\vec{u} = (u_x, u_y, u_z)^t$ - rotation axis, a unit vector
$\vec{r} = (r_x, \ r_y, \ r_z)^t$ - vector to be rotated
$\vec{r'} = (r_x', \ r_y', \ r_z')^t$ - vector after rotation
θ - rotation angle



Show that if $[0, \vec{r'}] = [cos\left(\frac{\theta}{2}\right), \ sin\left(\frac{\theta}{2}\right)\vec{u}] \cdot [0, \ \vec{r}] \cdot [cos\left(\frac{\theta}{2}\right), -sin\left(\frac{\theta}{2}\right)\vec{u}]$ then we have

$$\begin{bmatrix} r_x' \\ r_y' \\ r_z' \\ 1 \end{bmatrix} = \begin{bmatrix} cos\theta + (1-cos\theta)u_xu_x & (1-cos\theta)u_xu_y - sin\theta u_z & (1-cos\theta)u_xu_z + sin\theta u_y & 0 \\ (1-cos\theta)u_xu_y + sin\theta u_z & cos\theta + (1-cos\theta)u_yu_y & (1-cos\theta)u_yu_z - sin\theta u_x & 0 \\ (1-cos\theta)u_xu_z + sin\theta u_y & (1-cos\theta)u_yu_z + sin\theta u_x & cos\theta + (1-cos\theta)u_zu_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ r_z \\ 1 \end{bmatrix}$$

What does this mean?

Solution:

If we use q to represent $\left[cos\left(\frac{\theta}{2}\right),\ sin\left(\frac{\theta}{2}\right)\vec{u} \right]$ and q^{-1} to represent its conjugate $\left[cos\left(\frac{\theta}{2}\right), -sin\left(\frac{\theta}{2}\right)\vec{u} \right]$, then from slides 35 – 38 of the notes “Introduction and Technical Background”, we have

$$[0, \overrightarrow{r'}] = q \cdot [0, \vec{r}\,] \cdot q^{-1} = [0, cos\theta\vec{r} + (1-cos\theta)(\vec{r}\cdot\vec{u})\vec{u} + sin\theta(\vec{u}\otimes\vec{r})].$$

Hence

$$\overrightarrow{r'} = cos\theta\vec{r} + (1-cos\theta)(\vec{r}\cdot\vec{u})\vec{u} + sin\theta(\vec{u}\otimes\vec{r}) \qquad (1)$$

Using the properties that

$$(\vec{r}\cdot\vec{u})\vec{u} = \begin{bmatrix} (r_xu_x + r_yu_y + r_zu_z)u_x \\ (r_xu_x + r_yu_y + r_zu_z)u_y \\ (r_xu_x + r_yu_y + r_zu_z)u_z \end{bmatrix} = \begin{bmatrix} u_xu_x & u_xu_y & u_xu_z \\ u_xu_y & u_yu_y & u_yu_z \\ u_xu_z & u_yu_z & u_zu_z \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$$

and

$$\vec{u}\otimes\vec{r} = \begin{bmatrix} u_yr_z - u_zr_y \\ u_zr_x - u_xr_z \\ u_xr_y - u_yr_x \end{bmatrix} = \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$$

we can put (1) in matrix form as follows:

$$\overrightarrow{r'} = \begin{bmatrix} r_x' \\ r_y' \\ r_z' \end{bmatrix} = cos\theta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} + (1-cos\theta) \begin{bmatrix} u_xu_x & u_xu_y & u_xu_z \\ u_xu_y & u_yu_y & u_yu_z \\ u_xu_z & u_yu_z & u_zu_z \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$$

$$+ sin\theta \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$$

$$= \begin{bmatrix} cos\theta + (1 - cos\theta)u_x u_x & (1 - cos\theta)u_x u_y - sin\theta u_z & (1 - cos\theta)u_x u_z + sin\theta u_y \\ (1 - cos\theta)u_x u_y + sin\theta u_z & cos\theta + (1 - cos\theta)u_y u_y & (1 - cos\theta)u_y u_z - sin\theta u_x \\ (1 - cos\theta)u_x u_z - sin\theta u_y & (1 - cos\theta)u_y u_z + sin\theta u_x & cos\theta + (1 - cos\theta)u_z u_z \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} \qquad (2)$$

Hence by putting (2) in homogeneous coordinate form, we get

$$\begin{bmatrix} r_x' \\ r_y' \\ r_z' \\ 1 \end{bmatrix} = \begin{bmatrix} cos\theta + (1 - cos\theta)u_x u_x & (1 - cos\theta)u_x u_y - sin\theta u_z & (1 - cos\theta)u_x u_z + sin\theta u_y & 0 \\ (1 - cos\theta)u_x u_y + sin\theta u_z & cos\theta + (1 - cos\theta)u_y u_y & (1 - cos\theta)u_y u_z - sin\theta u_x & 0 \\ (1 - cos\theta)u_x u_z + sin\theta u_y & (1 - cos\theta)u_y u_z + sin\theta u_x & cos\theta + (1 - cos\theta)u_z u_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ r_z \\ 1 \end{bmatrix}.$$

This result shows that a rotation performed in quaternion form can be expressed as a vector-matrix multiplication in homogeneous coordinate form as well and, therefore, can also be accumulated with other geometric transformations such as translation and scaling.