3.1.3 Cubic Uniform B-Spline Curves

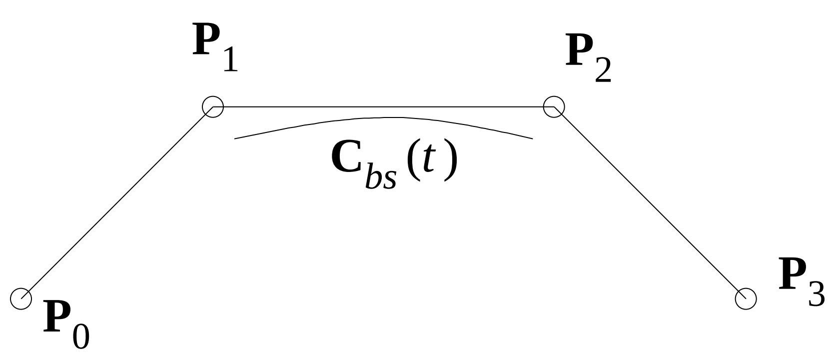
- A curve representation with **local property**

A Cubic Uniform B-Spline Curve segment

For four given control points $\mathbf{P}_0$, $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{P}_3$, a cubic uniform B-spline curve segment is defined as follows:

$$\mathbf{C}_{bs}(t) = \frac{(1-t)^3}{6}\mathbf{P}_0 + \frac{(4-6t^2+3t^3)}{6}\mathbf{P}_1 + \frac{(1+3t+3t^2-3t^3)}{6}\mathbf{P}_2 + \frac{t^3}{6}\mathbf{P}_3$$

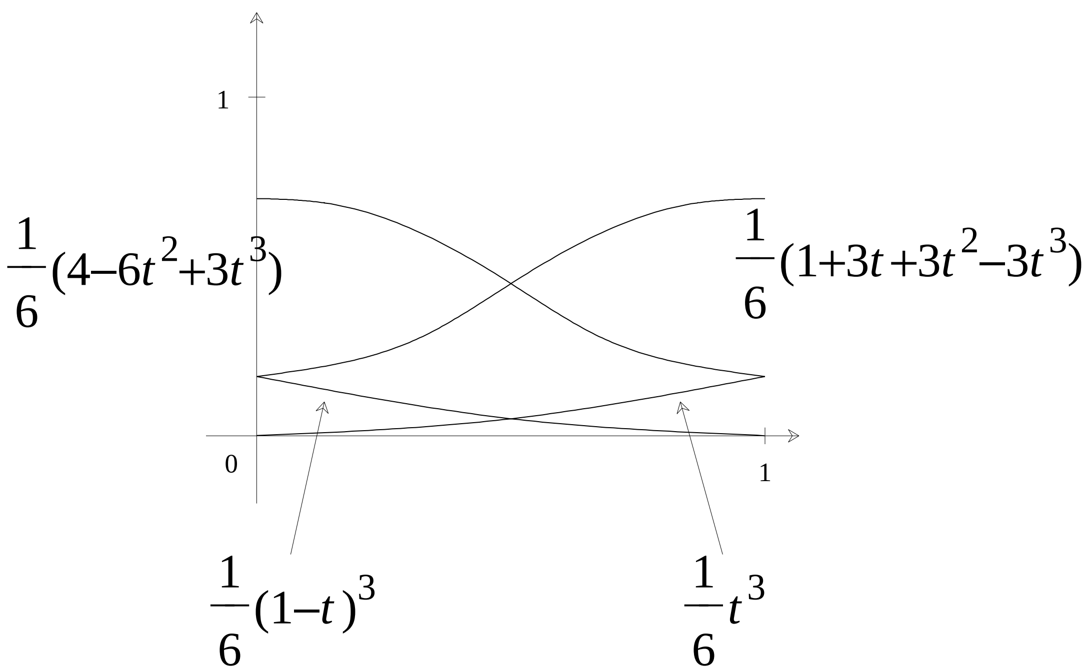
$$0 \le t \le 1$$



Matrix form

$$\mathbf{C}_{bs}(t) = [1, t, t^2, t^3] \, \frac{1}{6} \begin{bmatrix} 1 & 4 & 1 & 0 \\ -3 & 0 & 3 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{bmatrix} = \mathbf{T} \cdot \mathbf{M}_{bs} \cdot \mathbf{G}$$

Blending functions



Properties of B-spline blending fuunctions

- Non-negative
- Sum = 1
- Hence, again, a B-spline curve segment is always contained in the **convex hull** of its control points.
- However, $\mathbf{C}_{bs}(0) \neq \mathbf{P}_0$ and $\mathbf{C}_{bs}(1) \neq \mathbf{P}_3$. Actually

$$\mathbf{C}_{bs}(0) = \frac{1}{6}\mathbf{P}_0 + \frac{2}{3}\mathbf{P}_1 + \frac{1}{6}\mathbf{P}_2$$

$$\mathbf{C}_{bs}(1) = \frac{1}{6}\mathbf{P}_1 + \frac{2}{3}\mathbf{P}_2 + \frac{1}{6}\mathbf{P}_3$$

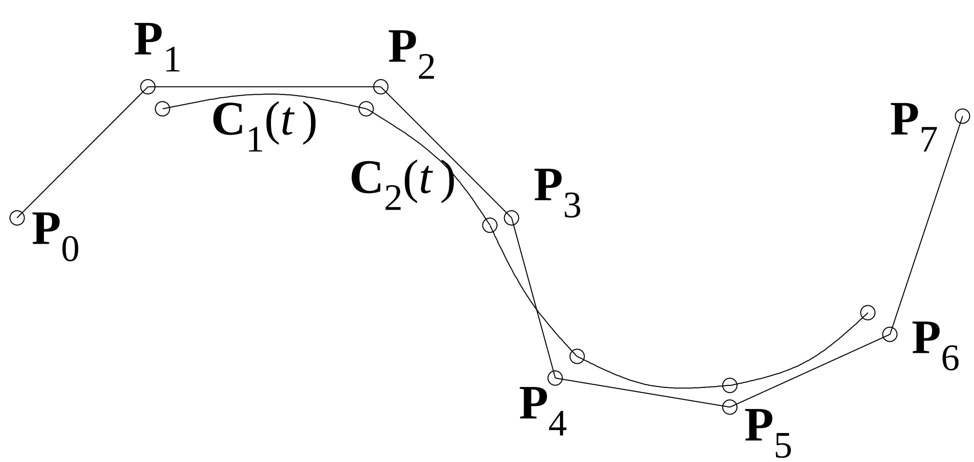
A Cubic Uniform B-Spline Curve

Given a set of n control points, one can define a cubic (uniform) B-spline curve with $(n-3)$ segments.

The first segment, $\mathbf{C}_1(t)$, is defined by the first four control points: $\mathbf{P}_0$, $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$.

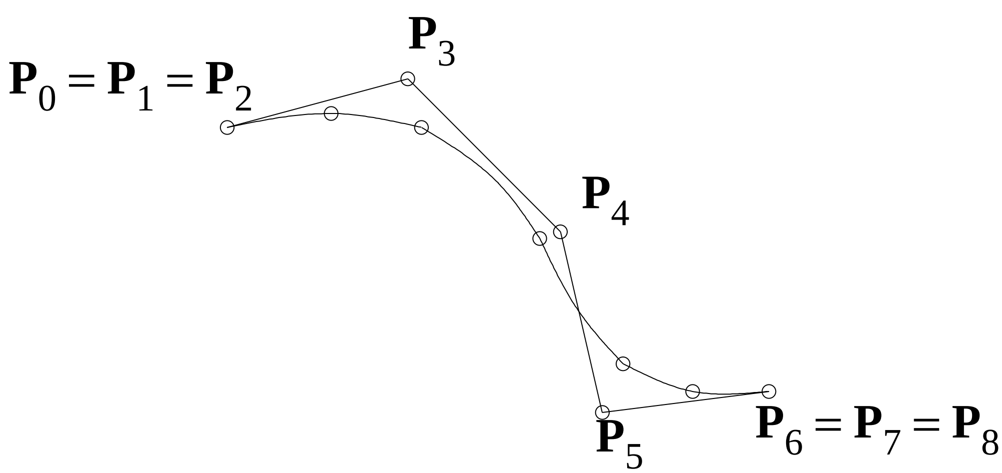
The second segment, $\mathbf{C}_2(t)$, is defined by the second four control points: $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, $\mathbf{P}_4$. ...

The last one, $\mathbf{C}_{n-3}(t)$, by $\mathbf{P}_{n-3}$, $\mathbf{P}_{n-2}$, $\mathbf{P}_{n-1}$, $\mathbf{P}_n$.



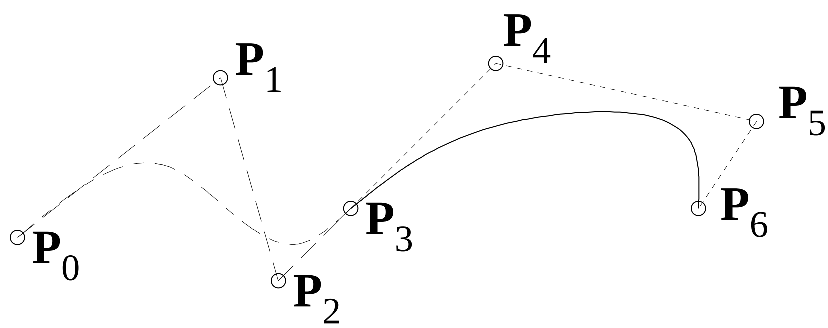
Properties/Advantages of a B-spline curve

- Local property (changing one control point will affect at most four segments)
- C^2 continuity at the joints
- Compact form for multiple segments
- Can use multiple control points to achieve exact point interpolation



3.1.4 Composite Bezier Curves

- Bezier curve segments can be joined together to form complicated shapes



$\mathbf{P}_0$, $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$ are control points of the 1st segment

$\mathbf{P}_3$, $\mathbf{P}_4$, $\mathbf{P}_5$, and $\mathbf{P}_6$ are control points of the 2nd segment

$\mathbf{P}_2$, $\mathbf{P}_3$, and $\mathbf{P}_4$ are collinear (to guarantee smooth joint)

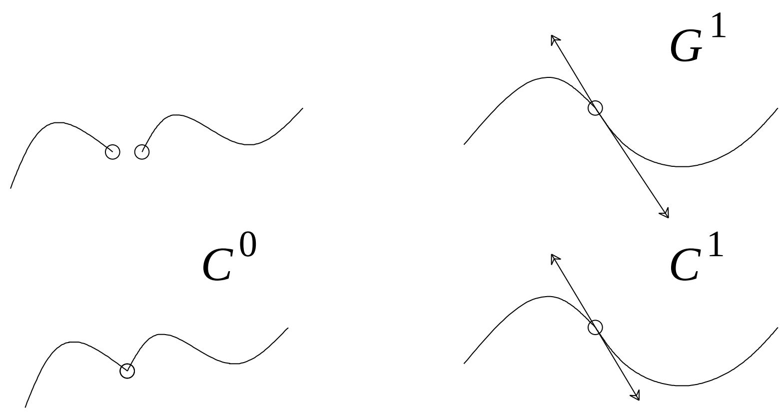
- Smoothness (continuity) at Join Points:

  C^0: the endpoints coincide

  G^1: tangents have the same slope

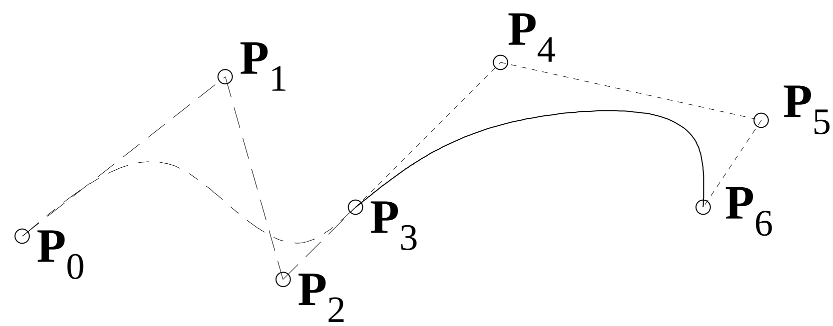
  C^1: the first derivatives on both segments match at join point

  C^2: nth derivatives on both segments match at join point



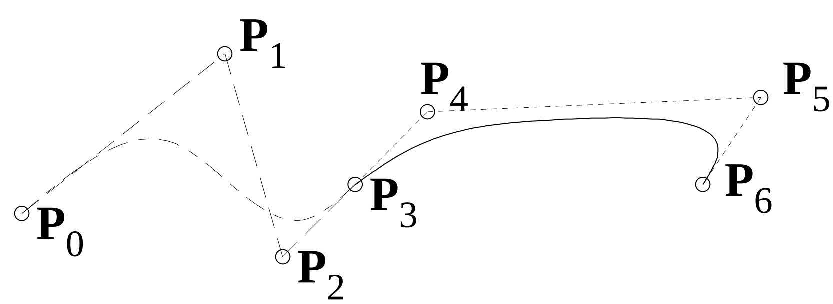
$\mathbf{P}_0$, $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$: control points of the 1st segment

$\mathbf{P}_3$, $\mathbf{P}_4$, $\mathbf{P}_5$, and $\mathbf{P}_6$: control points of the 2nd segment

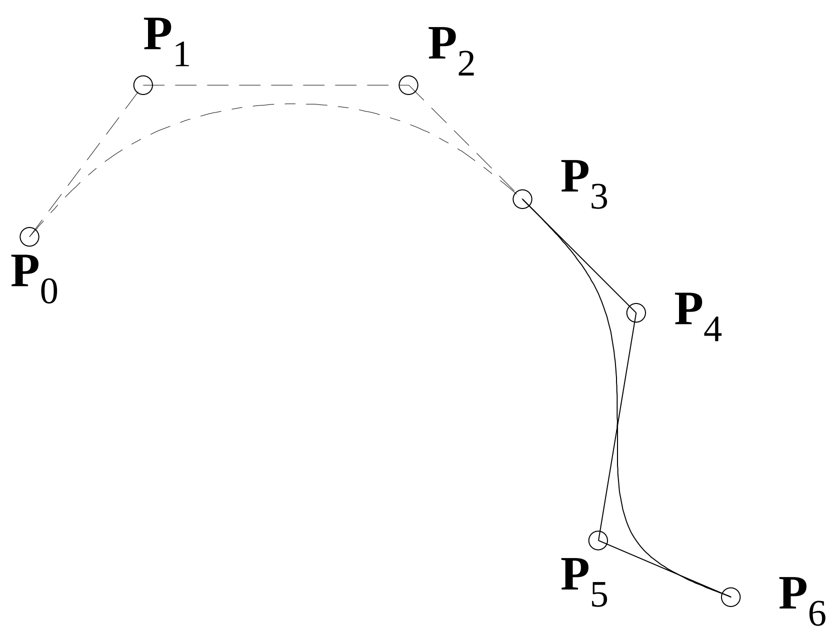


- G^1-continuity: $\mathbf{P}_2$, $\mathbf{P}_3$, and $\mathbf{P}_4$ are collinear (See the above example)

- C^1-continuity: $\mathbf{P}_2$, $\mathbf{P}_3$, and $\mathbf{P}_4$ are collinear and $\mathbf{P}_3$ is the midpoint of $\mathbf{P}_2\mathbf{P}_4$

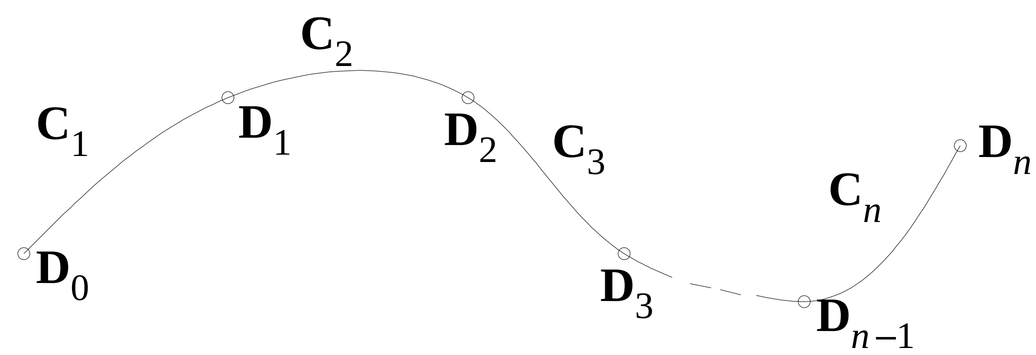


- C^2-continuity:
  - * $\mathbf{P}_2$, $\mathbf{P}_3$, and $\mathbf{P}_4$ are collinear
  - * $\mathbf{P}_3$ is the midpoint of $\mathbf{P}_2\mathbf{P}_4$
  - * $\mathbf{P}_5 = \mathbf{P}_1 + 4(\mathbf{P}_3 - \mathbf{P}_2)$



3.1.5 Curve Fitting using Composite Bezier Curves

- Give a set of data points $\mathbf{D}_0$, $\mathbf{D}_1$, ..., $\mathbf{D}_n$ $(n \geq 2)$, how can a composite cubic Bezier curve that interpolates these points be constructed?



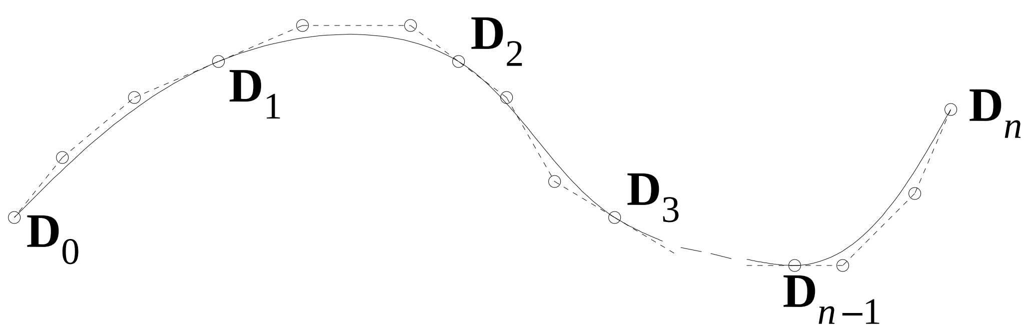
- The composite cubic Bezier curve has n segments $\mathbf{C}_1(t)$, $\mathbf{C}_2(t)$, ..., $\mathbf{C}_n(t)$ with $\mathbf{D}_{i-1}$ and $\mathbf{D}_i$ being the start and end points of $\mathbf{C}_i(t)$

- The composite cubic Bezier curve is C^2-continuous

An analysis of the problem:

- To get the curve constructed, how many control points are needed?

- But how many of them are known to us now?



- So, how many of them remain to be computed?

- And how should they be computed?

  (How should the C^1- and C^2-continuity conditions be used?)

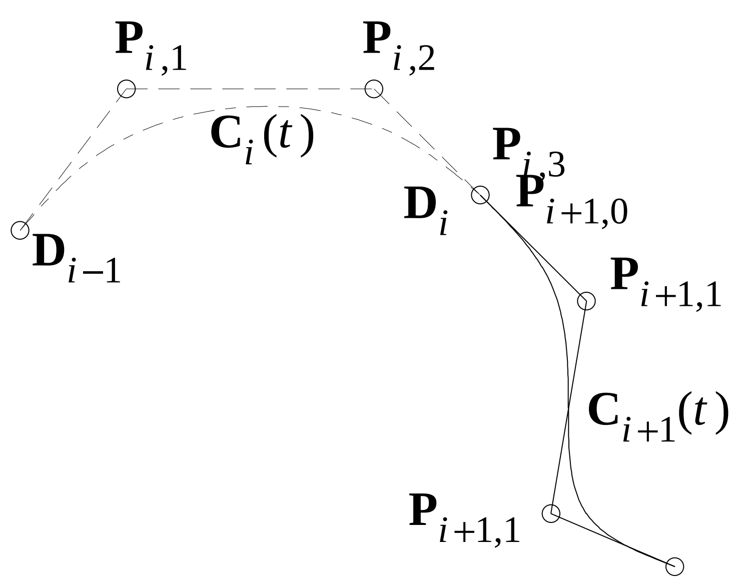
Let $\mathbf{P}_{i,0}$, $\mathbf{P}_{i,1}$, $\mathbf{P}_{i,2}$, $\mathbf{P}_{i,3}$ be the control points of the $\mathbf{C}_i(t)$.

Then for each two adjacent Bezier segments $\mathbf{C}_i(t)$ and $\mathbf{C}_{i+1}(t)$, we have

$$\mathbf{P}_{i,3} = \mathbf{D}_i = \mathbf{P}_{i+1,0}$$

$$\mathbf{P}_{i+1,1} - \mathbf{D}_i = \mathbf{D}_i - \mathbf{P}_{i,2}$$

$$\mathbf{P}_{i+1,2} - \mathbf{P}_{i,1} = 2(\mathbf{P}_{i+1,1} - \mathbf{P}_{i,2})$$



Hence, we have a system of $2(n-1)$ equations in $2n$ unknows $\left\{ \mathbf{P}_{i,1},\ \mathbf{P}_{i,2} \right\}_{i=1}^{n}$

$$\begin{cases} \mathbf{P}_{i,2} + \mathbf{P}_{i+1,1} = 2\mathbf{D}_i \\ \\ \mathbf{P}_{i,1} - 2\mathbf{P}_{i,2} + 2\mathbf{P}_{i+1,1} - \mathbf{P}_{i+1,2} = 0 \end{cases} \qquad i = 1, 2, ..., n-1 \tag{7.1}$$

Two **extra conditions** can be given as follows:

1. $\mathbf{P}_{1,1}$ and $\mathbf{P}_{n,2}$ are specified by the user, or
2. requiring the composite Bezier curve to have zero 2nd derivative at $\mathbf{D}_0$ and $\mathbf{D}_n$.

$$\begin{cases} \mathbf{C}_1''(0) = 6\,(\mathbf{P}_{1,2} - 2\mathbf{P}_{1,1} + \mathbf{P}_{1,0}) = 0 \\ \\ \mathbf{C}_n''(1) = 6\,(\mathbf{P}_{n,3} - 2\mathbf{P}_{n,2} + \mathbf{P}_{n,1}) = 0 \end{cases} \tag{7.2}$$

For instance, using the 2nd approach for the extra conditions, (7.2), together with (7.1), we get a system of $2n$ equations in $2n$ unknowns, as follows:

$$
\begin{bmatrix}
2 & -1 & & & & & & \\
0 & 1 & 1 & & & & & \\
1 & -2 & 2 & -1 & & & & \\
 & & & & \cdot & & & \\
 & & & & \cdot & & & \\
 & & & & 0 & 1 & 1 & \\
 & & & & 1 & -2 & 2 & -1 \\
 & & & & & & -1 & 2
\end{bmatrix}
\begin{bmatrix}
\mathbf{P}_{1,1} \\ \mathbf{P}_{1,2} \\ \mathbf{P}_{2,1} \\ \cdot \\ \cdot \\ \mathbf{P}_{n-1,2} \\ \mathbf{P}_{n,1} \\ \mathbf{P}_{n,2}
\end{bmatrix}
=
\begin{bmatrix}
\mathbf{D}_0 \\ 2\mathbf{D}_1 \\ 0 \\ \cdot \\ \cdot \\ 2\mathbf{D}_{n-1} \\ 0 \\ \mathbf{D}_n
\end{bmatrix}
$$

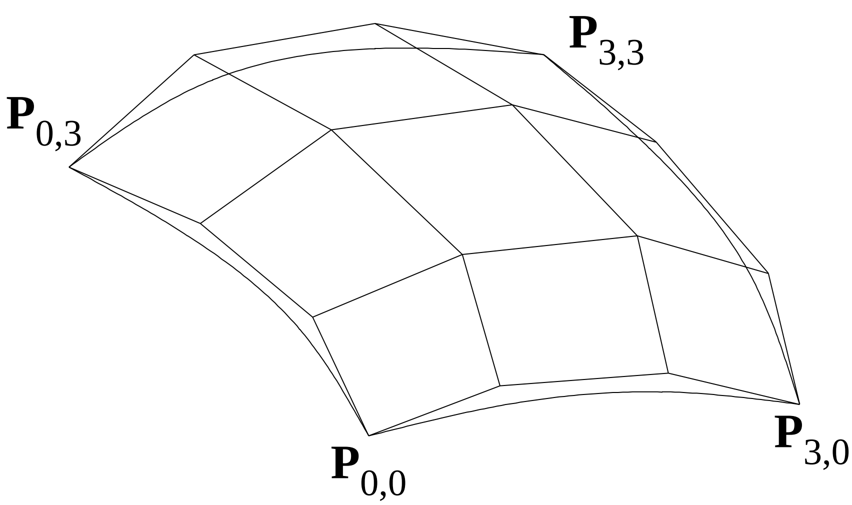
This system of equations can be solved using Gaussian elimination without pivoting.

3.1.6 Bicubic Bezier surface patches

$$\mathbf{S}(u,v) = \sum_{i=0}^{3} \sum_{j=0}^{3} B_{i,3}(u)\, B_{j,3}(v)\, \mathbf{P}_{i,j}$$

where

$$B_{k,3}(t) = \binom{3}{k} t^{k} (1-t)^{3-k}\ , \qquad 0 \le u,v \le 1$$



Matrix form

$$\mathbf{S}(u,v) = [\,1,\, u\,,\, u^2,\, u^3\,]\begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}\cdot$$

$$\begin{bmatrix} \mathbf{P}_{0,0} & \mathbf{P}_{0,1} & \mathbf{P}_{0,2} & \mathbf{P}_{0,3} \\ \mathbf{P}_{1,0} & \mathbf{P}_{1,1} & \mathbf{P}_{1,2} & \mathbf{P}_{1,3} \\ \mathbf{P}_{2,0} & \mathbf{P}_{2,1} & \mathbf{P}_{2,2} & \mathbf{P}_{2,3} \\ \mathbf{P}_{3,0} & \mathbf{P}_{3,1} & \mathbf{P}_{3,2} & \mathbf{P}_{3,3} \end{bmatrix}\begin{bmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ v \\ v^2 \\ v^3 \end{bmatrix}$$

$$= U \cdot M_b \cdot \mathbf{G} \cdot M_b^t \cdot V^t$$

- Satisfies **convex hull property**

- **Subdivision** process

  - Subdivide in u and then subdivide in v

- **Rendering** techniques

  - Wire frame: generate iso-parametric curves in both directions

  - Shaded images:

    - Ray tracing

    - Scan convert approximating polygons: approximate the surface patch by a set of fine polygons (triangles or quadri-terals) and then shade the polygons

- Patches can be joined together to form com-plicated shapes

3.1.7 Subdivision Techniques for Piecewise Surfaces

1. Midpoint subdivision
   (see previous section)
2. Adaptive subdivision
   (Cheng et al.)

Adaptive subdivision:

input: a piecewise surface P and a subdivision level assignment S

output: a triangular linear approximaiton P^{**} of P

<u>Three phases:</u>

Phase 1: define a label for each vertec of P

Phase 2: generate a gradrilateral subdivision mesh P^* of P

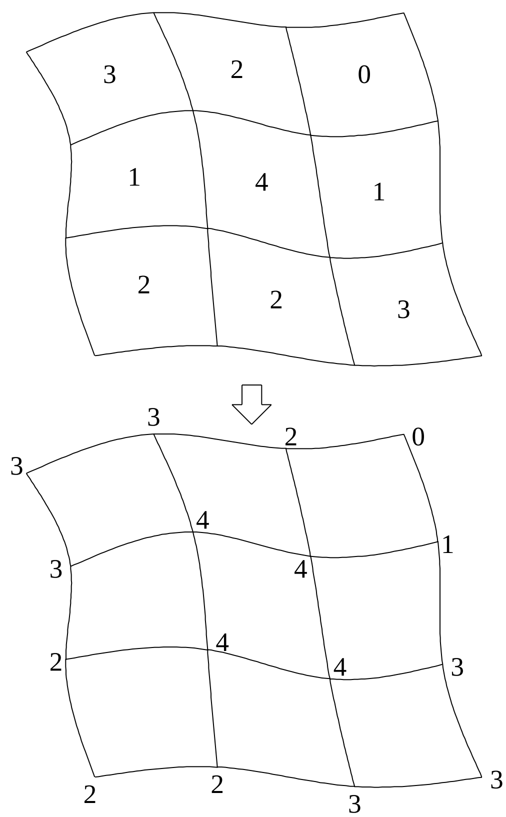
Phase 3: convert P^* to a triangular linear approximation P^{**} of P.

Phase 1:

/* $F \equiv \{\, f \mid f \textit{ is a patch of } P \,\}$ */

<u>for</u> each vertex v of P <u>do</u>

$$L(v) := \max\,(\,\{1\} \cup \{\, S(f) \mid f \in F,\ v \textit{ is a vertex of } f \,\}\,)$$



Phase 2:

1. <u>for</u> each vertex v of P <u>do</u>

   $LABEL(v) := L(v)$;

2. <u>for</u> each patch f of P <u>do</u>

   $Subdivide(f)$;

Subdivide(f: quadrilateral surface patch);

<u>if</u> ($LABEL(v) > 0$ for more than one vertex of f) <u>then</u>

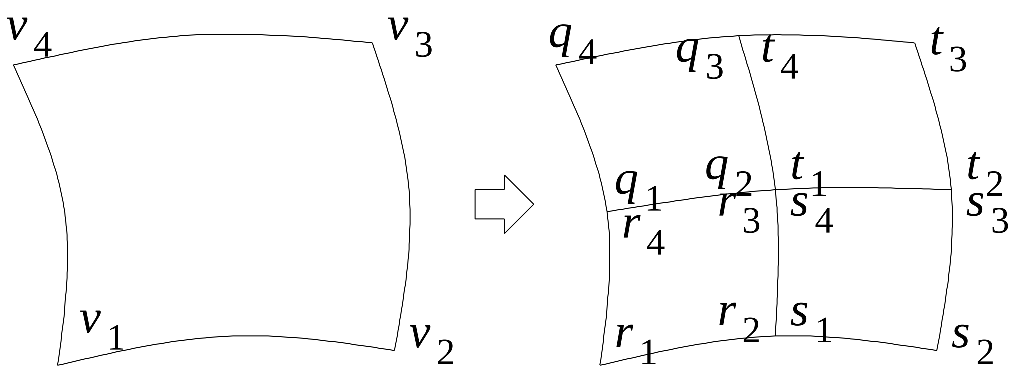
balanced_sub(f, f_1, f_2, f_3, f_4);

<u>for</u> $i := 1$ to 4 <u>do</u>
subdivide(f_i);

<u>else</u> <u>if</u> ($LABEL(v) > 0$ for only one vertex of f) <u>then</u>

unbalanced_sub(f, f_1, f_2, f_3);

<u>for</u> $i := 1$ to 3 <u>do</u>
subdivide(f_i);



balanced_sub(f):

Perform mid-point subdibision on f to get four new subpatches: $r_1r_2r_3r_4$, $s_1s_2s_3s_4$, $t_1t_2t_3t_4$, $q_1q_2q_3q_4$, and assign new labels as follows:

$LABEL(r_1) = \max\{0, LABEL(v_1) - 1\}$

$LABEL(s_2) = \max\{0, LABEL(v_2) - 1\}$

$LABEL(t_3) = \max\{0, LABEL(v_3) - 1\}$

$LABEL(q_4) = \max\{0, LABEL(v_4) - 1\}$

$LABEL(r_2) = LABEL(s_1) = \min\{LABEL(r_1), LABEL(s_2)\}$

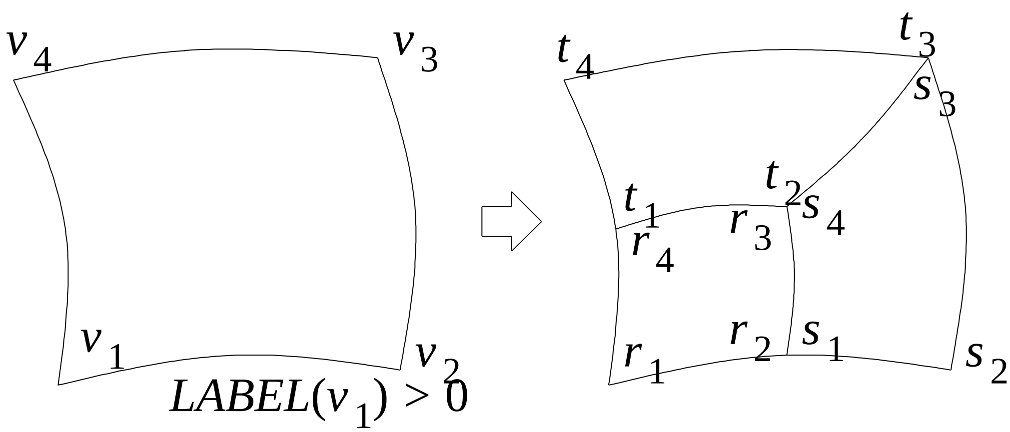
$LABEL(s_3) = LABEL(t_2) = \min\{LABEL(s_2), LABEL(t_3)\}$

$LABEL(t_4) = LABEL(q_3) = \min\{LABEL(t_3), LABEL(q_4)\}$

$LABEL(q_1) = LABEL(r_4) = \min\{LABEL(q_4), LABEL(r_1)\}$

$$LABEL(r_3) = LABEL(s_4) = LABEL(t_1) = LABEL(q_2)$$

$$= \begin{cases} 0, & \text{if } r_2,\ s_3,\ t_4,\ and\ q_1\ are\ assigned\ zero\ label \\ \min\{LABEL(v) \mid v \in \{r_2, t_3, t_4, q_1\}, LABEL(v) > 0\}, & otherwise \end{cases}$$



unbalanced_sub(f):

If $LABEL(v_1) > 0$, subdivide f as above to get three new subpatches: $r_1r_2r_3r_4$, $s_1s_2s_3s_4$, $t_1t_2t_3t_4$, and assign new labels as follows:

$LABEL(r_1) = LABEL(v_1) - 1$

$LABEL(r_i) = 0, \quad i = 2,3,4$; $\qquad LABEL(s_i) = 0, \quad i = 1,2,3,4$

$LABEL(t_i) = 0, \quad i = 1,2,3,4$.

The above algorithm guarantees that adjacent patches will generate the same vertices on the common edge. Hence, no cracks will be generated between adjacent patches. This follows from the following theorem:

Theorem: Let e be an edge of some patch and labels of the vertices of e be i and j, respectively. Let $V(i,j)$ denote the number of vertices created by the algorithm between these vertices. Then $V(i,j)$ depends on i and j only. Actually, if $j \ge i$ then $V(i,j) = 2^i + j - i - 1$.

For a proof, see "A Parallel mesh Generation Algorithm Based on the Vertex Label Assignment Scheme", International Journal for Numerical Methods in Engineering, Vol 28 (1989), 1429-1448.

3.1.8 Non-Uniform B-Spline Curves

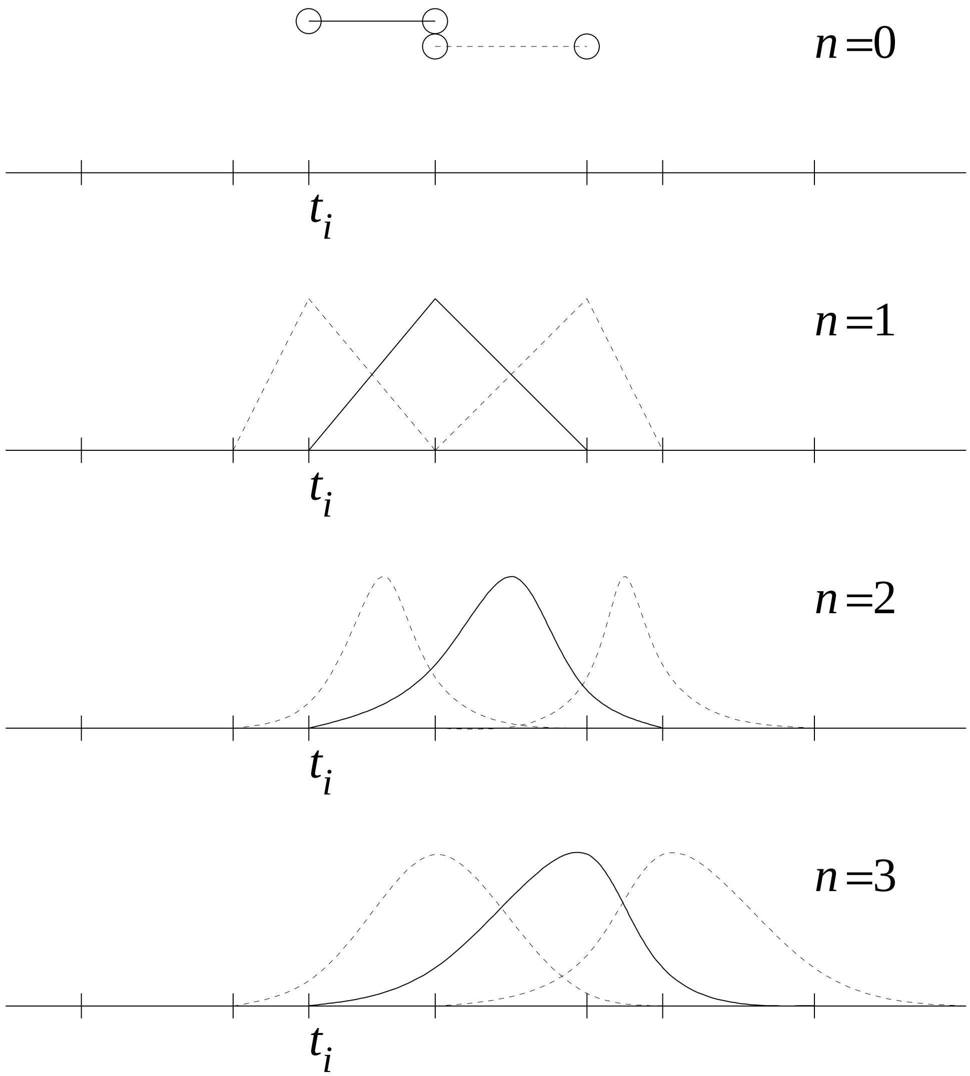
- **Definition:** Let $\{ t_i \}$ be an infinite sequence of points (called *knots*) on the real axis. The B-spline basis function $N_{i,n}(t)$ of degree n with *support* $[t_i, t_{i+m+1}]$ is defined by the following recursive procedure:

$$N_{i,0} = \begin{cases} 1, & t_i \le t < t_{i+1} \\ 0, & otherwise \end{cases}$$

and for $m \ge 1$

$$N_{i,m}(t) = \frac{t - t_i}{t_{i+m} - t_i} N_{i,m-1}(t) + \frac{t_{i+m+1} - t}{t_{i+m+1} - t_{i+1}} N_{i+1,m-1}(t)$$

Intuitively, B-splines of degree n (order $n+1$) are piecewise polynomial curves that are zero at all subintervals but $n+1$ of them and have continuous $(n-1)$st derivative. The following are examples of b-splines of degree 0, 1, 2, and 3.


n=0
t_i
n=1
t_i
n=2
t_i
n=3
t_i

Explicit forms of low degree B-splines:

1. Linear B-splines:

$$N_{i,1}(t)=\begin{cases}\dfrac{t-t_i}{t_{i+1}-t_i}, & t_i \le t < t_{i+1}\\[2ex] \dfrac{t_{i+2}-t}{t_{i+2}-t_{i+1}}, & t_{i+1} \le t < t_{i+2}\\[2ex] 0, & elsewhere\end{cases}$$

2. Quadratic B-splines:

$$N_{i,2}(t)=\begin{cases}\dfrac{(t-t_i)^2}{(t_{i+1}-t_i)(t_{i+2}-t_i)}, & t_i \le t < t_{i+1}\\[2ex] \dfrac{(t-t_i)(t_{i+2}-t)}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})}+\dfrac{(t_{i+3}-t)(t-t_{i+1})}{(t_{i+3}-t_{i+1})(t_{i+2}-t_{i+1})}, & t_{i+1} \le t < t_{i+2}\\[2ex] \dfrac{(t_{i+3}-t)^2}{(t_{i+3}-t_{i+1})(t_{i+3}-t_{i+2})}, & t_{i+2} \le t < t_{i+3}\\[2ex] 0, & elsewhere\end{cases}$$

3. Cubic B-splines:

$$N_{i,3}(t) = \frac{(t - t_i)^3}{(t_{i+1} - t_i)(t_{i+2} - t_i)(t_{i+3} - t_i)}, \quad t_i \le t < t_{i+1}$$

$$= \frac{(t - t_i)^2 (t_{i+2} - t)}{(t_{i+2} - t_i)(t_{i+2} - t_{i+1})(t_{i+3} - t_i)} + \frac{(t - t_i)(t_{i+3} - t)(t - t_{i+1})}{(t_{i+2} - t_{i+1})(t_{i+3} - t_{i+1})(t_{i+3} - t_i)}$$

$$+ \frac{(t_{i+4} - t)(t - t_{i+1})^2}{(t_{i+4} - t_{i+1})(t_{i+3} - t_{i+1})(t_{i+2} - t_{i+1})}, \quad t_{i+1} \le t < t_{i+2}$$

$$= \frac{(t - t_i)(t_{i+3} - t)^2}{(t_{i+3} - t_i)(t_{i+3} - t_{i+1})(t_{i+3} - t_{i+2})} + \frac{(t_4 - t)(t_{i+3} - t)(t - t_{i+1})}{(t_{i+4} - t_{i+1})(t_{i+3} - t_{i+1})(t_{i+3} - t_{i+2})}$$

$$+ \frac{(t_{i+4} - t)^2 (t - t_{i+2})}{(t_{i+4} - t_{i+1})(t_{i+4} - t_{i+2})(t_{i+3} - t_{i+2})}, \quad t_{i+2} \le t < t_{i+3}$$

$$= \frac{(t_{i+4} - t)^3}{(t_{i+4} - t_{i+1})(t_{i+4} - t_{i+2})(t_{i+4} - t_{i+3})}, \quad t_{i+3} \le t < t_{i+4}$$

$$= 0, \qquad \textit{elsewhere}$$

What happens if the knots are uniformly distributed, for instance, $t_i = i$ for all i?

In this case, we have

1. Uniform Linear B-splines:

$$N_{i,1}(t) = \begin{cases} t - i, & i \le t < i+1 \\ i+2-t, & i+1 \le t < i+2 \\ 0, & \text{elsewhere} \end{cases}$$

2. Uniform Quadratic B-splines:

$$N_{i,2}(t) = \begin{cases} \dfrac{(t-i)^2}{2,} & i \le t < i+1 \\[2ex] \dfrac{(t-i)(i+2-t)}{2} + \dfrac{(i+3-t)(t-i-1)}{2}, & i+1 \le t < i+2 \\[2ex] \dfrac{(i+3-t)^2}{2}, & i+2 \le t < i+3 \\[2ex] 0, & \text{elsewhere} \end{cases}$$

3. Uniform cubic B-splines:

$$N_{i,3}(t) = \begin{cases} \dfrac{(t-i)^3}{6}, & i \le t < i+1 \\[2ex] \dfrac{(t-i)^2(i+2-t)}{6} + \dfrac{(t-i)(i+3-t)(t-i-1)}{6} & \\[2ex] \quad + \dfrac{(i+4-t)(t-i-1)^2}{6}, & i+1 \le t < i+2 \\[2ex] \dfrac{(t-i)(i+3-t)^2}{6} + \dfrac{(i+4-t)(i+3-t)(t-i-1)}{6} & \\[2ex] \quad + \dfrac{(i+4-t)^2(t-i-2)}{6}, & i+2 \le t < i+3 \\[2ex] \dfrac{(i+4-t)^3}{6,} & i+3 \le t < i+4 \end{cases}$$

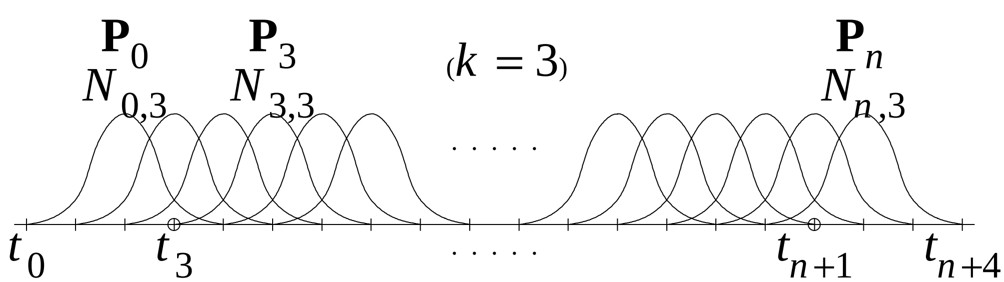
$$0, \qquad elsewhere$$

What are the relationship between the uniform cubic B-splines defined here and the cubic B-spline blending functions defined on page 60?

- **Definition:** A B-spline curve of degree k is defined as follows

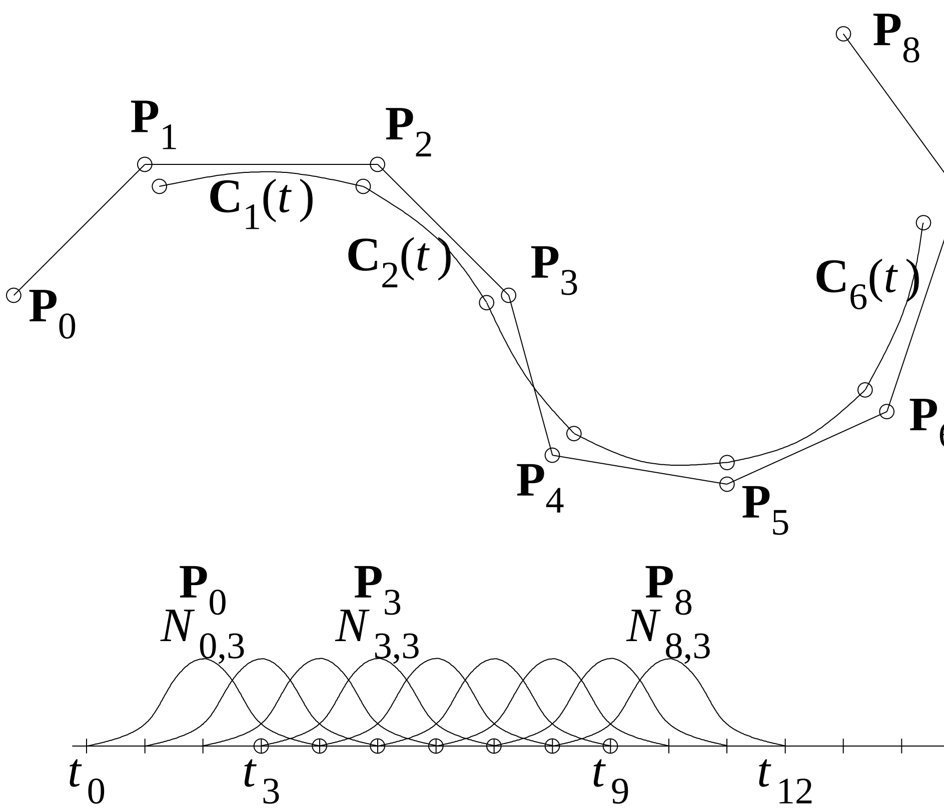
$$C(t) = \sum_{i=0}^{n} N_{i,k}(t)\,\mathbf{P}_i$$

where $N_{i,k}(t)$ are B-spline basis functions of degree k defined by the knot vector $\{\, t_i \mid 0 \le i \le n+k+1 \,\}$ and $\mathbf{P}_i$, $0 \le i \le n$, are 2D or 3D control points. The parameter space of this curve is the interval between t_k and t_{n+1} .



Each interval $[t_i, t_{i+1}]$, of the parameter space $[t_k, t_{n+1}]$ is called a *span*. The portion of the curve defined by a span is called a *segment*. So, $C(t)$ is a curve with $n-k+1$ segments defined by $n+1$ control points.

Example of a cubic B-spline curve:



If knot $t_i = i$ for all i then we get a uniform cubic B-spline curve. In that case, would the curve defined the above way be the same as the one given on page 63?

Questions:

1. Would a non-uniform cubic B-spline curve satisfy convex hull property?

2. What would happen if $t_0 = t_1 = t_2 = t_3$ and $t_{n+1} = t_{n+2} = t_{n+3} = t_{n+4}$?

3. What is the relationship between a composite cubic Bezier curve and a cubic B-spline curve?

Theorem: Let $\{ t_i \}$ be an infinite sequence of *knots* on the real axis and $N_{i,n}(t)$ be the corresponding B-spline basis functions of degree n. Then the summation of $N_{i,n}(t)$ for any t of the real axis is always equal to 1, i.e.,

$$\sum_i N_{i,n}(t) = 1, \qquad t \in R$$

It is okay that the bounds of the index are not given explicitly because the sum has only $n+1$ non-zero terms for each value of t.

Proof. If $t \in [t_k, t_{k+1})$, then

$$\sum_i N_{i,n}(t) = \sum_{i=k-n}^{k} N_{i,n}(t).$$

The definition of the B-spline basis functions shows that

$$N_{i-1,n}(t) = \frac{t - t_{i-1}}{t_{i+n-1} - t_{i-1}} N_{i-1,n-1}(t) + \frac{t_{i+n} - t}{t_{i+n} - t_i} N_{i,n-1}(t)$$

and

$$N_{i-1,n}(t) + N_{i,n}(t) = \frac{t - t_{i-1}}{t_{i+n-1} - t_{i-1}} N_{i-1,n-1}(t)$$

$$+ N_{i,n-1}(t) + \frac{t_{i+n+1} - t}{t_{i+n+1} - t_{i+1}} N_{i+1,n-1}(t)$$

Hence,

$$\sum_i N_{i,n}(t) = \frac{t - t_{k-n}}{t_k - t_{k-n}} N_{k-n,n-1}(t) + \sum_{i=k-n+1}^{k} N_{i,n-1}(t)$$

$$+\frac{t_{k+n+1}-t}{t_{k+n+1}-t_{k+1}}N_{k+1,n-1}(t)$$

$$=\sum_{i=k-n+1}^{k} N_{i,n-1}(t)$$

since $N_{k-n,n-1}$ and $N_{k+1,n-1}$ are both zero on $[t_k, t_{k+1})$. Iteratively repeat this process, we get

$$\sum_{i} N_{i,n}(t) = \sum_{i=k-n}^{k} N_{i,n}(t)$$

$$= \sum_{i=k-n+1}^{k} N_{i,n-1}(t)$$

$$= \sum_{i=k-n+2}^{k} N_{i,n-2}(t)$$

$$......$$

$$= \sum_{i=k}^{k} N_{i,0}(t)$$

$$= N_{k,0}(t)$$

$$= 1.$$

<u>Answer</u> <u>to</u> <u>Question</u> <u>1:</u>

The above theorem shows that a cubic B-spline curve satisfies a stronger convex hull property: each segment of a (non-uniform) cubic B-spline curve is contained in the convex hull of the four control points that determine the segment.

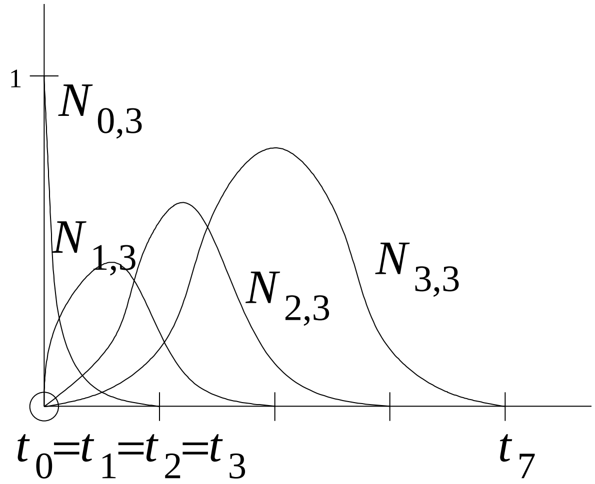
<u>Answer</u> <u>to</u> <u>Question</u> <u>2:</u>

The resulting cubic B-spline curve interpolate the first and last control points.

Why?

When $t_0 = t_1 = t_2 = t_3$, the first three cubic B-spline basis functions are of the following forms:

$$N_{0,3}(t) = \begin{cases} \dfrac{(t_4 - t)^3}{(t_4 - t_3)^3}, & t_3 \le t < t_4 \\ 0, & elsewhere \end{cases}$$



$$N_{1,3}(t) = \begin{cases} (t-t_3)\left[\dfrac{(t_4-t)^2}{(t_4-t_3)^3} + \dfrac{(t_5-t)(t_4-t)}{(t_5-t_3)(t_4-t_3)^2} + \dfrac{(t_5-t)^2}{(t_5-t_3)^2(t_4-t_3)}\right], & \\ & t_3 \le t < t_4 \\ \dfrac{(t_5-t)^3}{(t_5-t_3)^2(t_5-t_4)}, & t_4 \le t < t_5 \\ 0, & elsewhere \end{cases}$$

$$N_{2,3}(t) = \frac{(t_4-t)(t-t_3)^2}{(t_5-t_3)(t_4-t_3)^2} + \frac{(t_5-t)(t-t_3)^2}{(t_5-t_3)^2(t_4-t_3)}$$

$$+\frac{(t_6-t)(t-t_3)^2}{(t_6-t_3)(t_5-t_3)(t_4-t_3)}, \qquad t_3 \le t < t_4$$

$$= \frac{(t_5-t)^2(t-t_3)}{(t_5-t_3)^2(t_5-t_4)} + \frac{(t_6-t)(t-t_3)(t_5-t)}{(t_6-t_3)(t_5-t_3)(t_5-t_4)}$$

$$+\frac{(t_6-t)^2(t-t_4)}{(t_6-t_3)(t_6-t_4)(t_5-t_4)}, \qquad t_4 \le t < t_5$$

$$= \frac{(t_6-t)^3}{(t_6-t_3)(t_6-t_4)(t_6-t_5)}, \qquad t_5 \le t < t_6$$

$$= 0, \qquad \textit{elsewhere}$$

Hence, when $t = t_3$, we have $C(t_3) = N_{0,3}(t_3)\mathbf{P}_0 = \mathbf{P}_0$. Similarly, $C(t_{n+1}) = N_{n,3}(t_{n+1})\mathbf{P}_n = \mathbf{P}_n$.

How are $N_{0,3}$, $N_{1,3}$, and $N_{2,3}$ computed?

The Cox-de Boor recurrence formula shows that $N_{i,3}$ may be computed using the following chart:

$$
\begin{tikzcd}[row sep=tiny]
N_{i,0} \arrow[dr] & & & \\
& N_{i,1} \arrow[dr] & & \\
N_{i+1,0} \arrow[ur] \arrow[dr] & & N_{i,2} \arrow[dr] & \\
& N_{i+1,1} \arrow[ur] \arrow[dr] & & N_{i,3} \\
N_{i+2,0} \arrow[ur] \arrow[dr] & & N_{i+1,2} \arrow[ur] \arrow[dr] & \\
& N_{i+2,1} \arrow[ur] \arrow[dr] & & N_{i+1,3} \\
N_{i+3,0} \arrow[ur] \arrow[dr] & & N_{i+2,2} \arrow[ur] & \\
& N_{i+3,1} \arrow[ur] & & \\
N_{i+4,0} \arrow[ur] & & \vdots &
\end{tikzcd}
$$

The above chart can be simplified if one observes that on any given interval, $[t_i, t_{i+1})$, there are only $n+1$ B-splines of degree n that are non-zero. On that interval, $N_{i,n}$ depends on $N_{i,n-1}$ only, while $N_{i-l,n}$, $0 < l \le n$, depends on both $N_{i-l+1,n-1}$ and $N_{i-l,n-1}$. Therefore, for $t \in [t_i, t_{i+1})$, we have the following chart:

$$\begin{tikzcd}
 & & & N_{i-3,3} \\
 & & N_{i-2,2} \arrow[ur] \arrow[dr] & \\
 & N_{i-1,1} \arrow[ur] \arrow[dr, dash] & & N_{i-2,3} \\
N_{i,0} \arrow[ur] \arrow[dr] & & N_{i-1,2} \arrow[ur] \arrow[dr] & \\
 & N_{i,1} \arrow[ur] \arrow[dr] & & N_{i-1,3} \\
 & \vdots & N_{i,2} \arrow[ur] \arrow[dr] & \\
 & & & N_{i,3}
\end{tikzcd}$$

This chart will be used in the solution for question 3.

To answer Quesiton 3, we need to study the following problem:

> If a cubic B-spline curve has only one segment, and its knots satisfy the condition: $t_0 = t_1 = t_2 = t_3$ and $t_4 = t_5 = t_6 = t_7$, then what would happen ?

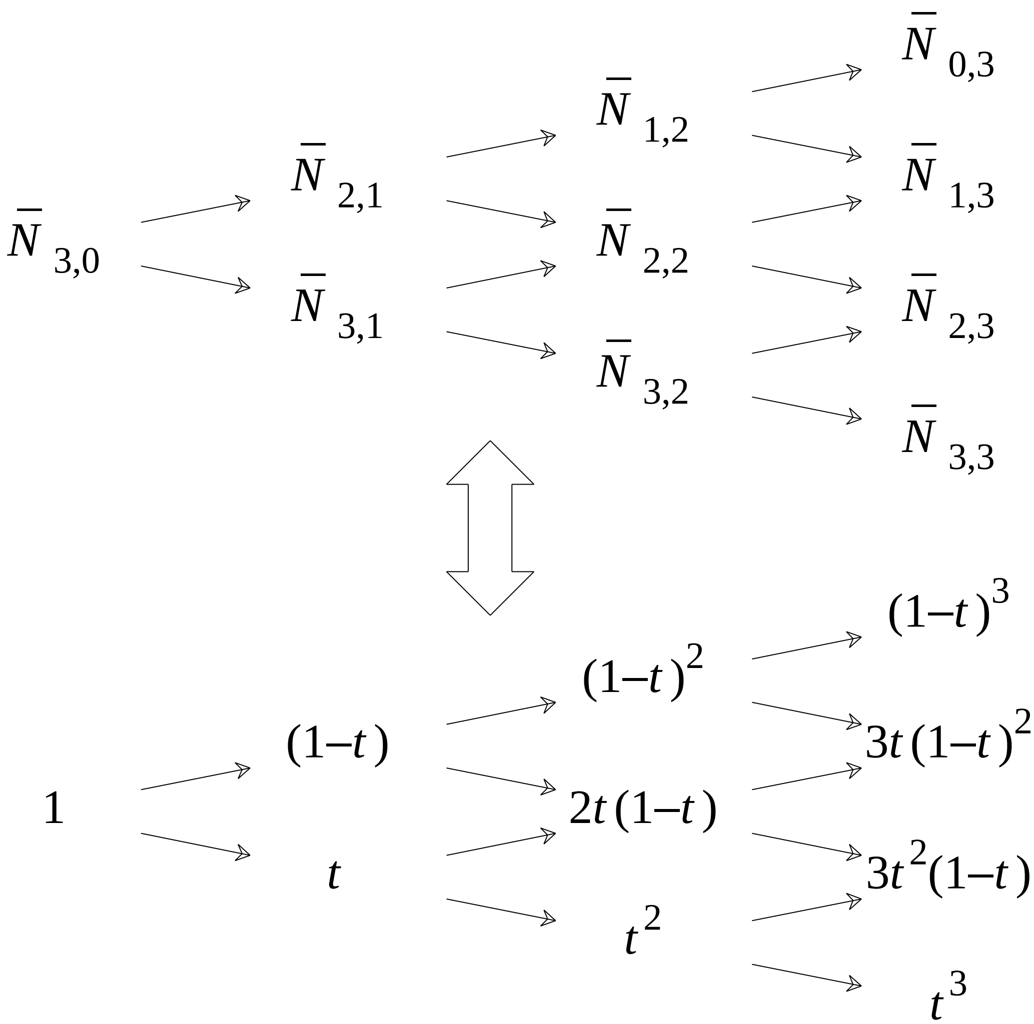
For simplicity, we shall consider the simple case $t_0 = t_1 = t_2 = t_3 = 0$ and $t_4 = t_5 = t_6 = t_7 = 1$ first. The corresponding cubic B-spline basis functions will be denoted $\bar{N}_{i,k}$, $0 \le k \le 3$.

This special cubic B-spline curve segment, denoted $\bar{C}(t)$, is defined as follows:

$$\bar{C}(t) = \sum_{i=0}^{3} \bar{N}_{i,3}(t)\, \bar{\mathbf{P}}_i, \qquad t \in [t_3, t_4] = [0, 1]$$

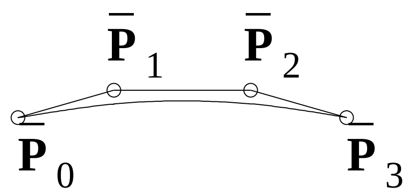
where $t_0 = t_1 = t_2 = t_3 = 0$ and $t_4 = t_5 = t_6 = t_7 = 1$, and $\bar{\mathbf{P}}_i$ are the control points of the curve segment.

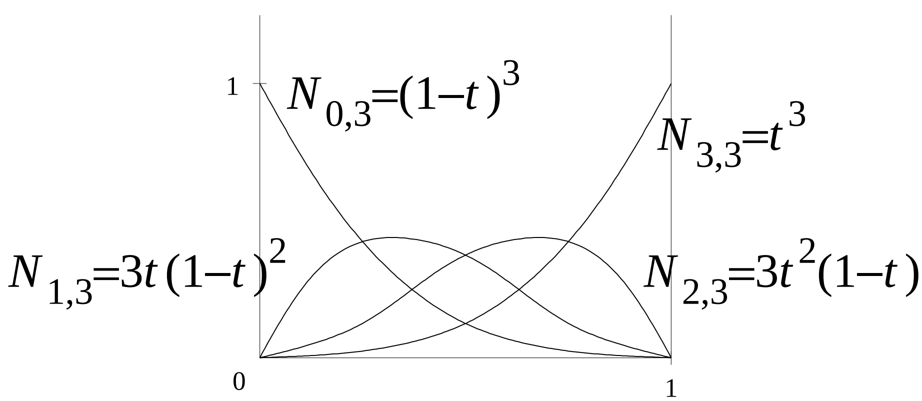
The property of a parametric curve depends on the definition of its blending functions. For $\bar{C}(t)$, $\bar{N}_{i,3}(t)$ can be computed using the chart on page 98. When $t \in [t_3, t_4) = [0, 1)$, we have



Hence, when $t_0 = t_1 = t_2 = t_3 = 0$ and $t_4 = t_5 = t_6 = t_7 = 1$, the corresponding cubic B-spline curve segment defined on page 100 is a cubic Bezier curve segment with control points $\bar{\mathbf{P}}_i$:

$$\bar{C}(t) = \sum_{i=0}^{3} \bar{N}_{i,3}(t)\,\bar{\mathbf{P}}_i = \sum_{i=0}^{3} B_{i,3}(t)\,\bar{\mathbf{P}}_i$$

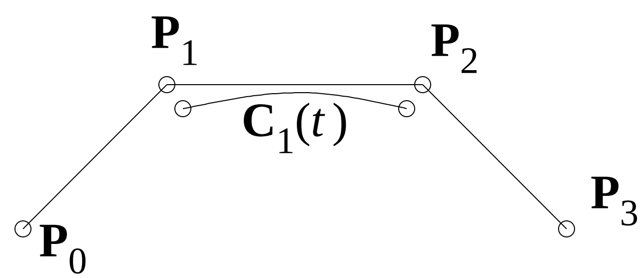


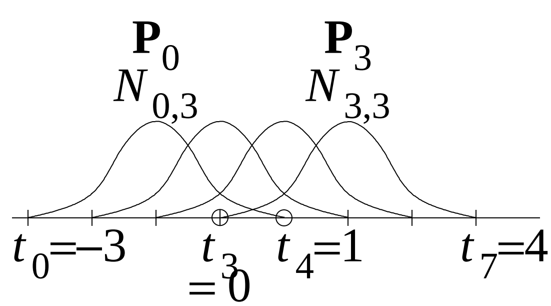


On the other hand, let $C(t)$ be a cubic B-spline curve with one segment too

$$C(t) = \sum_{i=0}^{3} N_{i,3}(t)\, P_i, \qquad t \in [t_3, t_4)$$

but the knots are all distinct and $t_i = i - 3$, for $i = 0,1,\ldots,7$. (Hence, $[t_3, t_4] = [0, 1]$)





If $\bar{C}(t)$ and $C(t)$ represent the same cueve, then what is the relationship between control points of $C(t)$, $\mathbf{P}_i$, and control points of $\bar{C}(t)$, $\bar{\mathbf{P}}_i$?

<u>Solution:</u>

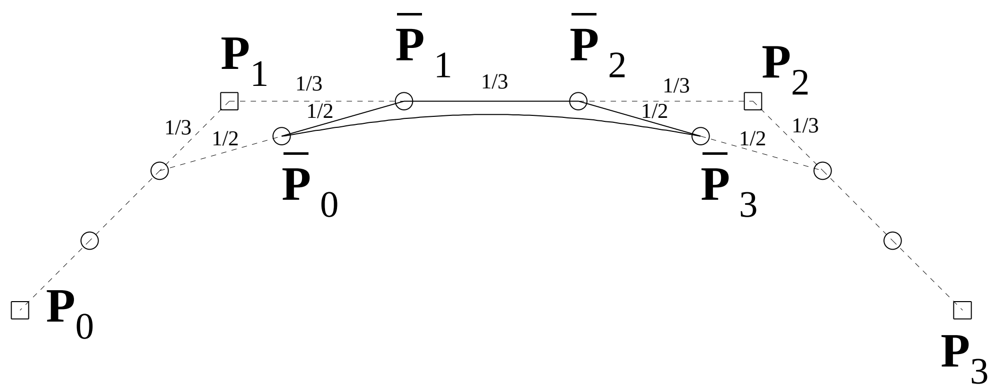
$$\bar{\mathbf{P}}_1 = \mathbf{P}_1 + \frac{1}{3}(\mathbf{P}_2 - \mathbf{P}_1), \qquad \bar{\mathbf{P}}_2 = \mathbf{P}_1 + \frac{2}{3}(\mathbf{P}_2 - \mathbf{P}_1),$$

$$\bar{\mathbf{P}}_0 = \frac{[\,\mathbf{P}_0 + \frac{2}{3}(\mathbf{P}_1 - \mathbf{P}_0)\,] + \bar{\mathbf{P}}_1}{2} = \frac{1}{6}\mathbf{P}_0 + \frac{4}{6}\mathbf{P}_1 + \frac{1}{6}\mathbf{P}_2,$$

$$\bar{\mathbf{P}}_3 = \frac{\bar{\mathbf{P}}_2 + [\,\mathbf{P}_2 + \frac{1}{3}(\mathbf{P}_3 - \mathbf{P}_2)\,]}{2} = \frac{1}{6}\mathbf{P}_1 + \frac{4}{6}\mathbf{P}_2 + \frac{1}{6}\mathbf{P}_3.$$

This means a uniform cubic B-spline curve segment can be converted to a cubic Bezier curve segment, and vice versa.

The relationship between there control points is as follows:



$\mathbf{P}_i$: B-spline control points
$\bar{\mathbf{P}}_i$: Bezier control points

Why?

If the cubic Bezier curve $\bar{C}(t)$ defined on page 100 is the same as the uniform cubic B-spline curve segement defined on page 102 are the same,

$$\bar{C}(t) = \sum_{i=0}^{3} B_{i,3}(t)\,\bar{\mathbf{P}}_i = C(t) = \sum_{i=0}^{3} N_{i,3}(t)\,P_i,$$

$$t \in [0, 1],$$

then we must have

$$\bar{C}(0) = C(0), \qquad \bar{C}(1) = C(1)$$

$$\bar{C}'(0) = C'(0), \qquad \bar{C}'(1) = C'(1)$$

i.e.,

$$\bar{\mathbf{P}}_0 = \frac{1}{6}\mathbf{P}_0 + \frac{4}{6}\mathbf{P}_1 + \frac{1}{6}\mathbf{P}_2, \quad \bar{\mathbf{P}}_3 = \frac{1}{6}\mathbf{P}_1 + \frac{4}{6}\mathbf{P}_2 + \frac{1}{6}\mathbf{P}_3$$

$$3(\bar{\mathbf{P}}_1 - \bar{\mathbf{P}}_0) = -\frac{1}{2}\mathbf{P}_0 + \frac{1}{2}\mathbf{P}_2, \quad 3(\bar{\mathbf{P}}_3 - \bar{\mathbf{P}}_2) = -\frac{1}{2}\mathbf{P}_1 + \frac{1}{2}\mathbf{P}_3$$

equivalent to the conditons on page 102.

Now, consider the general case.

Let $\bar{C}(s)$ be a cubic Bezier curve defined as follows:

$$\bar{C}(s) = \sum_{i=0}^{3} B_{i,3}(s)\bar{\mathbf{P}}_i, \qquad s \in [0, 1]$$

where $B_{i,3}(s)$ are Bezier blending functions and $\bar{\mathbf{P}}_i$ are control points of the curve.

Let $C(t)$ be a non-uniform cubic B-spline curve segment defined as follows:

$$C(t) = \sum_{i=0}^{3} N_{i,3}(t)\,\mathbf{P}_i, \qquad t \in [\,t_3, t_4\,]$$

where $N_{i,3}(t)$ are cubic B-spline basis functions defined by the knot sequence $\{\, t_i \mid 0 \le i \le 7 \,\}$ and $\mathbf{P}_i$ are contrl points of the curve.

If $\bar{C}(s)$ and $C(t)$ represent the same curve, then what is the relationship between the control points of $\bar{C}(s)$ and the control points of $C(t)$?

Solutions:

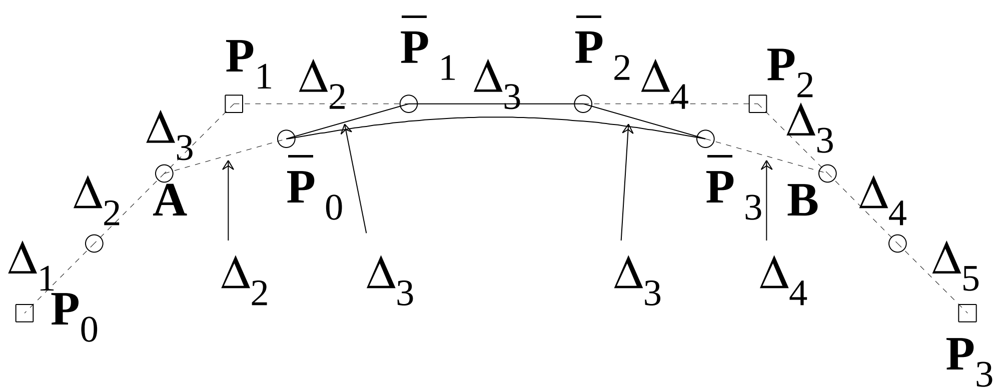
$$\bar{\mathbf{P}}_1 = \frac{t_5 - t_3}{t_5 - t_2}\mathbf{P}_1 + \frac{t_3 - t_2}{t_5 - t_2}\mathbf{P}_2, \qquad \bar{\mathbf{P}}_2 = \frac{t_5 - t_4}{t_5 - t_2}\mathbf{P}_1 + \frac{t_4 - t_2}{t_5 - t_2}\mathbf{P}_2,$$

$$\bar{\mathbf{P}}_0 = \frac{t_4 - t_3}{t_4 - t_2}\mathbf{A} + \frac{t_3 - t_2}{t_4 - t_2}\bar{\mathbf{P}}_1, \qquad \bar{\mathbf{P}}_3 = \frac{t_5 - t_4}{t_5 - t_3}\bar{\mathbf{P}}_2 + \frac{t_4 - t_3}{t_5 - t_3}\mathbf{B}$$

where

$$\mathbf{A} = \frac{t_4 - t_3}{t_4 - t_1}\mathbf{P}_0 + \frac{t_3 - t_1}{t_4 - t_1}\mathbf{P}_1, \qquad \mathbf{B} = \frac{t_6 - t_4}{t_6 - t_3}\mathbf{P}_2 + \frac{t_4 - t_3}{t_6 - t_3}\mathbf{P}_3,$$

i.e.,



where $\Delta_i \equiv t_{i+1} - t_i$. Why?

We need the following Lemma first.

Lemma: The derivative of a B-spline basis function of degree $m \geq 1$ is the linear difference of two B-spline basis functions of degree $m-1$, as follows:

$$N_{i,m}'(t) = m \left(\frac{N_{i,m-1}(t)}{t_{i+m} - t_i} - \frac{N_{i+1,m-1}(t)}{t_{i+m+1} - t_{i+1}} \right)$$

Proof: By induction.

First, show that

$$N_{i,1}'(t) = 1 \left(\frac{N_{i,0}(t)}{t_{i+1} - t_i} - \frac{N_{i+1,0}(t)}{t_{i+2} - t_{i+1}} \right).$$

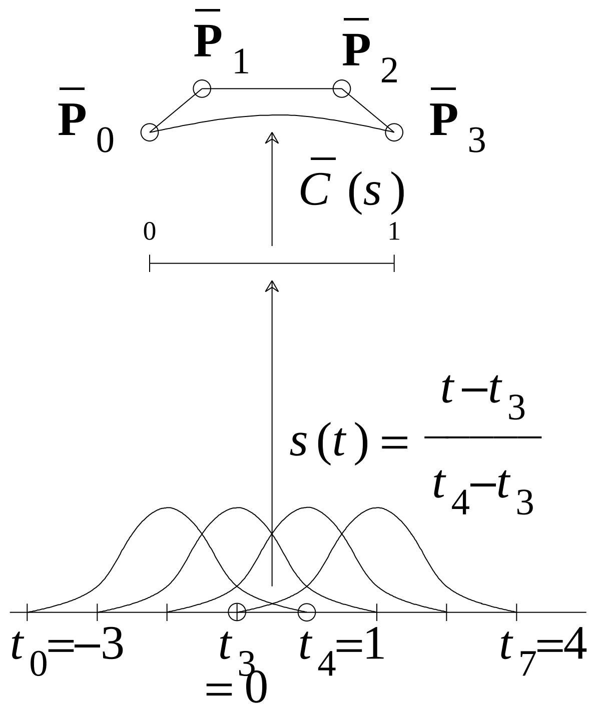
Then prove that if the formula is true for all degrees $\leq m-1$ then the above formula would hold for degree m too.

Now the proof of the results on page 107.

First, define a map from $[t_3, t_4]$ to $[0, 1]$

$$s(t) = \frac{t - t_3}{t_4 - t_3} = \frac{t - t_3}{\Delta_3}, \qquad t \in [t_3, t_4]$$

so $\bar{C}(s(t))$ is a function defined on $[t_3, t_4]$ too.



Since $\bar{C}(s(t)) = C(t)$, we have

$$\bar{C}(s(t_3)) = \bar{C}(0) = C(t_3), \qquad \bar{C}(s(t_4)) = \bar{C}(1) = C(t_4)$$

$$\bar{C}'(s(t_3)) = \frac{\bar{C}'(0)}{\Delta_3} = C'(t_3), \qquad \bar{C}'(s(t_4)) = \frac{\bar{C}'(1)}{\Delta_3} = C'(t_4)$$

or

$$\bar{\mathbf{P}}_0 = N_{0,3}(t_3)\mathbf{P}_0 + N_{1,3}(t_3)\mathbf{P}_1 + N_{2,3}(t_3)\mathbf{P}_2 \qquad \text{(B1)}$$

$$\bar{\mathbf{P}}_3 = N_{1,3}(t_4)\mathbf{P}_1 + N_{2,3}(t_4)\mathbf{P}_2 + N_{3,3}(t_4)\mathbf{P}_3 \qquad \text{(B2)}$$

$$\frac{3(\bar{\mathbf{P}}_1 - \bar{\mathbf{P}}_0)}{\Delta_3} = N_{0,3}'(t_3)\mathbf{P}_0 + N_{1,3}'(t_3)\mathbf{P}_1 + N_{2,3}'(t_3)\mathbf{P}_2 \qquad \text{(B3)}$$

$$\frac{3(\bar{\mathbf{P}}_3 - \bar{\mathbf{P}}_2)}{\Delta_3} = N_{1,3}'(t_4)\mathbf{P}_1 + N_{2,3}'(t_4)\mathbf{P}_2 + N_{3,3}'(t_4)\mathbf{P}_3 \qquad \text{(B4)}$$

Using the recursive formulas for B-spline basis functions and their derivatives, we have from (B1), (B2), (B3) and (B4) the following relations:

$$\bar{\mathbf{P}}_0 = \frac{N_{1,2}(t_3)}{t_4-t_1}\left[(t_4-t_3)\mathbf{P}_0 + (t_3-t_1)\mathbf{P}_1\right] + \frac{N_{2,2}(t_3)}{t_5-t_2}\left[(t_5-t_3)\mathbf{P}_1 + (t_3-t_2)\mathbf{P}_2\right] \qquad \text{(B5)}$$

$$\bar{\mathbf{P}}_3 = \frac{N_{2,2}(t_4)}{t_5-t_2}\left[(t_5-t_4)\mathbf{P}_1 + (t_4-t_2)\mathbf{P}_2\right] + \frac{N_{3,2}(t_4)}{t_6-t_3}\left[(t_6-t_4)\mathbf{P}_2 + (t_4-t_3)\mathbf{P}_3\right] \qquad \text{(B6)}$$

$$\bar{\mathbf{P}}_1-\bar{\mathbf{P}}_0 = (t_4-t_3)\left[\frac{N_{1,2}(t_3)}{t_4-t_1}(\mathbf{P}_1-\mathbf{P}_0) + \frac{N_{2,2}(t_3)}{t_5-t_2}(\mathbf{P}_2-\mathbf{P}_1)\right] \qquad \text{(B7)}$$

$$\bar{\mathbf{P}}_3-\bar{\mathbf{P}}_2 = (t_4-t_3)\left[\frac{N_{2,2}(t_4)}{t_5-t_2}(\mathbf{P}_2-\mathbf{P}_1) + \frac{N_{3,2}(t_4)}{t_6-t_3}(\mathbf{P}_3-\mathbf{P}_2)\right] \qquad \text{(B8)}$$

From (B5) and (B7), we have

$$\bar{\mathbf{P}}_1 = \left[N_{1,2}(t_3) + \frac{t_5 - t_4}{t_5 - t_2} N_{2,2}(t_3) \right] \mathbf{P}_1 + \left[\frac{t_4 - t_2}{t_5 - t_2} N_{2,2}(t_3) \right] \mathbf{P}_2$$

From the definition of $N_{i,2}(t)$ (page 85), we have

$$N_{1,2}(t_3) = \frac{t_4 - t_3}{t_4 - t_2}; \qquad N_{2,2}(t_3) = \frac{t_3 - t_2}{t_4 - t_2}$$

Substituting these expressions for $N_{1,2}(t_3)$ and $N_{2,2}(t_3)$ into the above equation, we get

$$\bar{\mathbf{P}}_1 = \frac{t_5 - t_3}{t_5 - t_2} \mathbf{P}_1 + \frac{t_3 - t_2}{t_5 - t_2} \mathbf{P}_2$$

$$= \mathbf{P}_1 + (\mathbf{P}_2 - \mathbf{P}_1) \frac{t_3 - t_2}{t_5 - t_2} \qquad \text{(B9)}$$

Similarly, From (B6) and (B8), we have

$$\bar{\mathbf{P}}_2 = \left[\frac{t_5 - t_3}{t_5 - t_2} N_{2,2}(t_4)\right]\mathbf{P}_1 + \left[N_{3,2}(t_4) + \frac{t_3 - t_2}{t_5 - t_2} N_{2,2}(t_4)\right]\mathbf{P}_2$$

From the definition of $N_{i,2}(t)$ (page 85), we have

$$N_{2,2}(t_4) = \frac{t_5 - t_4}{t_5 - t_3}; \qquad N_{3,2}(t_4) = \frac{t_4 - t_3}{t_5 - t_3}$$

Substituting these expressions for $N_{2,2}(t_4)$ and $N_{3,2}(t_4)$ into the above equation, we get

$$\bar{\mathbf{P}}_2 = \frac{t_5 - t_4}{t_5 - t_2}\mathbf{P}_1 + \frac{t_4 - t_2}{t_5 - t_2}\mathbf{P}_2$$

$$= \mathbf{P}_1 + (\mathbf{P}_2 - \mathbf{P}_1)\frac{t_4 - t_2}{t_5 - t_2} \tag{B10}$$

Finally, if we set

$$\mathbf{A} = \frac{t_4 - t_3}{t_4 - t_1}\mathbf{P}_0 + \frac{t_3 - t_1}{t_4 - t_1}\mathbf{P}_1, \qquad \mathbf{B} = \frac{t_6 - t_4}{t_6 - t_3}\mathbf{P}_2 + \frac{t_4 - t_3}{t_6 - t_3}\mathbf{P}_3,$$

then from (B5) and (B9), we have

$$\begin{aligned} \overline{\mathbf{P}}_0 &= N_{1,2}(t_3)\mathbf{A} + N_{2,2}(t_3)\overline{\mathbf{P}}_1 \\ &= \frac{t_4 - t_3}{t_4 - t_2}\mathbf{A} + \frac{t_3 - t_2}{t_4 - t_2}\overline{\mathbf{P}}_1 \end{aligned} \qquad \text{(B11)}$$

and from (B6) and (B10), we have

$$\begin{aligned} \overline{\mathbf{P}}_3 &= N_{2,2}(t_4)\overline{\mathbf{P}}_2 + N_{3,2}(t_4)\mathbf{B} \\ &= \frac{t_5 - t_4}{t_5 - t_3}\overline{\mathbf{P}}_2 + \frac{t_4 - t_3}{t_5 - t_3}\mathbf{B}. \end{aligned} \qquad \text{(B12)}$$

So the proof is completed.

In general, we have the following relationship between cubic B-spline curves and composite cubic Bezier curves:

Let $C(t)$ be a non-uniform cubic B-spline curve

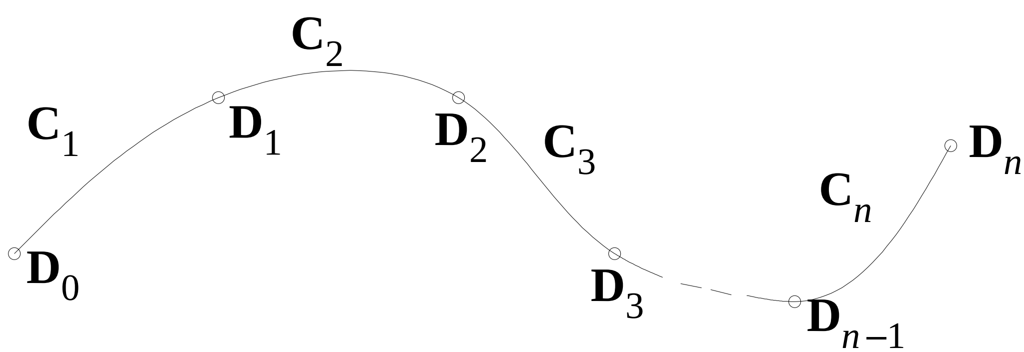
$$C(t) = \sum_{i=0}^{n} N_{i,3}(t)\,\mathbf{P}_i \qquad t \in [t_3, t_{n+1})$$

with control points $\{\, \mathbf{P}_i \mid 0 \le i \le n \,\}$ and knot sequence $\{\, t_i \mid 0 \le i \le n+4 \,\}$

First, for each leg of the control polygon, $\mathbf{P}_i\mathbf{P}_{i+1}$, divide it into three subsegments at the points $\mathbf{Q}_{3i-2}$ and $\mathbf{Q}_{3i-1}$ in the ratio $\Delta_{i+1} : \Delta_{i+2} : \Delta_{i+3}$ with $\Delta_i \equiv t_{i+1} - t_i$. Next split the line segment $\mathbf{Q}_{3i-4}\mathbf{Q}_{3i-2}$ into two subsegments at the point $\mathbf{Q}_{3(i-1)}$ in the ratio $\Delta_{i+1} : \Delta_{i+2}$. Then $\{\mathbf{Q}_{3(i-1)}, \mathbf{Q}_{3i-2}, \mathbf{Q}_{3i-1}, \mathbf{Q}_{3i}\,\}$ are the Bezier control points of the ith segment of the B-spline curve.

3.1.9 Curve Fitting using Uniform Cubic B-Spline Curves

- Given a set of data points $\mathbf{D}_i = (x_i, y_i)$, $i = 0, 1, \cdots, n$, $(n \geq 2)$, how can a cubic B-spline curve that interpolates these points be constructed?

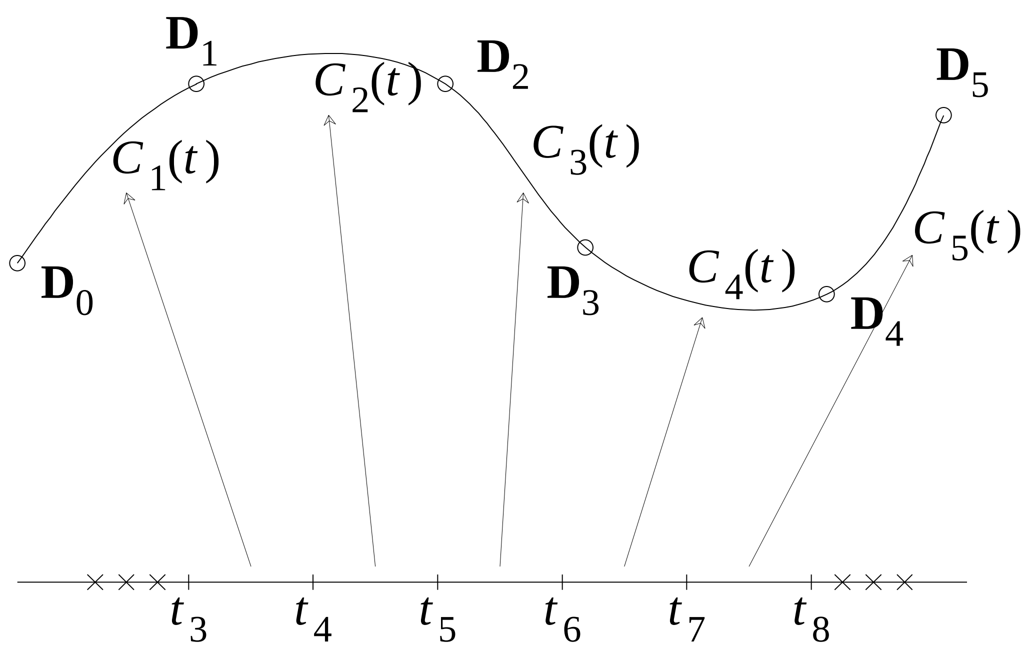


- The cubic B-spline curve has n segments $\mathbf{C}_1(t)$, $\mathbf{C}_2(t)$, ..., $\mathbf{C}_n(t)$ with $\mathbf{D}_{i-1}$ and $\mathbf{D}_i$ being the start and end points of $\mathbf{C}_i(t)$

An analysis of the problem:

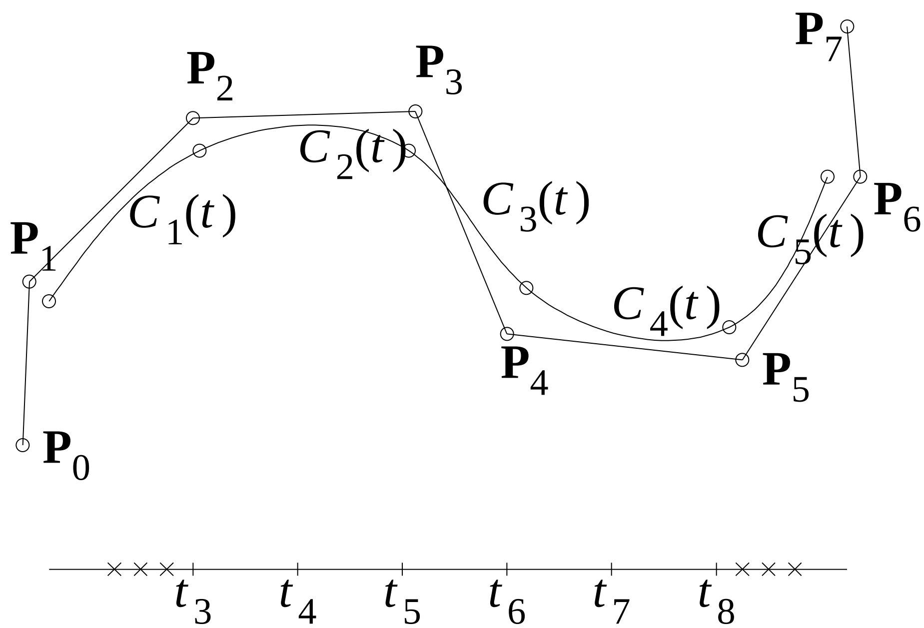
- To get the curve constructed, how many knots are needed?

  Consider the following case:



So, to interpolate $(n+1)$ data points, one needs $(n+7)$ knots, t_0, t_1, ..., t_{n+6}, for a uniform cubic B-spline interpolating curve.

- To get the curve constructed, how many control points are needed?
  Consider the following case:



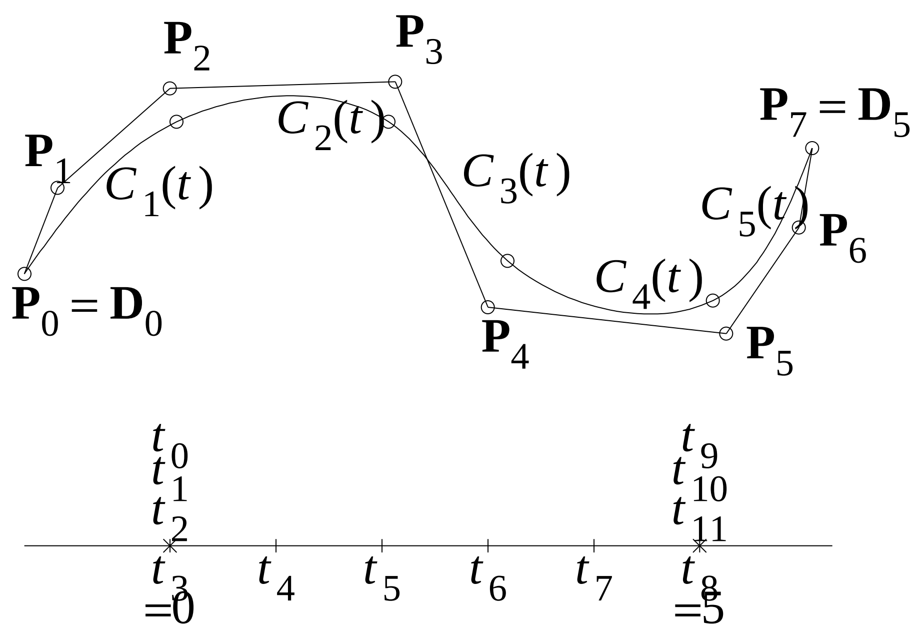
So, to interpolate $(n+1)$ data points, one needs $(n+3)$ control points, $\mathbf{P}_0$, $\mathbf{P}_1$, ..., $\mathbf{P}_{n+2}$, for a uniform cubic B-spline interpolating curve.

- To make things easier, we shall assume that

$$t_i = i-3, \qquad i = 3,4,\ldots,n+3$$

with $t_0 = t_1 = t_2 = t_3$ and $t_{n+3} = t_{n+4} = t_{n+5} = t_{n+6}$.

Consequently, we have $\mathbf{P}_0 = \mathbf{D}_0$ and $\mathbf{P}_{n+2} = \mathbf{D}_n$.



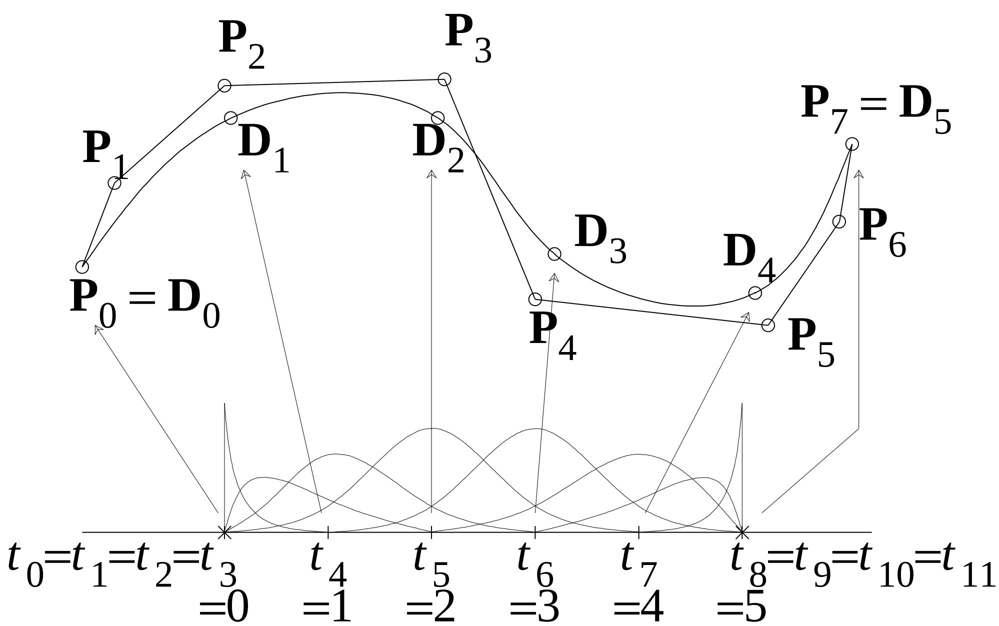
Still, we need to find $\mathbf{P}_1$, $\mathbf{P}_2$, ... $\mathbf{P}_{n+1}$. How?

The interpolating curve to be constructed must be of the following form:

$$C(t) = \sum_{i=0}^{n+2} N_{i,3}(t)\,\mathbf{P}_i, \qquad t \in [0, n]$$

and satisfies the following conditions:

$$C(j) = \sum_{i=0}^{n+2} N_{i,3}(j)\,\mathbf{P}_i = \mathbf{D}_j, \qquad j=0,1,\ldots,n \qquad (*)$$



Note that at each knot there are at most 3 cubic B-spline basis functions which are non-zero. Therefore, equations in (*) are of the following form:

$$N_{i,3}(i)\mathbf{P}_i + N_{i+1,3}(i)\mathbf{P}_{i+1} + N_{i+2,3}(i)\mathbf{P}_{i+2} = \mathbf{D}_i, \quad i = 0,1,\ldots,n$$

or

$$\mathbf{P}_0 = \mathbf{D}_0$$

$\frac{1}{4}\mathbf{P}_1 + \frac{7}{12}\mathbf{P}_2 + \frac{1}{6}\mathbf{P}_3 = \mathbf{D}_1$

$\frac{1}{6}\mathbf{P}_2 + \frac{2}{3}\mathbf{P}_3 + \frac{1}{6}\mathbf{P}_4 = \mathbf{D}_2$

...

$\frac{1}{6}\mathbf{P}_{n-2} + \frac{2}{3}\mathbf{P}_{n-1} + \frac{1}{6}\mathbf{P}_n = \mathbf{D}_{n-2}$

$\frac{1}{6}\mathbf{P}_{n-1} + \frac{7}{12}\mathbf{P}_n + \frac{1}{4}\mathbf{P}_{n+1} = \mathbf{D}_{n-1}$

$$\mathbf{P}_{n+2} = \mathbf{D}_n$$

So, actually, only $\mathbf{P}_1$, $\mathbf{P}_2$, ..., $\mathbf{P}_{n+1}$ are unknown. By ignoring the first and the last equations, we have a system of $n-1$ equations (those marked with "#") in $n+1$ unknowns. We need two extra conditions to get this system solved.

One option is to set the second derivative of the curve at the start and end points to zero:

$$C''(0) = N_{0,3}''(0)\mathbf{P}_0 + N_{1,3}''(0)\mathbf{P}_1 + N_{2,3}''(0)\mathbf{P}_2 = 0$$

$$C''(n) = N_{n,3}''(n)\mathbf{P}_n + N_{n+1,3}''(n)\mathbf{P}_n + N_{n+2,3}''(n)\mathbf{P}_{n+2} = 0$$

or

$$6\mathbf{P}_0 - 9\mathbf{P}_1 + 3\mathbf{P}_2 = 0$$

$$3\mathbf{P}_n - 9\mathbf{P}_{n+1} + 6\mathbf{P}_{n+2} = 0$$

Note that $\mathbf{P}_0$ and $\mathbf{P}_{n+2}$ are known to us ($\mathbf{P}_0 = \mathbf{D}_0$ and $\mathbf{P}_{n+2} = \mathbf{D}_n$). Hence, the above equations can be written as:

$$3\mathbf{P}_1 - \mathbf{P}_2 = 2\mathbf{D}_0$$

$$-\mathbf{P}_n + 3\mathbf{P}_{n+1} = 2\mathbf{D}_n$$

By combining these two equations with the equations on page 121 marked with #, we have a system of $n+1$ equations in $n+1$ unknows:

$$3\mathbf{P}_1 - \mathbf{P}_2 = 2\mathbf{D}_0$$

$$\frac{1}{4}\mathbf{P}_1 + \frac{7}{12}\mathbf{P}_2 + \frac{1}{6}\mathbf{P}_3 = \mathbf{D}_1$$

$$\frac{1}{6}\mathbf{P}_2 + \frac{2}{3}\mathbf{P}_3 + \frac{1}{6}\mathbf{P}_4 = \mathbf{D}_2$$

$$\ldots$$

$$\frac{1}{6}\mathbf{P}_{n-2} + \frac{2}{3}\mathbf{P}_{n-1} + \frac{1}{6}\mathbf{P}_n = \mathbf{D}_{n-2}$$

$$\frac{1}{6}\mathbf{P}_{n-1} + \frac{7}{12}\mathbf{P}_n + \frac{1}{4}\mathbf{P}_{n+1} = \mathbf{D}_{n-1}$$

$$-\mathbf{P}_n + 3\mathbf{P}_{n+1} = 2\mathbf{D}_n$$

The matrix form of this system is:

$$
\begin{bmatrix}
3 & -1 & & & & & \\
1/4 & 7/12 & 1/6 & & & & \\
 & 1/6 & 2/3 & 1/6 & & & \\
 & & & \cdot & & & \\
 & & & \cdot & & & \\
 & & & 1/6 & 2/3 & 1/6 & \\
 & & & & 1/6 & 7/12 & 1/4 \\
 & & & & & -1 & 3
\end{bmatrix}
\begin{bmatrix}
\mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \\ \cdot \\ \cdot \\ \mathbf{P}_{n-1} \\ \mathbf{P}_n \\ \mathbf{P}_{n+1}
\end{bmatrix}
=
\begin{bmatrix}
2\mathbf{D}_0 \\ \mathbf{D}_1 \\ \mathbf{D}_2 \\ \cdot \\ \cdot \\ \mathbf{D}_{n-2} \\ \mathbf{D}_{n-1} \\ 2\mathbf{D}_n
\end{bmatrix}
$$

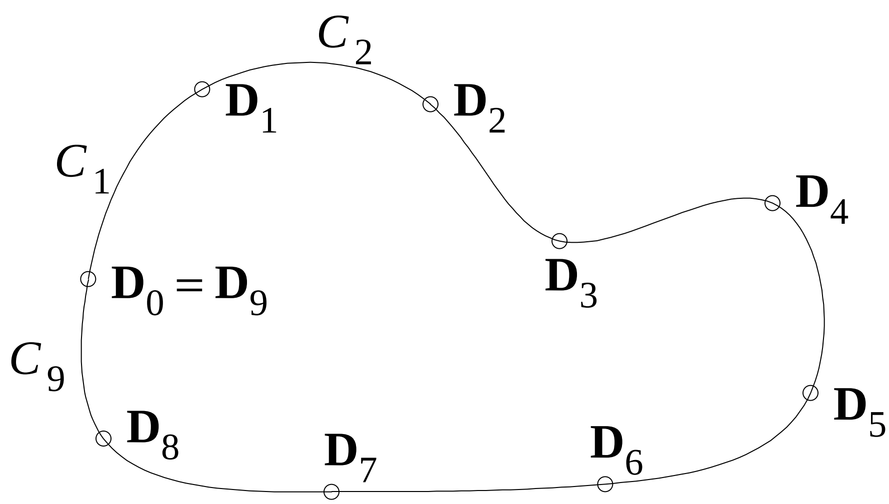
This system of equaitons can be solved using Gaussian elimination without pivoting.

Most of the curve drawing programs in 2D drawing software packages (such as Gremlin, xfig, ...) are implemented using this approach.

Question:

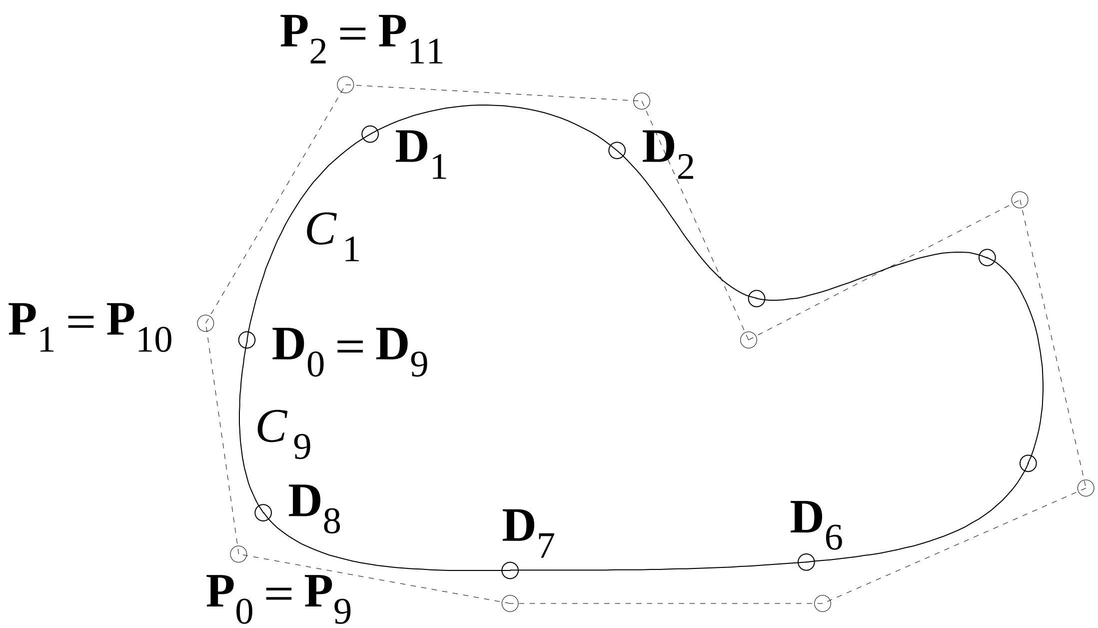
- What should we do to generate smooth, closed interpolating curves?

  For instance, give a set of data points $\mathbf{D}_i = (x_i, y_i)$, $i = 0, 1, \cdots, n$, $(n \geq 2)$, with $\mathbf{D}_0 = \mathbf{D}_n$, how can a closed, smooth (C^2 continuous) cubic B-spline curve that interpolates these points be constructed?



- The closed cubic B-spline curve has n segments $\mathbf{C}_1(t)$, $\mathbf{C}_2(t)$, ..., $\mathbf{C}_n(t)$ with $\mathbf{D}_{i-1}$ and $\mathbf{D}_i$ being the start and end points of $\mathbf{C}_i(t)$

- From previous discussion, we know such a curve must have $(n+3)$ control points: $\mathbf{P}_0$, $\mathbf{P}_1$, ..., $\mathbf{P}_{n+2}$.



- To guarantee C^2 continuity at $\mathbf{C}_0 = \mathbf{C}_n$, control points must satisfy the following conditions:

$$\mathbf{P}_n = \mathbf{P}_0, \quad \mathbf{P}_{n+1} = \mathbf{P}_1, \quad \mathbf{P}_{n+2} = \mathbf{P}_2$$

- Such a curve needs $(n+7)$ knots: $t_0, t_1, \ldots, t_{n+6}$.

- To make things easier, we shall assume that

$$t_i = i - 3, \qquad i = 0,1,\ldots,n+6$$

- Such a (cyclic) curve can be defined as follows:

$$C(t) = \sum_{i=0}^{n+2} N_{i,3}(t)\,\mathbf{P}_{(i \ mod \ n)}, \qquad t \in [t_3, t_{n+3}] = [0, n]$$

such that

$$C(t_{i+3}) = C(i) = \sum_{i=0}^{n+2} N_{i,3}(i)\,\mathbf{P}_{(i \ mod \ n)} = \mathbf{D}_i, \qquad \text{(C1)}$$

$$i = 0,1,\ldots,n$$

There are n unknows and n equations in the above system (C1):

$$\frac{1}{6}\mathbf{P}_0 + \frac{2}{3}\mathbf{P}_1 + \frac{1}{6}\mathbf{P}_2 = \mathbf{D}_0$$

$$\frac{1}{6}\mathbf{P}_1 + \frac{2}{3}\mathbf{P}_2 + \frac{1}{6}\mathbf{P}_3 = \mathbf{D}_1$$

$$\ldots$$

$$\frac{1}{6}\mathbf{P}_{n-2} + \frac{2}{3}\mathbf{P}_{n-1} + \frac{1}{6}\mathbf{P}_n = \mathbf{D}_{n-2}$$

$$\frac{1}{6}\mathbf{P}_{n-1} + \frac{2}{3}\mathbf{P}_n + \frac{1}{6}\mathbf{P}_{n+1} = \mathbf{D}_{n-1}$$

$$\frac{1}{6}\mathbf{P}_n + \frac{2}{3}\mathbf{P}_{n+1} + \frac{1}{6}\mathbf{P}_{n+2} = \mathbf{D}_n$$

(The last equation is the same as the first equation and, hence, can be ignored.)

The matrix form of this system is:

$$
\begin{bmatrix}
2/3 & 1/6 & & & & & & 1/6 \\
1/6 & 2/3 & 1/6 & & & & & \\
& 1/6 & 2/3 & 1/6 & & & & \\
& & & & . & & & \\
& & & & . & & & \\
& & & & 1/6 & 2/3 & 1/6 & \\
& & & & & 1/6 & 2/3 & 1/6 \\
1/6 & & & & & & 1/6 & 2/3
\end{bmatrix}
\begin{bmatrix}
\mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \\ . \\ . \\ \mathbf{P}_{n-1} \\ \mathbf{P}_n
\end{bmatrix}
=
\begin{bmatrix}
\mathbf{D}_0 \\ \mathbf{D}_1 \\ \mathbf{D}_2 \\ . \\ . \\ \mathbf{D}_{n-2} \\ \mathbf{D}_{n-1}
\end{bmatrix}
$$

This system of equaitons can be solved using Gaussian elimination without pivoting as well.