5.5 Subdivision Depth Computation Techniques for CCSS

For a given $\epsilon$, compute a subdivision depth $k$ for each patch of the given Catmull-Clark subdivision surface so that, after $k$ levels of recursive subdivision, distance between the control mesh and the limit surface is $\leq \epsilon$.
Basic idea: cubic B-spline curve segment case

If

$$L(t) = V_1 + (V_2 - V_1) \times t$$

$$C(t) = \text{cubic } B - \text{spline curve segment}$$

then, we have

$$\left\| L(t) - C(t) \right\| \leq \frac{1}{6} \max\{ \left\| 2V_1 - V_0 - V_2 \right\|, \left\| 2V_2 - V_1 - V_3 \right\| \}$$
\[ \| L(t) - C(t) \| \]

\[ = \| B_{03}(t)V_1 + B_{13}(t)B + B_{23}(t)C + B_{33}(t)V_2 - B_{03}(t)A - B_{13}(t)B - B_{23}(t)C - B_{33}(t)D \| \]

\[ = \| (1 - t)^3(V_1 - A) + t^3(V_2 - D) \| \]
After performing one subdivision, we have

\[ \|2V_1' - V_2' - V_0'\| \leq \frac{1}{4} \|2V_1 - V_2 - V_0\| \]

\[ \|2V_3' - V_2' - V_4'\| \leq \frac{1}{4} \|2V_2 - V_1 - V_3\| \]
Actually, we also have

We also have

$$\|2V_2' - V_1' - V_3'\| \leq \frac{1}{4} \max \{ \| 2V_1 - V_2 - V_0 \|, \| 2V_2 - V_1 - V_3 \| \}$$
If we define

\[ M_0 = \max_k \{ \|2V_k - V_{k-1} - V_{k+1}\| \} \]

\[ M_1 = \max_k \{ \|2V'_k - V'_{k-1} - V'_{k+1}\| \} \]

The above results show that

\[ M_1 \leq \frac{1}{4} M_0 \]
In general, after $i$ levels of recursive subdivision, if we use $\mathbf{V}_k^{(i)}$ to represent the new control points, and set

$$M_i = \max_k \left\{ \|2\mathbf{V}_k^{(i)} - \mathbf{V}_{k-1}^{(i)} - \mathbf{V}_{k+1}^{(i)}\| \right\}$$

we have

$$M_i \leq \left(\frac{1}{4}\right)^i M_0$$

$M_i$ is called the 2nd order norm of $\{\mathbf{V}_k^{(i)}\}$.
Subdivision depth computation for a cubic B-spline curve segment

After $i$ levels of recursive subdivision, for each segment, we have

$$\| L_i(t) - C_i(t) \| \leq \frac{1}{6} M_i \leq \frac{1}{6} \left( \frac{1}{4} \right)^i M_0 \leq \epsilon$$

if

$$i \geq \log_4 \left( \frac{M_0}{6\epsilon} \right)$$

i.e., for the control polygon to be close enough to the limit curve (within $\epsilon$), we need to perform at least $i$ levels of recursive subdivision.
Subdivision depth computation for an Ordinary patch

\[ S(u, v) = \text{bicubic } B - \text{ spline surface patch} \]
If we parametrize the center face bilinearly as follows

\[ \mathbf{L}(u, v) = (1 - v)[(1 - u)\mathbf{V}_{11} + u\mathbf{V}_{21}] + v[(1 - u)\mathbf{V}_{12} + u\mathbf{V}_{22}] \]

then

\[ \| \mathbf{L}(u, v) - \mathbf{S}(u, v) \| \leq \frac{1}{3} M_0 \]

where \( M_0 \) is the 2nd order norm of \( \{\mathbf{V}_{kl}\} \)

\[ M_0 = \max_{k,l} \left\{ \| 2\mathbf{V}_{k,l} - \mathbf{V}_{k-1,l} - \mathbf{V}_{k+1,l} \|, \right. \]

\[ \left. \| 2\mathbf{V}_{k,l} - \mathbf{V}_{k,l-1} - \mathbf{V}_{k,l+1} \| \right\} \]
Proof

$$\| \mathbf{L}(u, v) - \mathbf{S}(u, v) \|$$

$$= \| (1 - v)\mathbf{L}_1(u) + v\mathbf{L}_2(u)$$

$$- (1 - v) \sum_{i=0}^{3} N_{i3}(u) \mathbf{V}_{i1} - v \sum_{i=0}^{3} N_{i3}(u) \mathbf{V}_{i2}$$

$$+ (1 - v) \sum_{i=0}^{3} N_{i3}(u) \mathbf{V}_{i1} + v \sum_{i=0}^{3} N_{i3}(u) \mathbf{V}_{i2}$$

$$- \sum_{i=0}^{3} \sum_{j=0}^{3} N_{i3}(u) N_{j3}(v) \mathbf{V}_{ij} \|$$
\[ \leq \| (1 - v)(L_1(u) - C_1(u)) + v(L_2(u) - C_2(u)) \| \\
+ \| \sum_{i=0}^{3} N_{i3}(u)[((1 - v)V_{i1} + vV_{i2}) \\
- \sum_{j=0}^{3} N_{j3}(v)V_{ij}] \| \\
\leq (1 - v)\frac{1}{6} \max\left\{ \|2V_{11} - V_{01} - V_{21}\|, \\
\|2V_{21} - V_{11} - V_{31}\| \right\} \\
+ v\frac{1}{6} \max\left\{ \|2V_{12} - V_{02} - V_{22}\|, \\
\|2V_{22} - V_{12} - V_{32}\| \right\} \\
+ \sum_{i=0}^{3} N_{i3}(u)\frac{1}{6} \max\left\{ \|2V_{i1} - V_{i0} - V_{i2}\|, \\
\|2V_{i2} - V_{i1} - V_{i3}\| \right\} \]
Proof (conti)

Let $M_0$ be the 2nd order norm of $S(u, v)$. Then

$$
\| L(u, v) - S(u, v) \| \leq (1 - v) \frac{1}{6} M_0 + v \frac{1}{6} M_0
$$

$$
+ \Sigma_{i=0}^{3} N_{i3}(u) \frac{1}{6} M_0
$$

$$
\leq \frac{1}{6} (M_0 + M_0)
$$

$$
= \frac{1}{3} M_0
$$
In general, after $i$ levels of recursive subdivision, if we use $V_{kl}^{(i)}$ to represent the new control points, we have

$$M_i \leq \left(\frac{1}{4}\right)^i M_0$$

where $M_0$ is the 2nd order norm of $\{V_{kl}\}$ and $M_i$ is the 2nd order norm of $V_{kl}^{(i)}$

$$M_i = \max_{k,l} \left\{ \|2V_{k,l}^{(i)} - V_{k-1,l}^{(i)} - V_{k+1,l}^{(i)}\|, \right.$$

$$\left. \|2V_{k,l}^{(i)} - V_{k,l-1}^{(i)} - V_{k,l+1}^{(i)}\| \right\}$$
Subdivision depth computation for a bicubic B-spline surface patch

$S_k(u, v)$ is a subpatch after $i$ levels of recursive subdivision. If we parametrize the center face of the control mesh of $S_k(u, v)$ bilinearly and call it $L_k(u, v)$, we have

$$
\|L_k(u, v) - S_k(u, v)\| \leq \frac{1}{3} M_i \leq \frac{1}{3} \left(\frac{1}{4}\right)^i M_0 \leq \epsilon
$$

as long as

$$
i \geq \log_4 \left(\frac{M_0}{3\epsilon}\right)
$$
End of 5.5