15.6 Forward Differencing

- Another technique to render a (cubic) curve
- Each component of a cubic Bezier curve is a polynomial of degree 3. Hence, the question is: how to efficiently compute points of a cubic polynomial $f(t)$ at $0, \delta, 2\delta, 3\delta, \ldots, 1$?

- **Forward differencing**: only three additions are needed to compute a new point
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Let

\[ f(t) = a + bt + ct^2 + dt^3, \quad t \in [0, 1] \]  

and \( \delta > 0 \) given. Define

\[ \Delta f(t) = f(t + \delta) - f(t), \quad t \in [0, 1] \]

\( \Delta \) is called a forward differencing operator

If we know \( f(t) \) and \( \Delta f(t) \) then from (*) we have

\[ f(t + \delta) = f(t) + \Delta f(t) \] (**)
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Equation (**) shows that if we know \( f(0) \) and \( \Delta f(0) \) then we can compute \( f(\delta) \) as follows:

\[
f(\delta) = f(0) + \Delta f(0)
\]

If we know \( f(\delta) \) and \( \Delta f(\delta) \) then we can compute \( f(2\delta) \) as follows:

\[
f(2\delta) = f(\delta) + \Delta f(\delta)
\]
If we know $f(2\delta)$ and $\Delta f(2\delta)$ then we can compute $f(3\delta)$ as follows:

$$f(3\delta) = f(2\delta) + \Delta f(2\delta)$$

\[
\begin{align*}
&f(0) \quad \rightarrow \quad f(\delta) \quad \rightarrow \quad f(2\delta) \quad \rightarrow \quad f(3\delta) \quad \rightarrow \quad f(4\delta) \quad \ldots \\
&\quad \Delta f(0) \quad \rightarrow \quad \Delta f(\delta) \quad \rightarrow \quad \Delta f(2\delta) \quad \rightarrow \quad \Delta f(3\delta) \quad \rightarrow \quad \Delta f(4\delta)
\end{align*}
\]
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Note that

\[ \Delta f(t) = (b\delta + c\delta^2 + d\delta^3) + (2c\delta + 3d\delta^2)t + (3d\delta)t^2 \]

(***)

a polynomial of degree 2. So \( \Delta f(\delta), \Delta f(2\delta), \) ... are computable.
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But, instead of using (***) directly, is there a way to compute $\Delta f(\delta), \Delta f(2\delta), \Delta f(3\delta), \ldots$ more efficiently? **YES**

If we define

$$\Delta^2 f(t) \equiv \Delta(\Delta f(t)) = \Delta f(t + \delta) - \Delta f(t)$$  (#)

then $\Delta f(t + \delta)$ can be computed as follows if $\Delta f(t)$ and $\Delta^2 f(t)$ are known to us

$$\Delta f(t + \delta) = \Delta f(t) + \Delta^2 f(t)$$  (##)
For instance, if $\Delta f(0)$ and $\Delta^2 f(0)$ are known to us then we can compute $\Delta f(\delta)$ as follows:

$$\Delta f(\delta) = \Delta f(0) + \Delta^2 f(0)$$

If $\Delta f(\delta)$ and $\Delta^2 f(\delta)$ are known to us then we can compute $\Delta f(2\delta)$ as follows:

$$\Delta f(2\delta) = \Delta f(\delta) + \Delta^2 f(\delta)$$
If $\Delta f(2\delta)$ and $\Delta^2 f(2\delta)$ are known to us then we can compute $\Delta f(3\delta)$ as follows:

$$\Delta f(3\delta) = \Delta f(2\delta) + \Delta^2 f(2\delta)$$

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<table>
<thead>
<tr>
<th>$f(0)$</th>
<th>$\rightarrow$</th>
<th>$f(\delta)$</th>
<th>$\rightarrow$</th>
<th>$f(2\delta)$</th>
<th>$\rightarrow$</th>
<th>$f(3\delta)$</th>
<th>$\rightarrow$</th>
<th>$f(4\delta)$</th>
<th>$\ldots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta f(0)$</td>
<td>$\rightarrow$</td>
<td>$\Delta f(\delta)$</td>
<td>$\rightarrow$</td>
<td>$\Delta f(2\delta)$</td>
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<td>$\Delta f(3\delta)$</td>
<td>$\rightarrow$</td>
<td>$\Delta f(4\delta)$</td>
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<tr>
<td>$\Delta^2 f(0)$</td>
<td>$\rightarrow$</td>
<td>$\Delta^2 f(\delta)$</td>
<td>$\rightarrow$</td>
<td>$\Delta^2 f(2\delta)$</td>
<td>$\rightarrow$</td>
<td>$\Delta^2 f(3\delta)$</td>
<td>$\rightarrow$</td>
<td>$\Delta^2 f(4\delta)$</td>
<td></td>
</tr>
</tbody>
</table>
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Note that

$$\Delta^2 f(t) = (2c \delta^2 + 6d \delta^3) + (6d \delta^2)t,$$

a polynomial of degree 1. So $\Delta^2 f(0), \Delta^2 f(\delta), \Delta^2 f(2\delta), \ldots$ are computable.

But, again, is there a way to compute $\Delta^2 f(\delta), \Delta^2 f(2\delta), \Delta^2 f(3\delta), \ldots$ more efficiently, instead of using (###)? YES
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Define

\[ \Delta^3 f(t) \equiv \Delta(\Delta^2 f(t)) = \Delta^2 f(t + \delta) - \Delta^2 f(t) \quad (\&) \]

From (###), we have

\[ \Delta^3 f(t) \equiv 6d \delta^3 \quad (\&&) \]

a constant.
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Equation (15.6) shows that if $\Delta^2 f(t)$ and $\Delta^3 f(t)$ are known to us then $\Delta^2 f(t + \delta)$ can be computed as their sum.

$$\Delta^2 f(t + \delta) = \Delta^2 f(t) + \Delta^3 f(t) \quad (\&\&\&)$$

For instance, if $\Delta^2 f(0)$ and $\Delta^3 f(0)$ are known to us then we can compute $\Delta^2 f(\delta)$ as follows:

$$\Delta^2 f(\delta) = \Delta^2 f(0) + \Delta^3 f(0)$$
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If $\Delta^2 f(\delta)$ and $\Delta^3 f(\delta)$ are known to us then we can compute $\Delta^2 f(2\delta)$ as follows:

$$\Delta^2 f(2\delta) = \Delta^2 f(\delta) + \Delta^3 f(\delta)$$

If $\Delta^2 f(2\delta)$ and $\Delta^3 f(2\delta)$ are known to us then we can compute $\Delta^2 f(3\delta)$ as follows:

$$\Delta^2 f(3\delta) = \Delta^2 f(2\delta) + \Delta^3 f(2\delta)$$
15.6 Forward Differencing

One needs to compute the values of $f(0)$, $\Delta f(0)$, $\Delta^2 f(0)$, and $\Delta^3 f(0)$ using (\%), (%%%), (###), and (&&), respectively, first.

$$f(0) = a$$

$$\Delta f(0) = b\delta + c\delta^2 + d\delta^3$$

$$\Delta^2 f(0) = 2c\delta^2 + 6d\delta^3$$

$$\Delta^3 f(0) = 6d\delta^3$$

These terms requires several multiplications. But every subsequent point then requires 3 additions to compute only.
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Each of this iteration process requires three additions.

Note that only the items above the line in the above table are needed in the rendering process of the curve. The items below the line are used to find the items above the line.

**Forward differencing** is the most efficient curve rendering technique. However, since numerical errors will be propagated all the way from $f(0)$ to the last term, $f(n\delta) = f(1)$, it is **not numerically stable**.
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For instance, once we have the values of \( f(0), \Delta f(0), \Delta^2 f(0), \) and \( \Delta^3 f(0) \) in the first column of the above table, we can then compute the values of the items in the second column by adding \( \Delta f(0) \) to \( f(0) \) to get \( f(\delta) \), adding \( \Delta^2 f(0) \) to \( \Delta f(0) \) to get \( \Delta f(\delta) \), adding \( \Delta^3 f(0) \) to \( \Delta^2 f(0) \) to get \( \Delta^2 f(\delta) \), and setting \( \Delta^3 f(\delta) = \Delta^3 f(0) \).

The values of the items in the third column are determined using a similar approach, i.e., adding \( \Delta f(\delta) \) to \( f(\delta) \) to get \( f(2\delta) \), adding \( \Delta^2 f(\delta) \) to \( \Delta f(\delta) \) to get \( \Delta f(2\delta) \), adding \( \Delta^3 f(\delta) \) to \( \Delta^2 f(\delta) \) to get \( \Delta^2 f(2\delta) \), and setting \( \Delta^3 f(2\delta) = \Delta^3 f(\delta) \).
End of 15.6