CS375:
Logic and Theory of Computing

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8. Turing Machines and Equivalent Models – The Church-Turing Thesis

Goal: to show you construction of Turing Machines that can perform:

1. the addition function,
2. the subtraction function,
3. the multiplication function, and
4. the division function

for unary numbers

TM for the addition function for the unary number system

Notations:

The unary number is made up of only one character, i.e. The number 5 can be written in unary number system as 11111. In this TM, we are going to perform the addition of two unary numbers.

For example:

2 + 3

i.e., 11 + 111 = 11111

Solution:

If you observe this process of addition, you will find the resemblance with **string concatenation** function.

In this case, we simply replace ‘+’ by ‘1’ and move right to search for the right most ‘1’ and convert that ‘1’ to ‘Λ’.

Input: 4+2

The simulation for 111+11Λ can be shown as below:

1	1	1	1	+	1	1	Λ			

State=0

Move right up to the + sign:

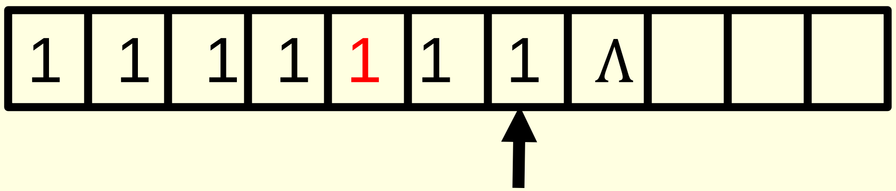
1	1	1	1	+	1	1	Λ			

Convert + to 1 and move right:

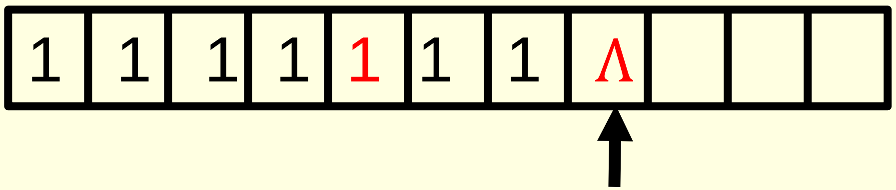
1	1	1	1	1	1	1	Λ			

Conti.

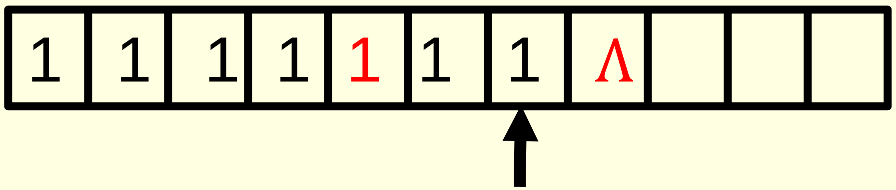
Keep moving right:



Until we reach Λ:

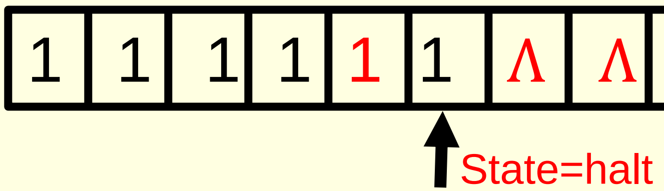


Move left :

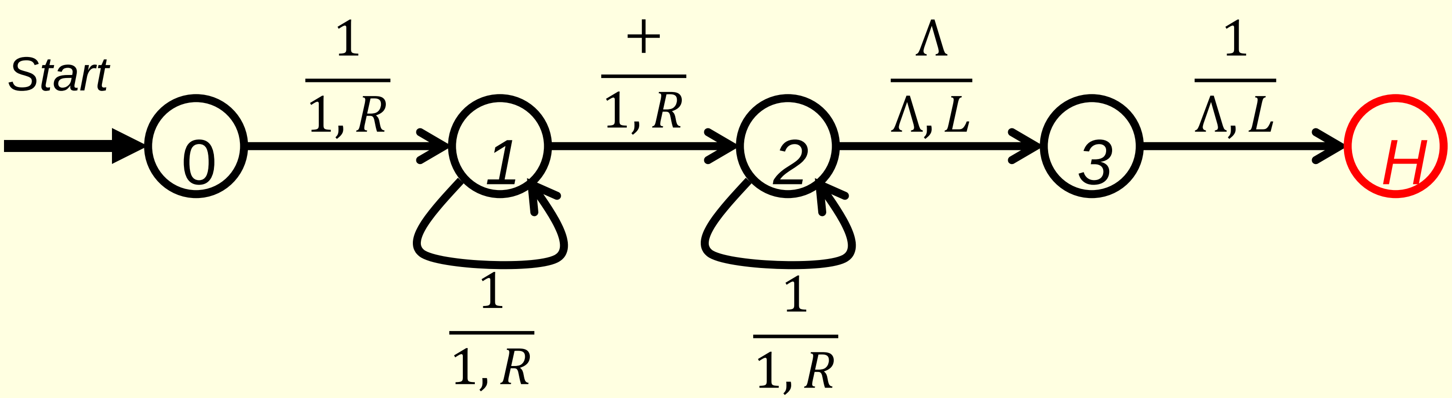


Conti.

Convert 1 to Λ, move left and stop :

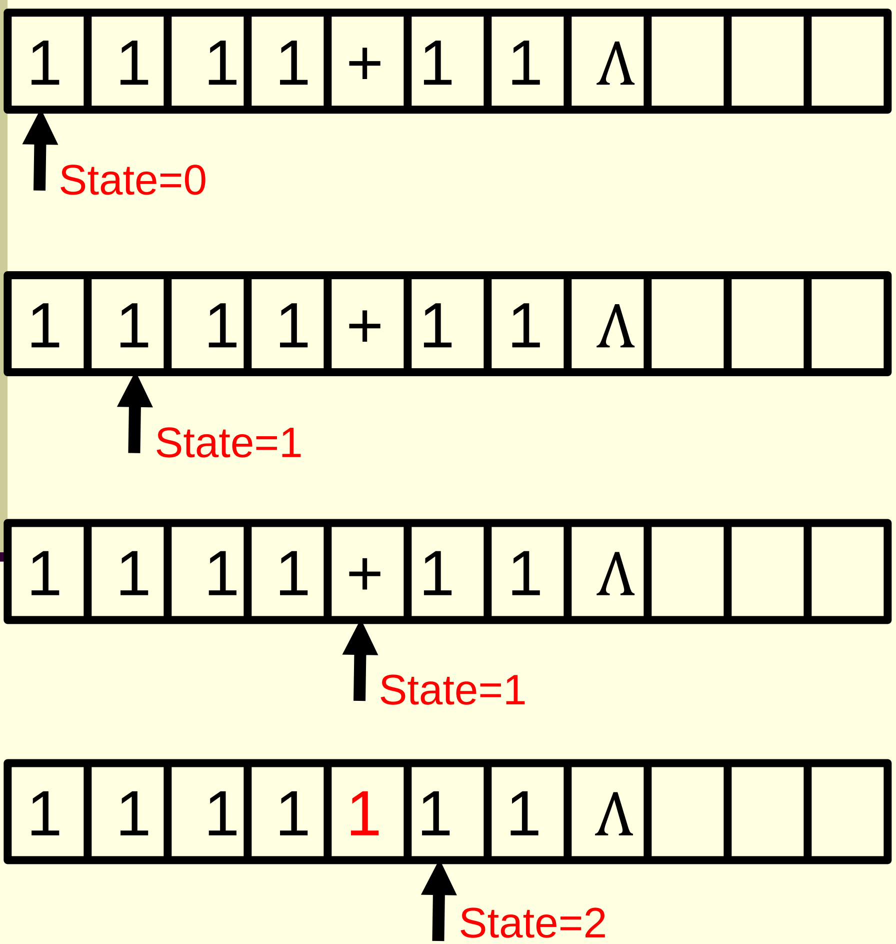


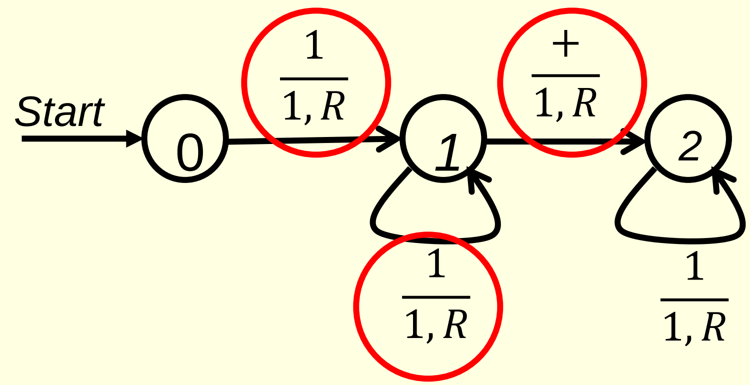
Such a TM would look like the one below :



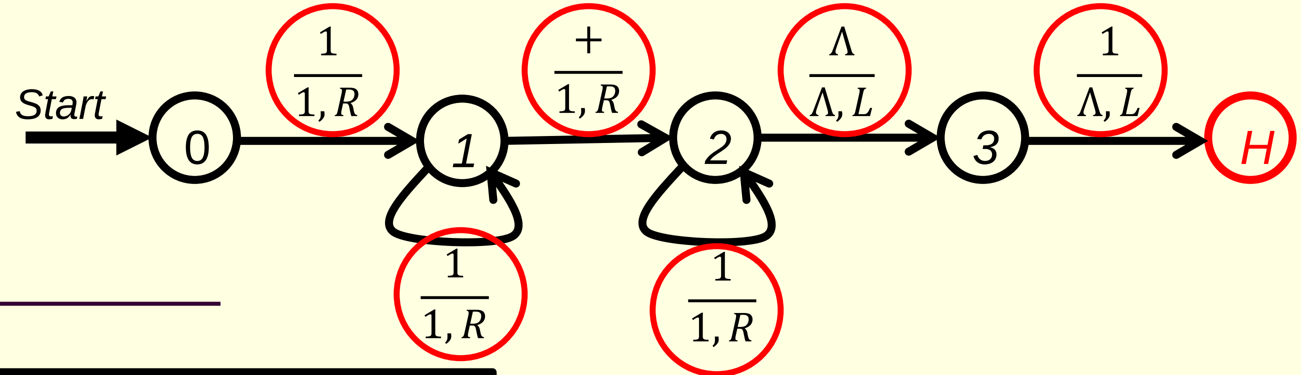
Input: 4+2

The simulation for 111+11Λ can be shown as below:





Conti.



1	1	1	1	1	1	1	Λ			

State=2

1	1	1	1	1	1	1	Λ			

State=2

1	1	1	1	1	1	1	Λ			

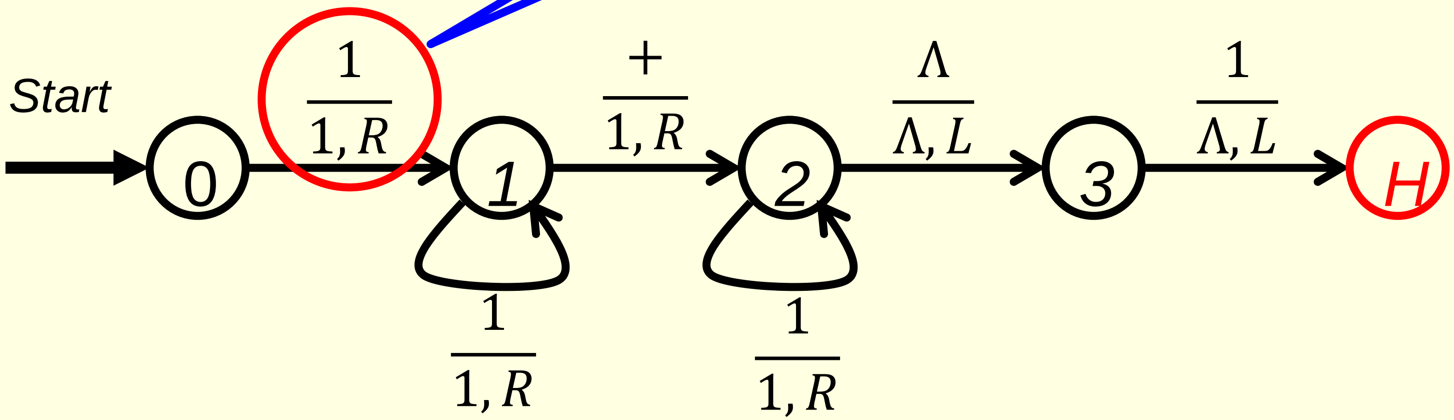
State=3

1	1	1	1	1	1	Λ	Λ			

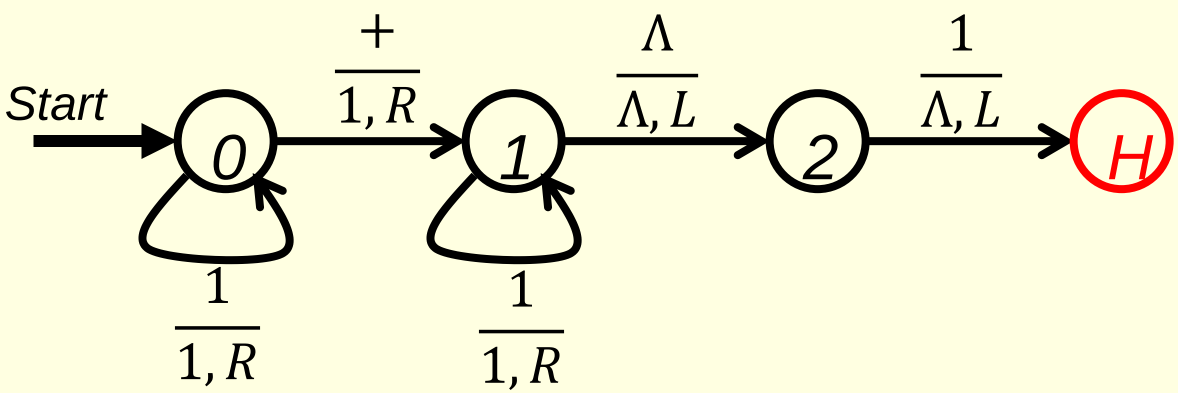
State=halt

Note:

This $\frac{1}{1,R}$ is to prevent having a '+' in the left most cell



Would the following TM work as well?



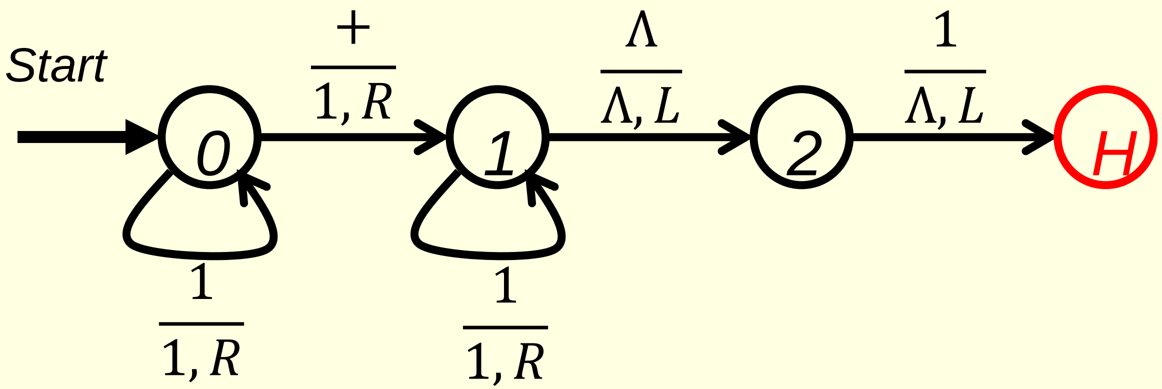
Can it handle the following input?

1	1	1	1	+	1	1	Λ			

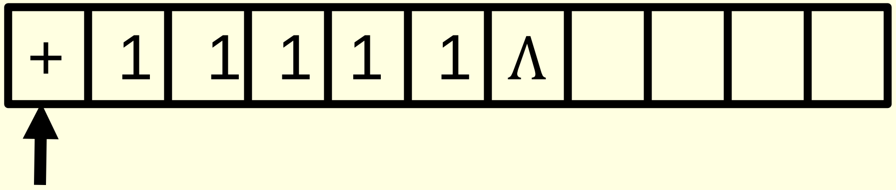
State=0

YES

Would the following TM work?



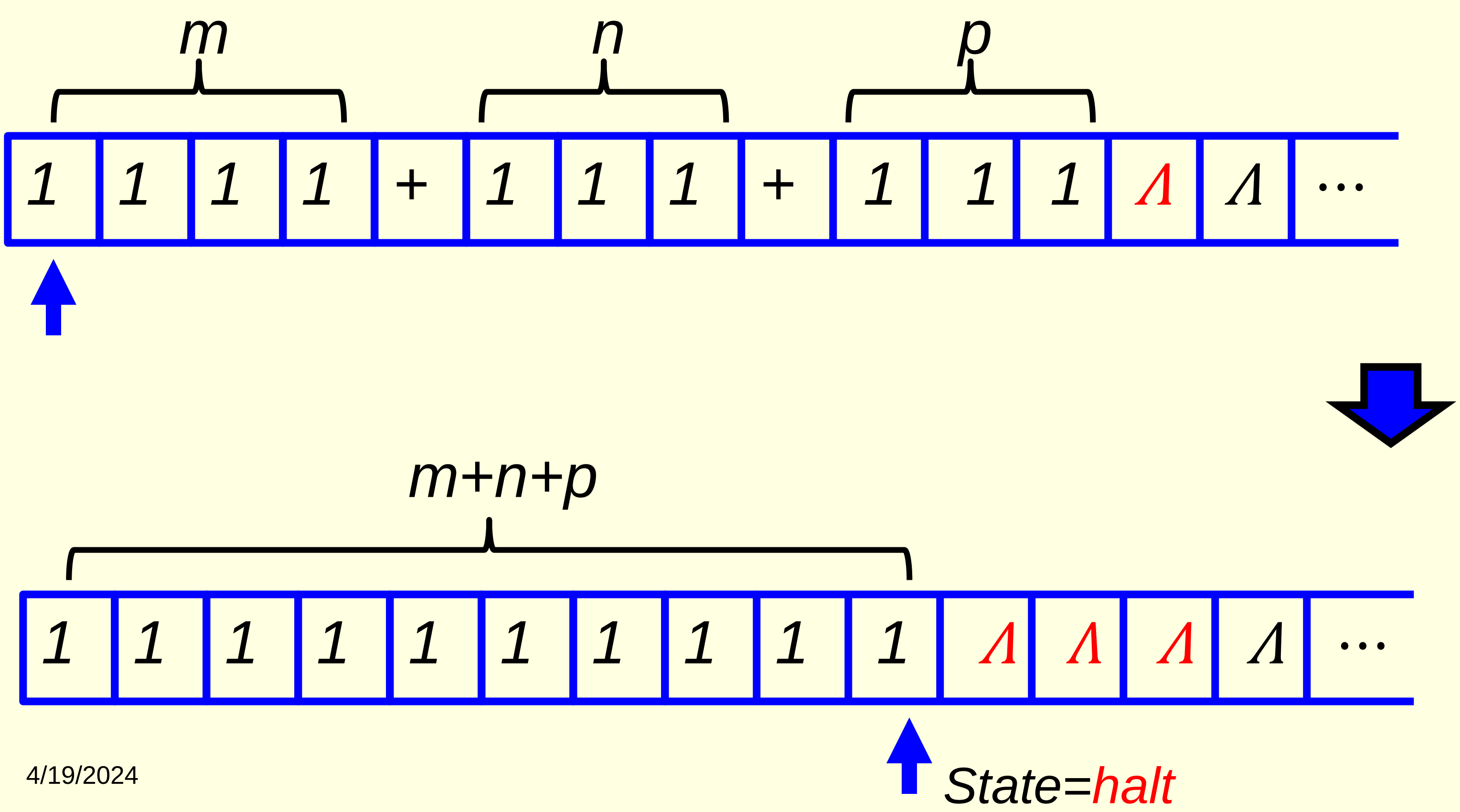
What if the input is like the one below?



This input would be rejected by the TM given in slide 9

Question:

Can the above TM be modified to do addition of three numbers, four numbers, ..., n numbers in unary form directly?



TM for the **subtraction function** for the unary number system

Notations and Assumption:

For example:

4 - 2

i.e., 1111 - 11 = 11

Develop a TM for the subtraction of two unary numbers f(a-b) = c where ‘a’ is greater than or equal to ‘b’.

Solution:

If there are n 1's in the unary representation of *b* and m 1's in the unary representation of *a*, the process is to reduct n 1's from the unary representation of *a*.

The reduction will be perform from the right side of the unary representation of *a*.

Input: 4-2

So the input tape will look like:

1	1	1	1	-	1	1	Λ			

State=0

Move right to the '–' symbol:

1	1	1	1	-	1	1	Λ			

Move right and convert 1 to '*' as:

1	1	1	1	-	*	1	Λ			

Move left:

1	1	1	1	-	*	1	Λ			

Conti.

Again move left:

1	1	1	1	-	*	1	Λ			

Convert 1 to '*' and move right:

1	1	1	*	-	*	1	Λ			

Keep moving right until a '1' is reached:

1	1	1	*	-	*	1	Λ			

Conti.

Convert 1 to * and move left:

1	1	1	*	-	*	*	Λ			

Keep moving left until a 1 is reached:

1	1	1	*	-	*	*	Λ			

Convert 1 to * and move right

1	1	*	*	-	*	*	Λ			

Conti.

Keep moving, ignore all *'s and '–' until we reach an '1' or a 'Λ'

1	1	*	*	-	*	*	Λ			

Find a 'Λ'. Turn left, ignore *'s and the '–', until an '1' is reached

1	1	*	*	-	*	*	Λ			

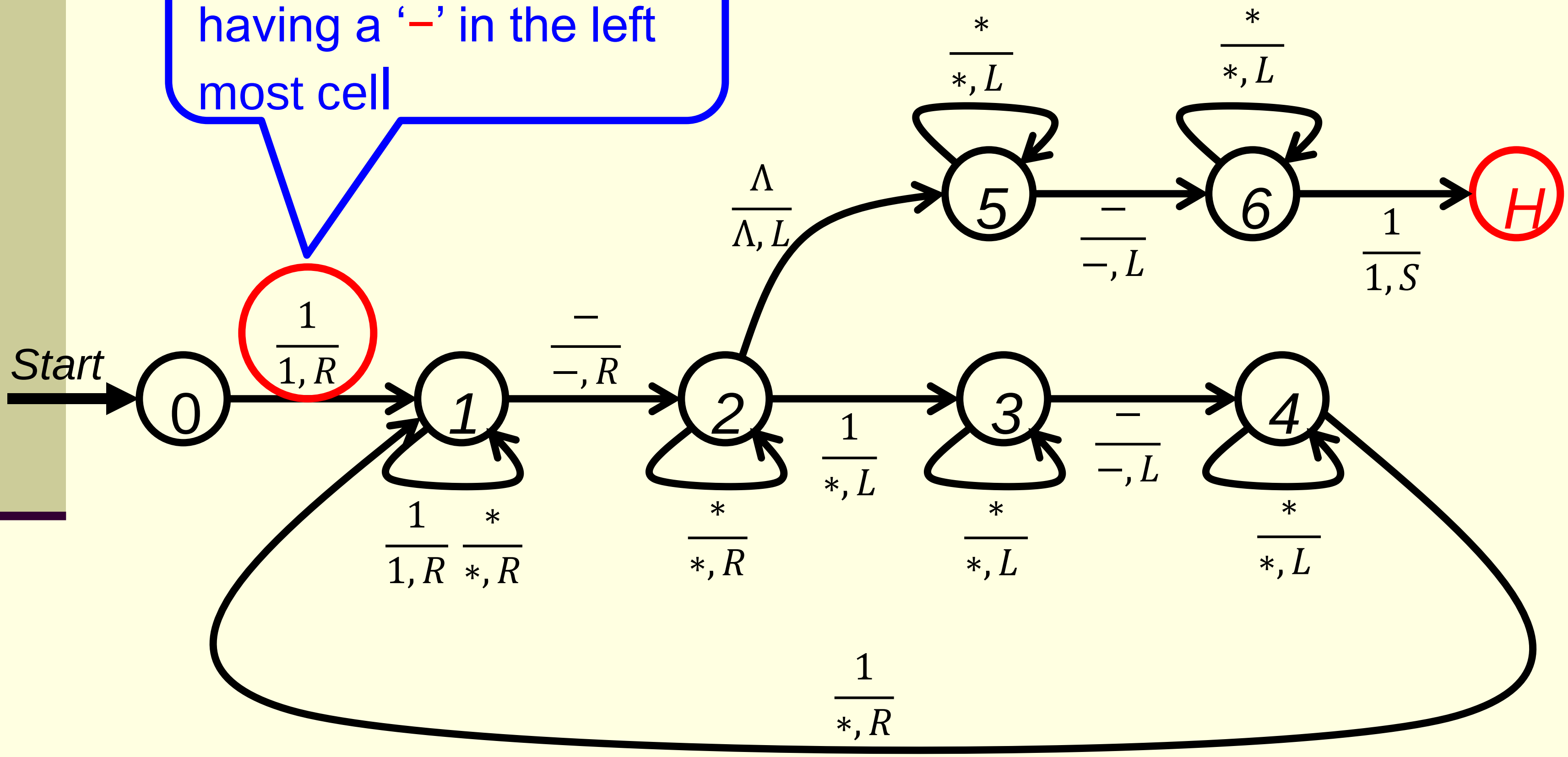
Change the state to 'halt' and stop.

1	1	*	*	-	*	*	Λ			

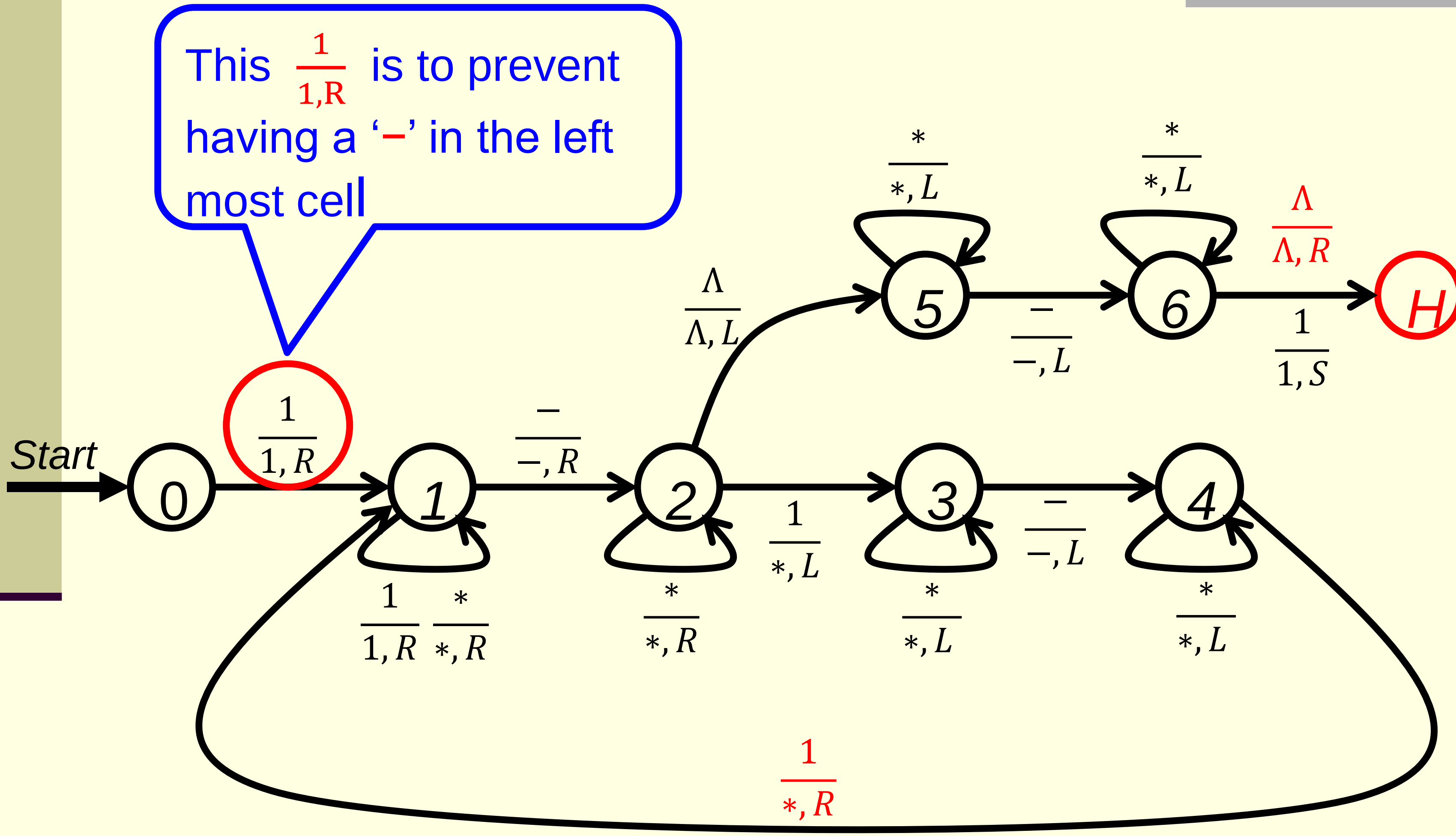
State=halt

The TM will look like the one below (when n < m):

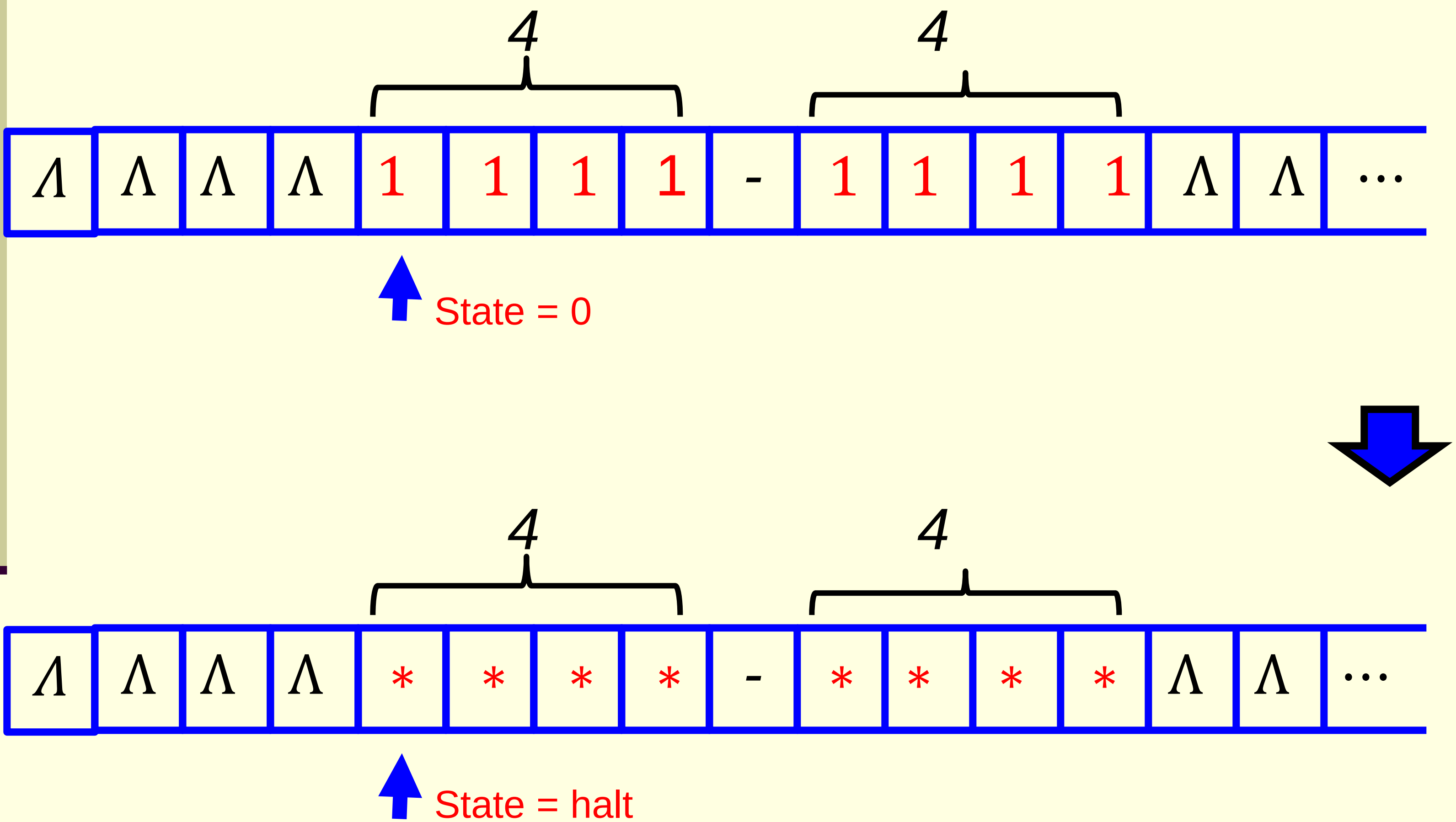
This $\frac{1}{1,R}$ is to prevent having a '$-$' in the left most cell



The TM will look like the one below (when n <= m):

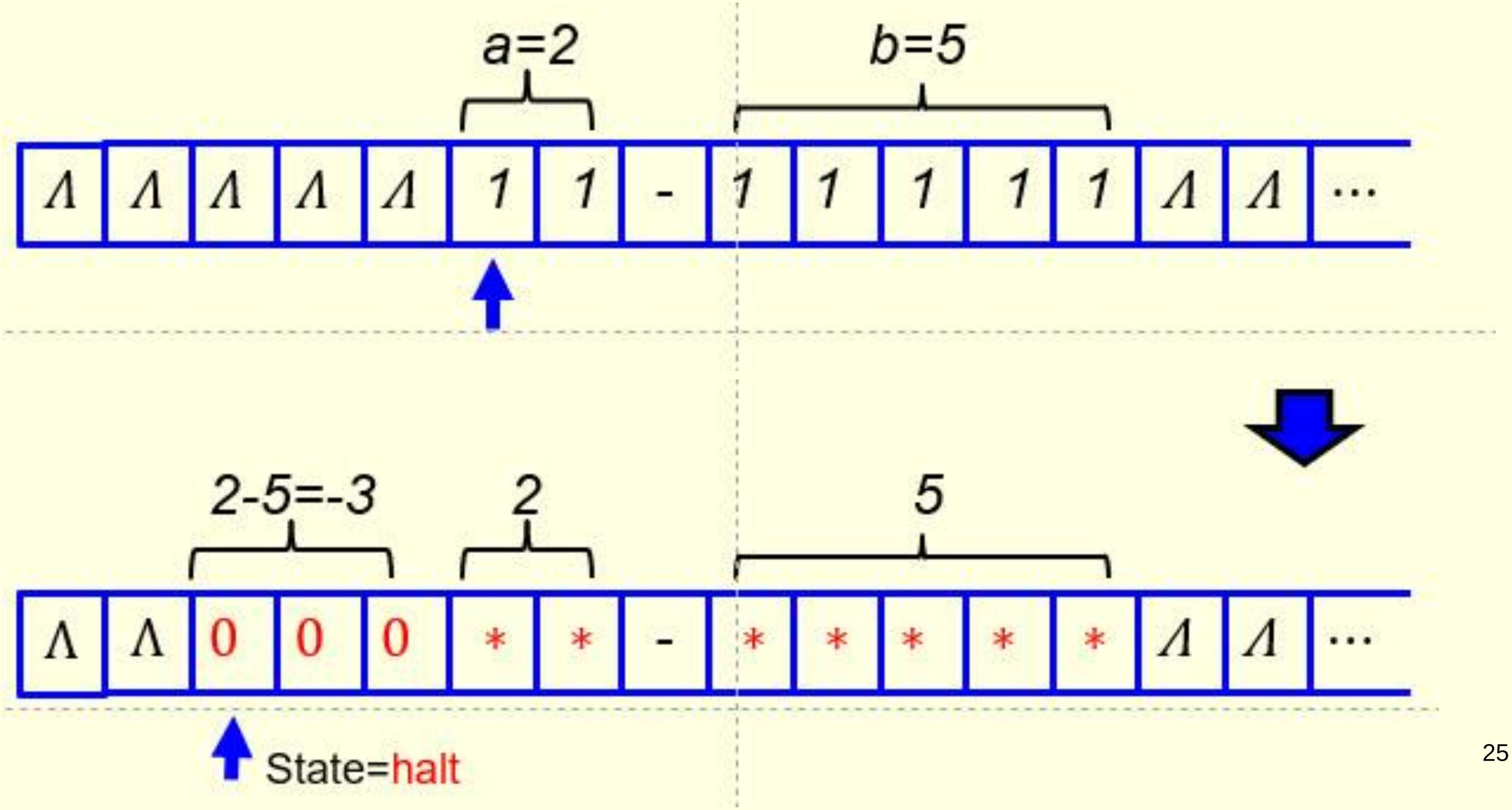


Note that in case *m=n*, the read/write head, at the end, will point at the left most '*'



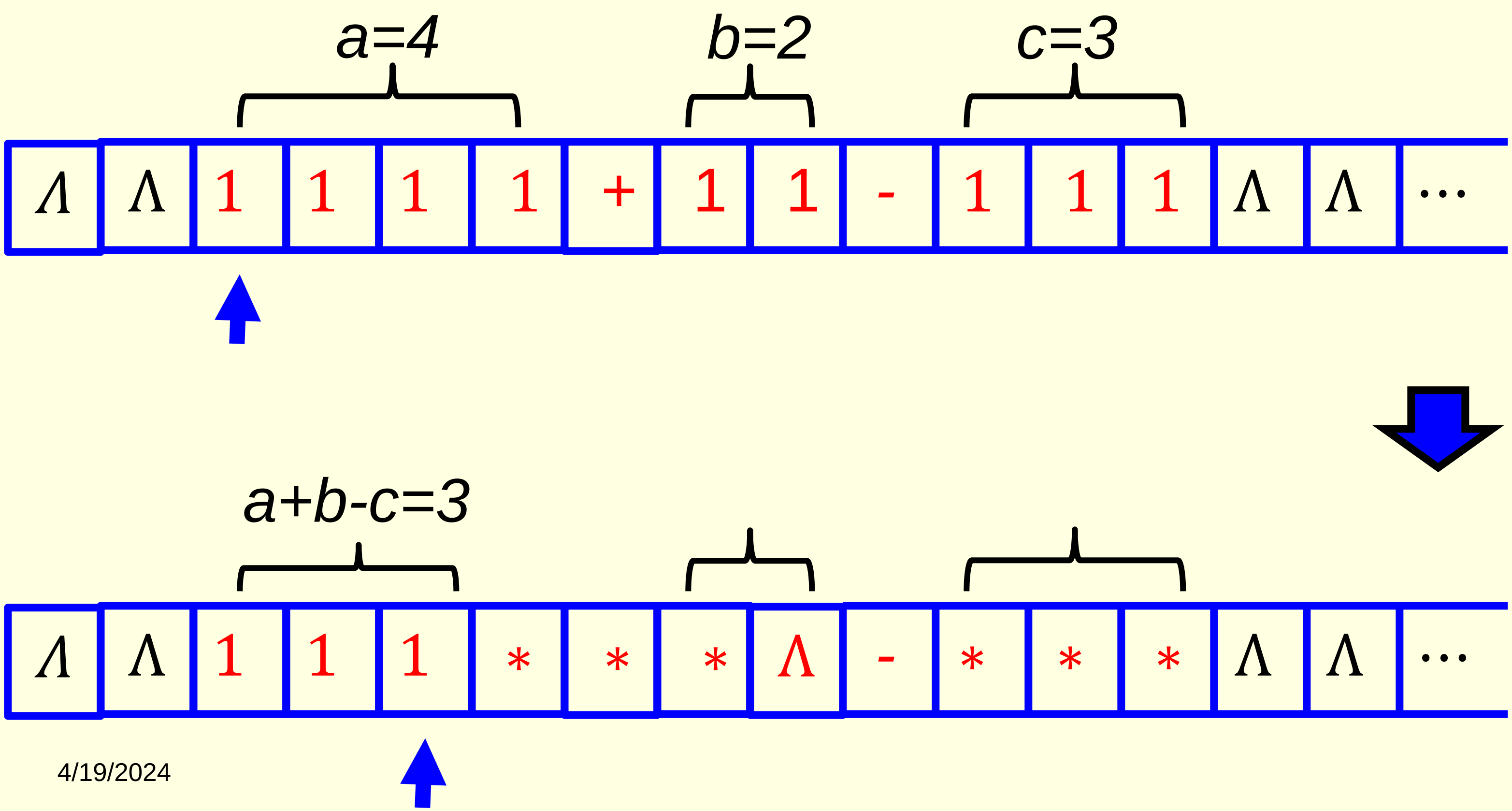
Question 1:

Can the above TM be modified to do subtraction $f(a-b)=c$ even when b is bigger than a?



Question 2:

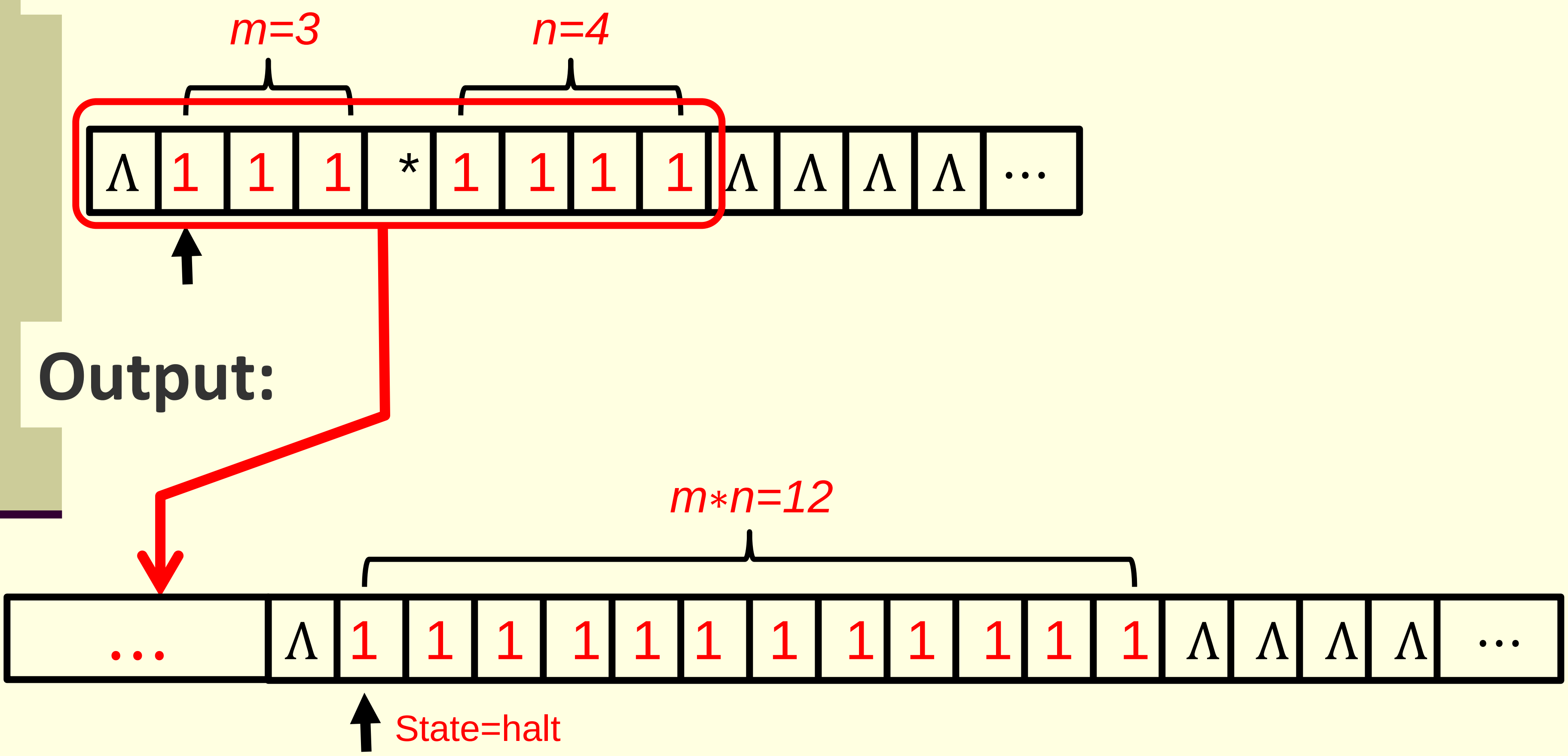
Given three non-zero positive unary numbers *a*, *b* and *c*, can a TM be built to carry out this function $f(a+b-c)$?



TM for the **multiplication function** for the unary number system

Input:

Output:



Basic concept: repeatedly attach *m* 1's to the end of the 2nd string until *n* iterations have been done

What is multiplication?

Extended addition. Why?

$$4 \times 5 = \left.\begin{array}{r} 4 \\ 4 \\ 4 \\ 4 \\ +\ 4 \\ \hline \end{array}\right\} \begin{array}{c} 5 \\ \text{copies} \end{array} = \left.\begin{array}{r} 1\,1\,1\,1 \\ 1\,1\,1\,1 \\ 1\,1\,1\,1 \\ 1\,1\,1\,1 \\ +\ 1\,1\,1\,1 \\ \hline \end{array}\right\} \begin{array}{c} 5 \\ \text{copies} \end{array}$$

Basic concept: repeatedly attach *m* 1's to the end of the 2nd string until *n* iterations have been done

What is multiplication?

Extended addition. **Why?**

```
  1 1 1 1                  1 1 1 1 1 1 1 1
  1 1 1 1                  1 1 1 1
  1 1 1 1        =         1 1 1 1              =
  1 1 1 1               +  1 1 1 1
+ 1 1 1 1               ___________
__________
```

Basic concept: repeatedly attach *m* 1's to the end of the 2nd string until *n* iterations have been done

What is multiplication?

Extended addition. Why?

1 1 1 1 1 1 1 1 1 1 1 1

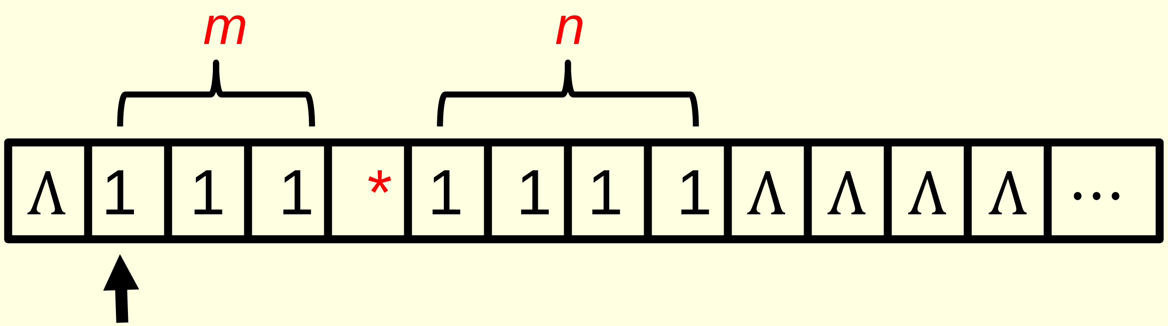
1 1 1 1

+ 1 1 1 1

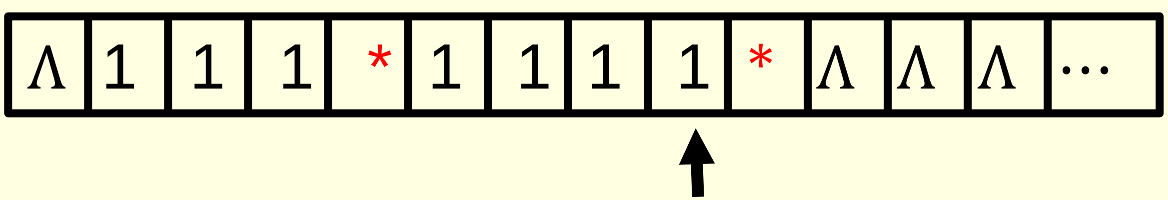
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1

Implementation:

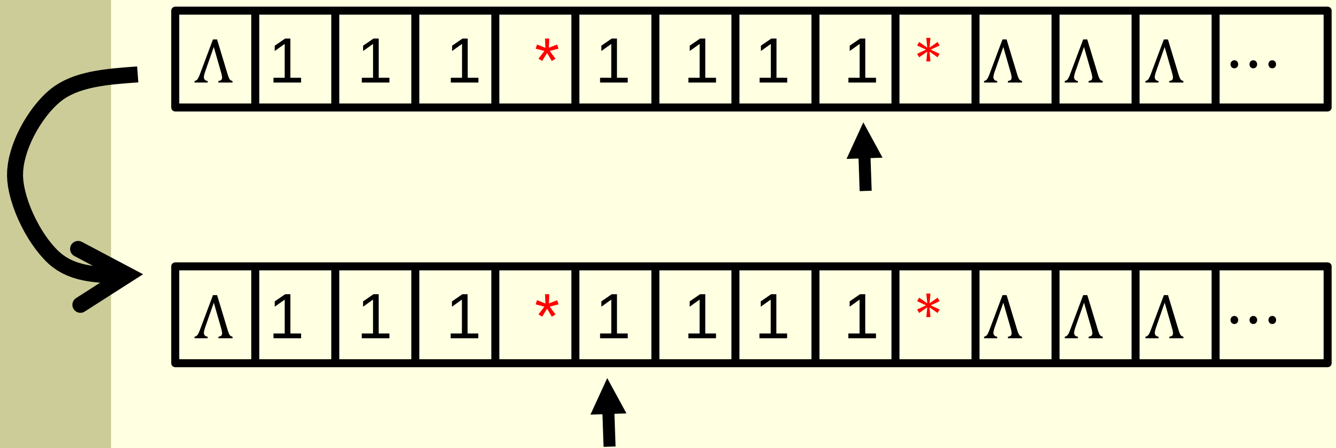
Input:



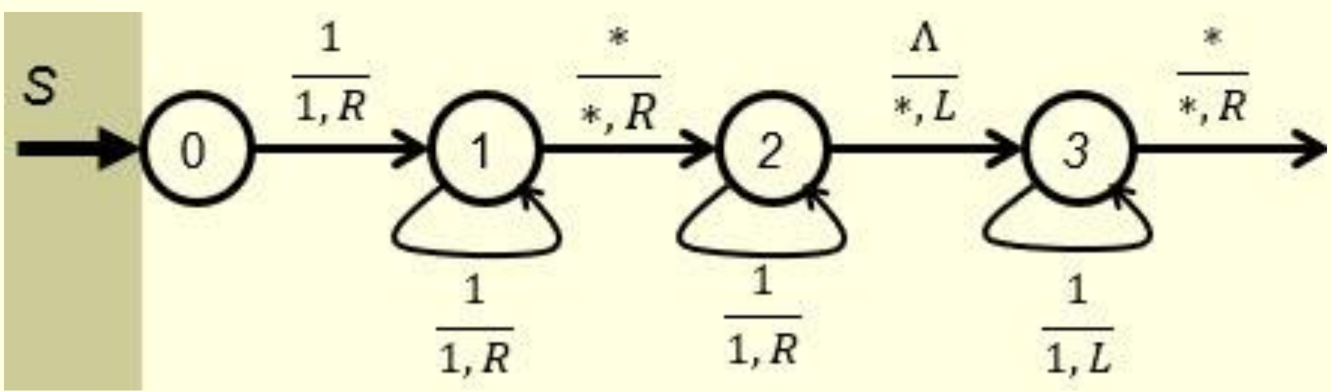
Step 1: move right, ignore all the 1’s and *, to find the first Λ, convert it to * and turn left



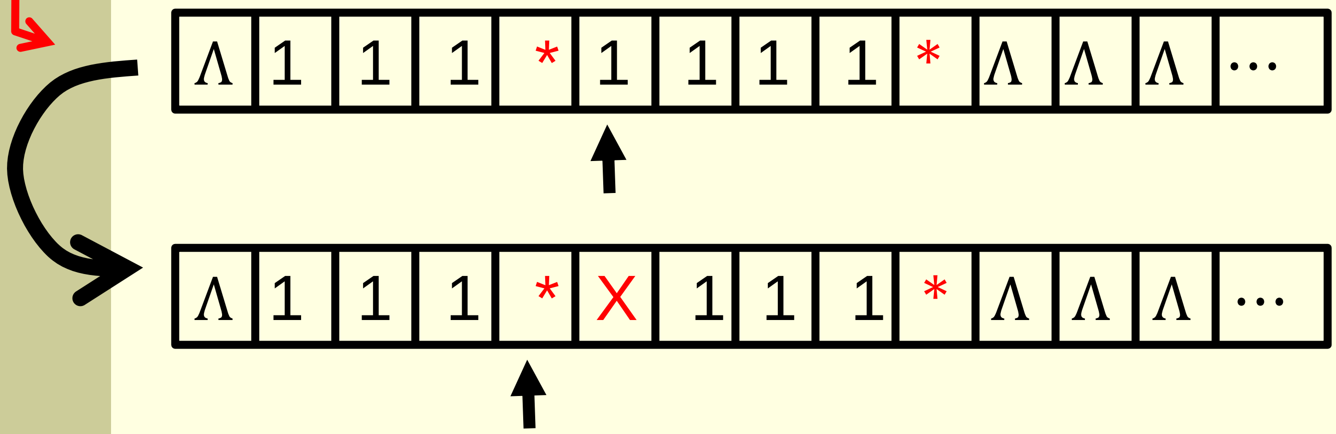
Step 2: move left, ignore all the 1's, until a * is found, then turn right



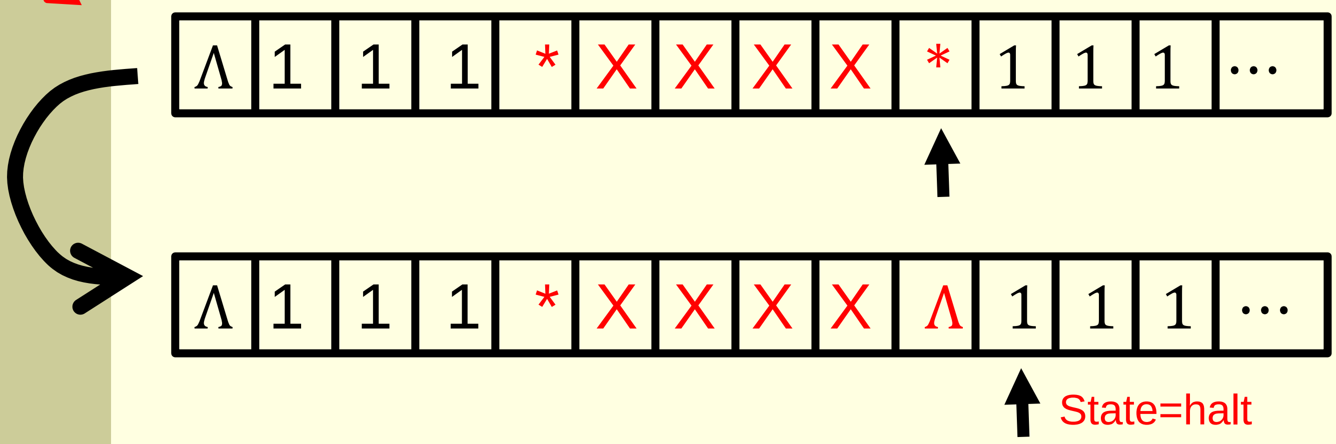
The portion of the TM for step 1 and step 2:



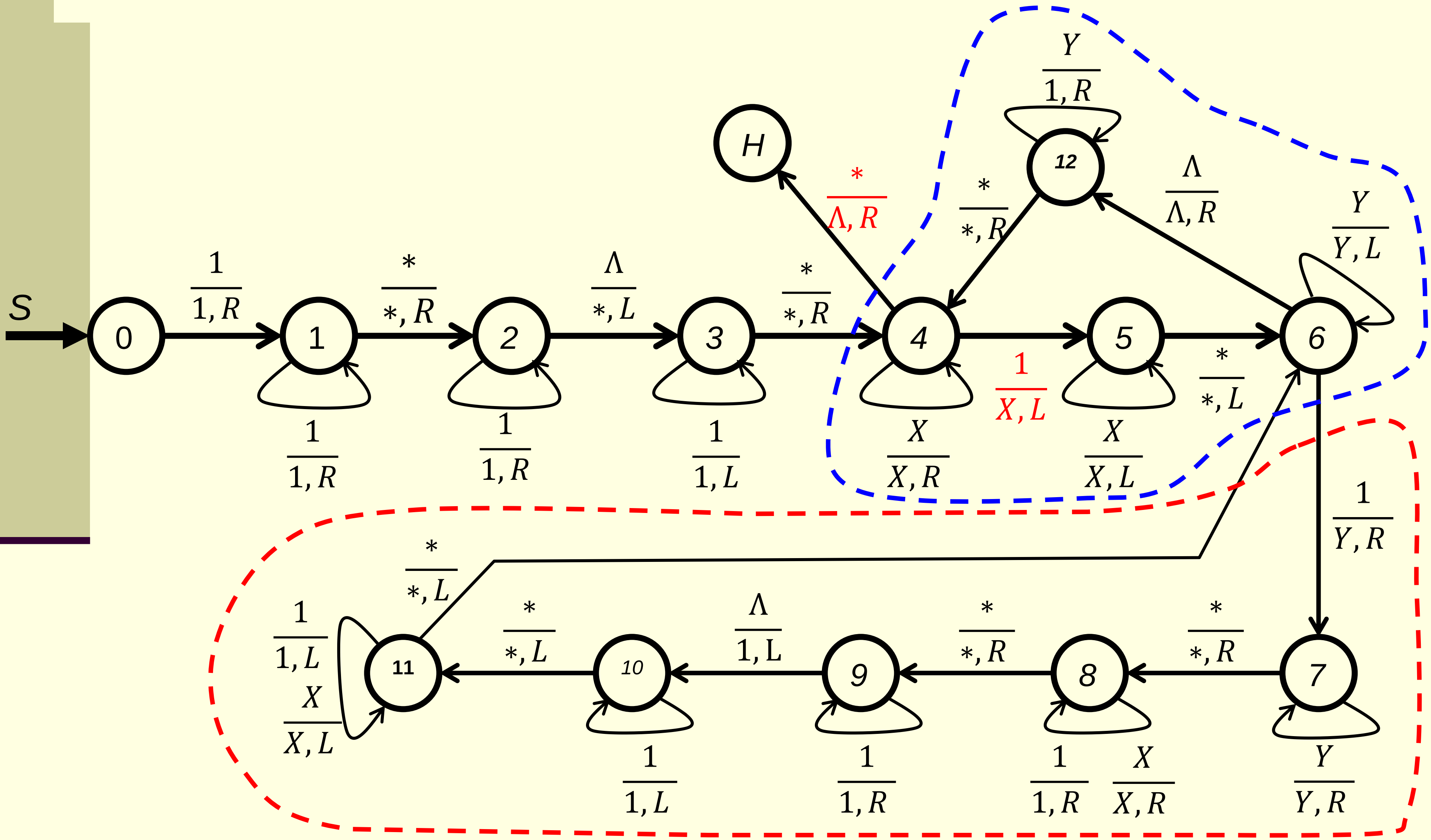
Step 3: move right, mark the first 1 reached with an ‘X’ and turn left. If no ‘1’ found but ‘*’ is reached, convert it to ‘Λ’, move one unit to the right and stop.



Step 3: move right, mark the first 1 reached with an ‘X’ and turn left. If no ‘1’ found but ‘*’ is reached, convert it to ‘Λ’, move one unit to the right and stop.



The TM will look like as follows :

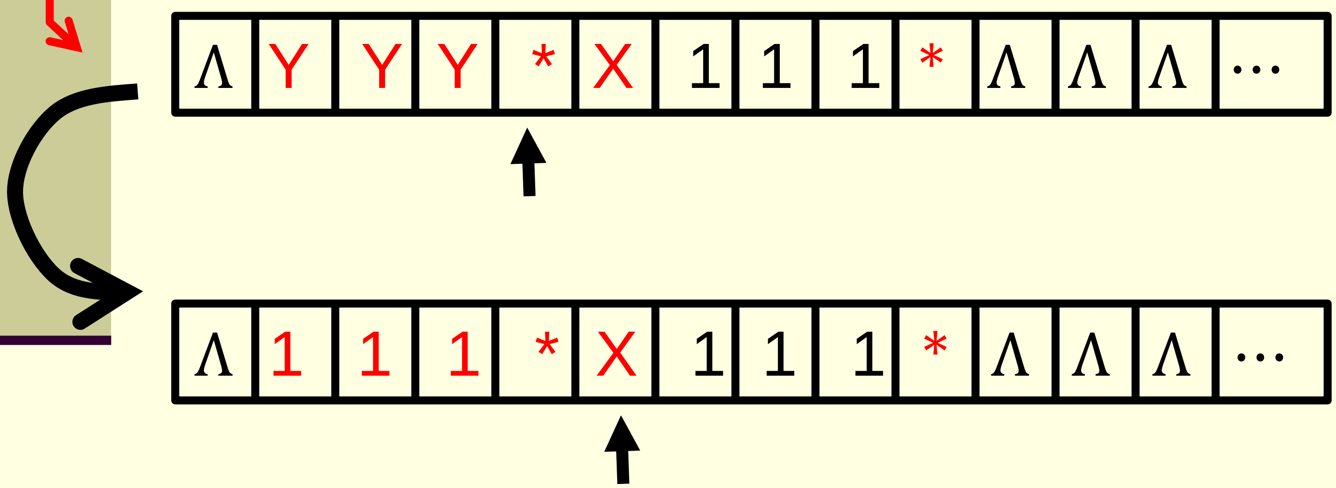


Step 4: move left, mark the first 1 reached with an ‘Y’ and turn right. If no ‘1’ found but a ‘Λ’ is reached, keep that ‘Λ’, turn right and change all the Y’s to 1, when we reach the ‘*’, keep that ‘*’, move one unit to the right and go to Step 3.

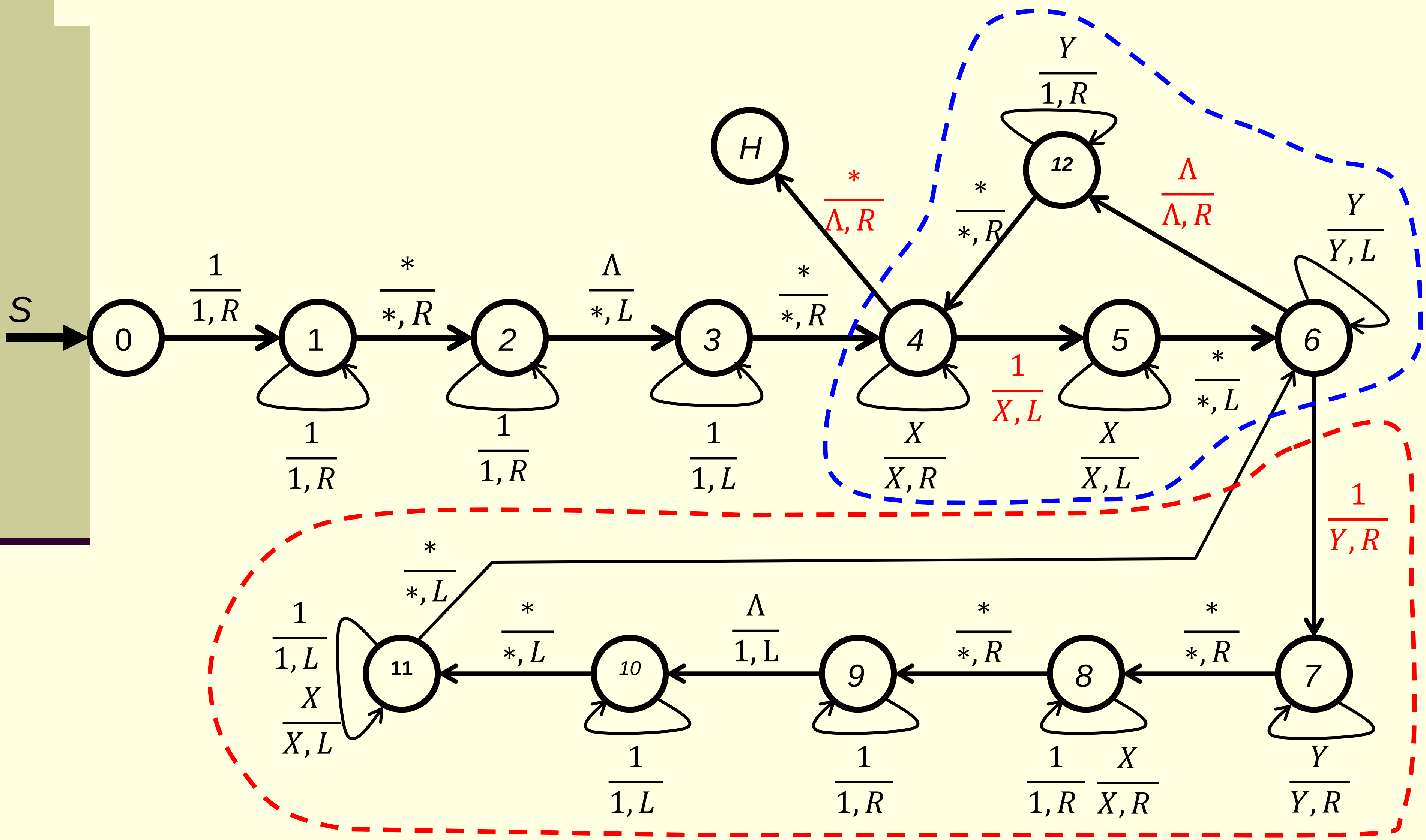
Λ	1	1	1	*	X	1	1	1	*	Λ	Λ	Λ	…

Λ	1	1	Y	*	X	1	1	1	*	Λ	Λ	Λ	…

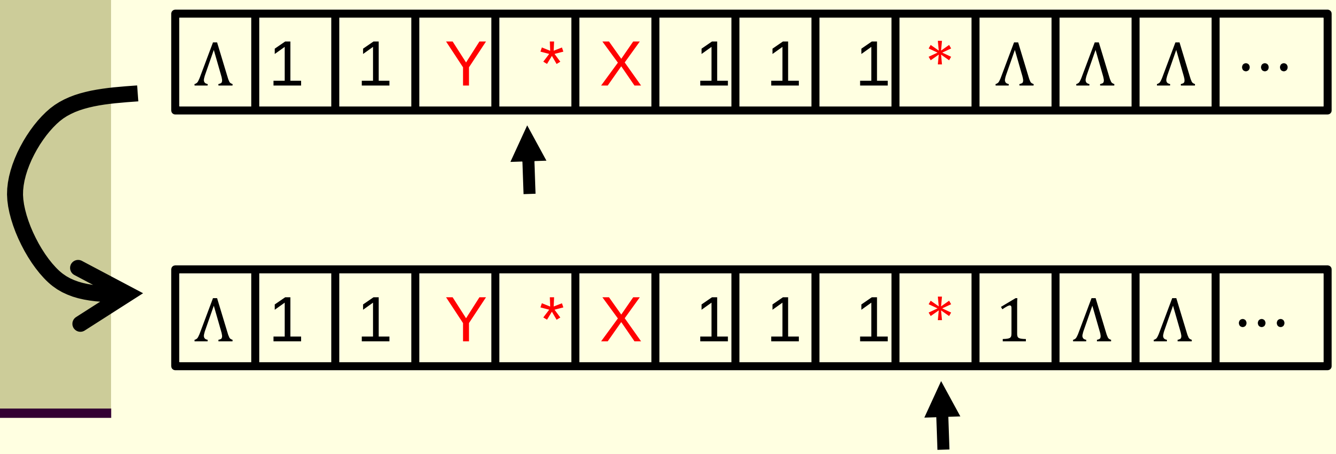
Step 4: move left, mark the first 1 reached with an ‘Y’ and turn right. If no ‘1’ found but a ‘Λ’ is reached, keep that ‘Λ’, turn right and change all the Y’s to 1, when we reach the ‘*’, keep that ‘*’, move one unit to the right and go to Step 3.



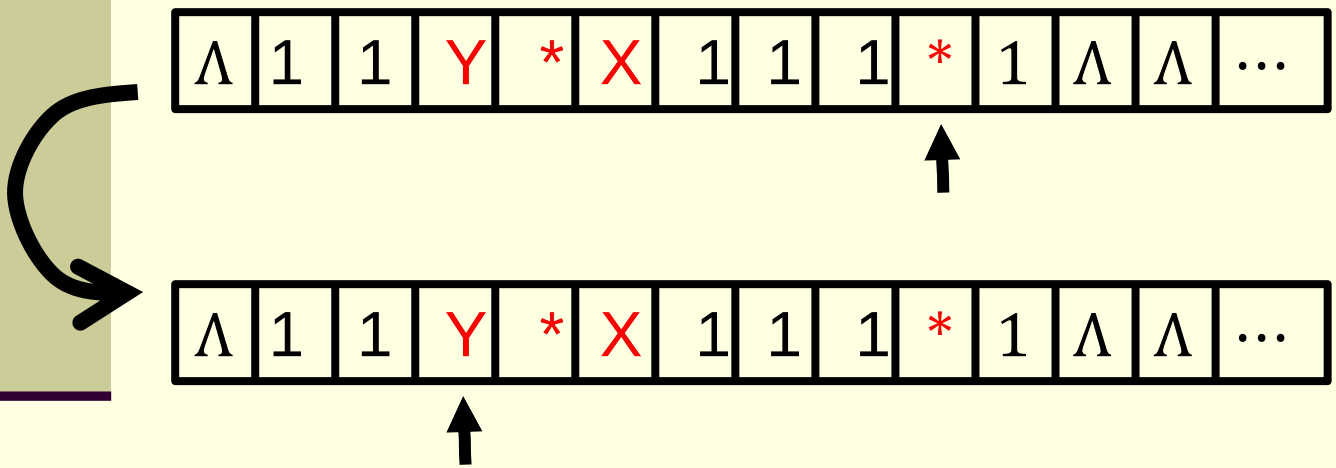
The TM will look like as follows :



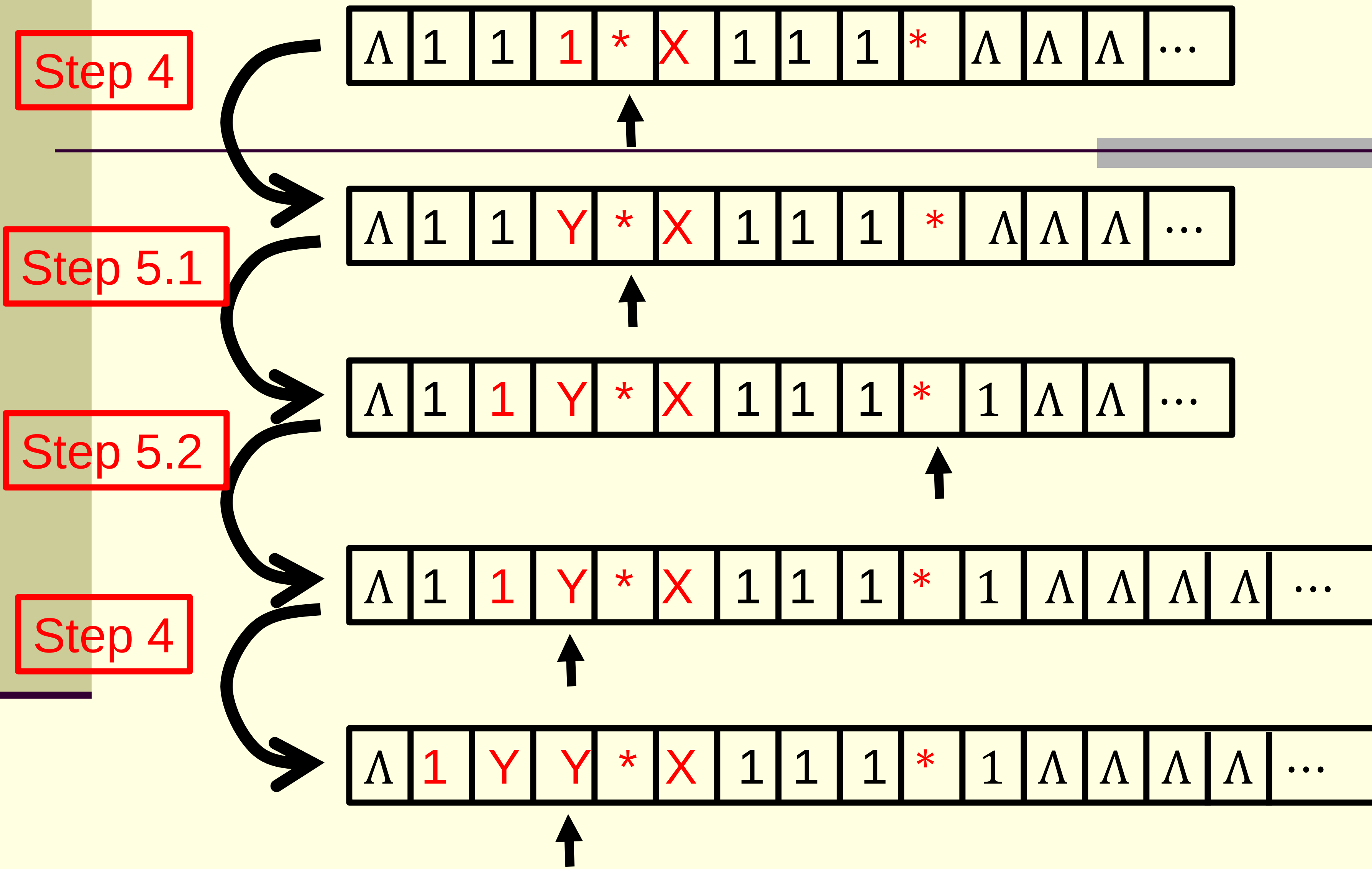
Step 5.1: move right, ignore all the Y’s, the first ‘*’ , all the X’s, all the 1’s, the second ‘*’, all the 1’s until we reach a ‘Λ’. Convert that Λ to a 1 and then turn left.

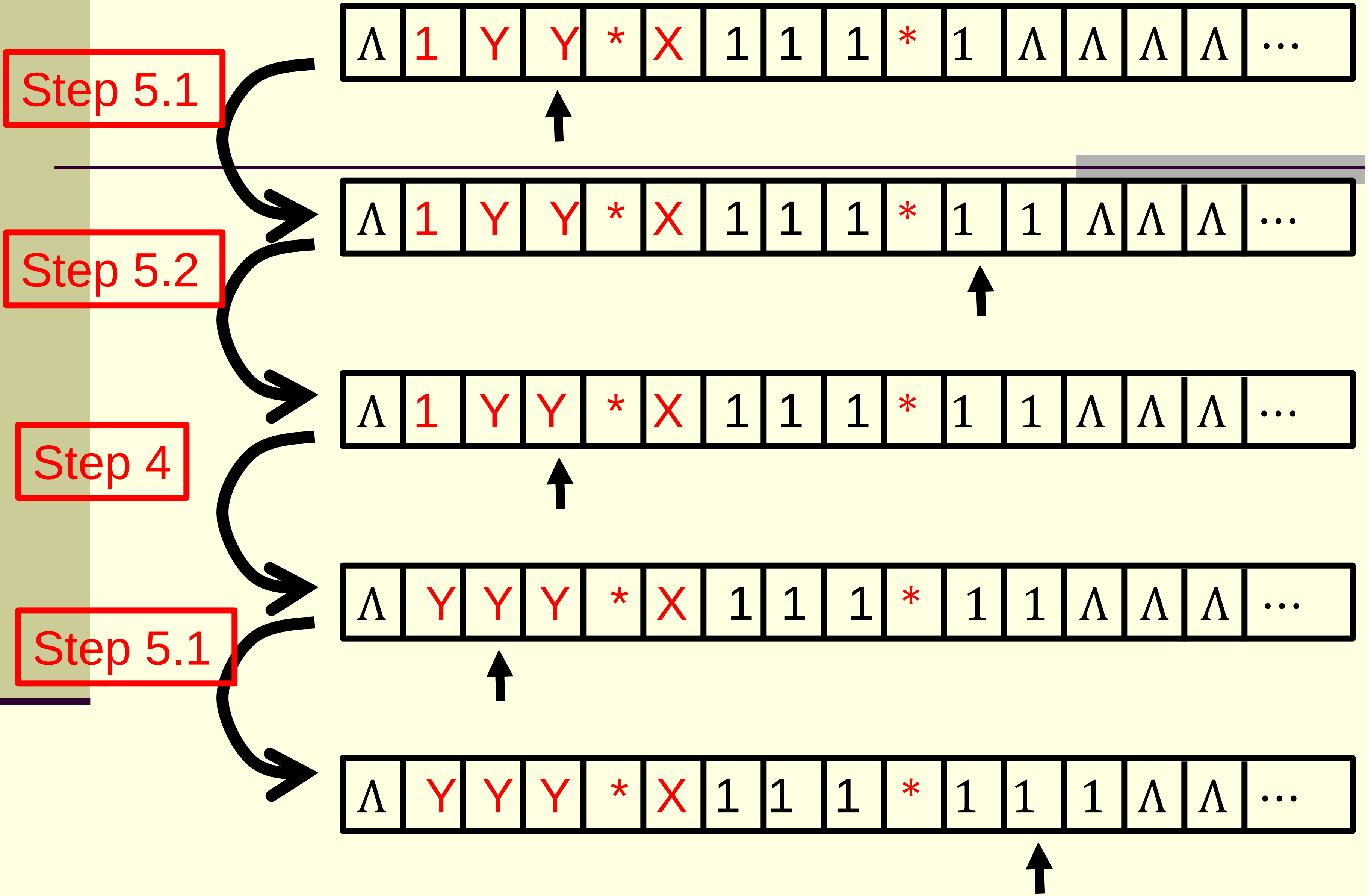


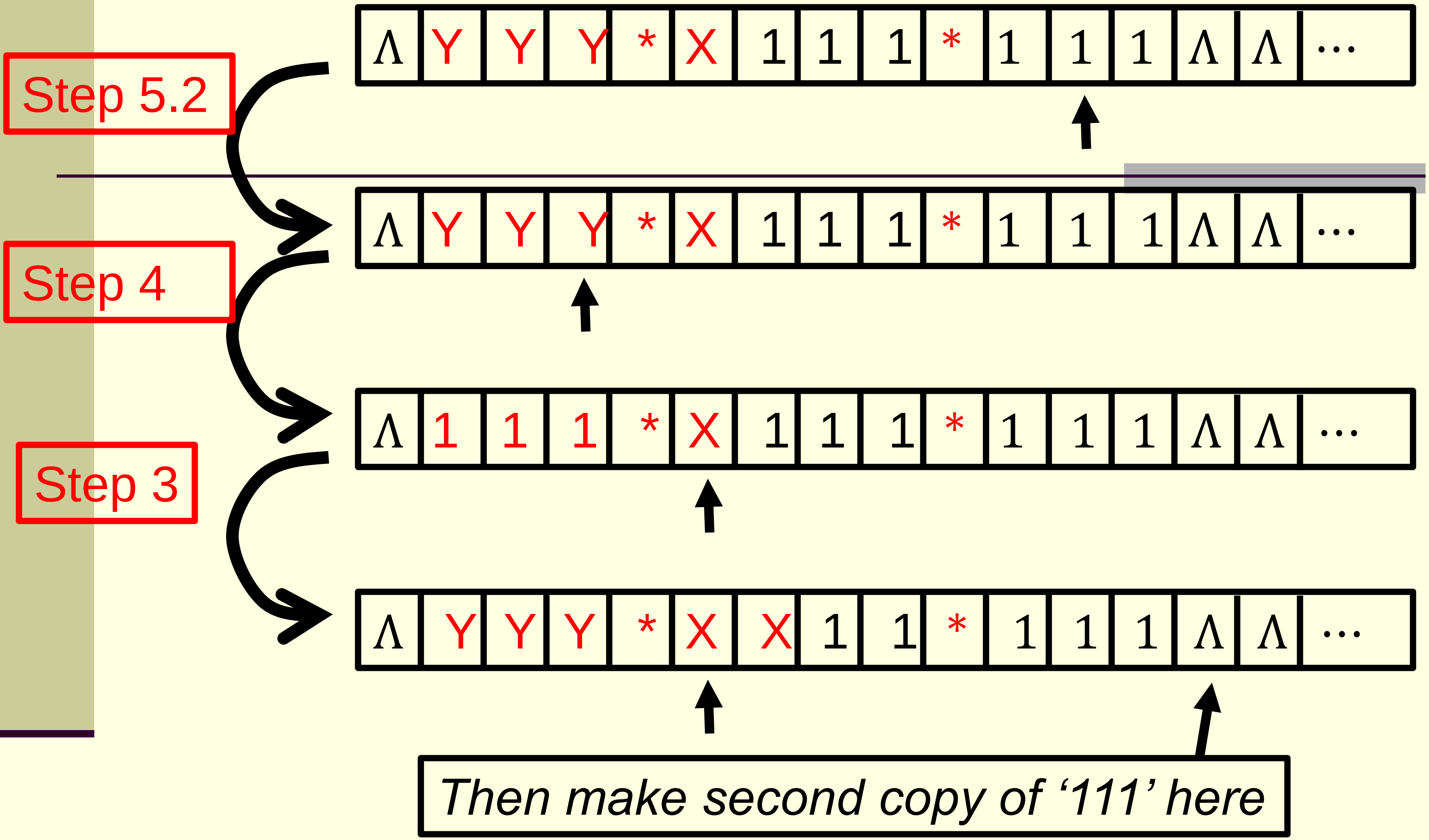
Step 5.2: Move left, ignore the first '*', all the 1's, all the X's, until the second '*' is reached. Keep that '*' and move one unit to the left and go to Step 4.



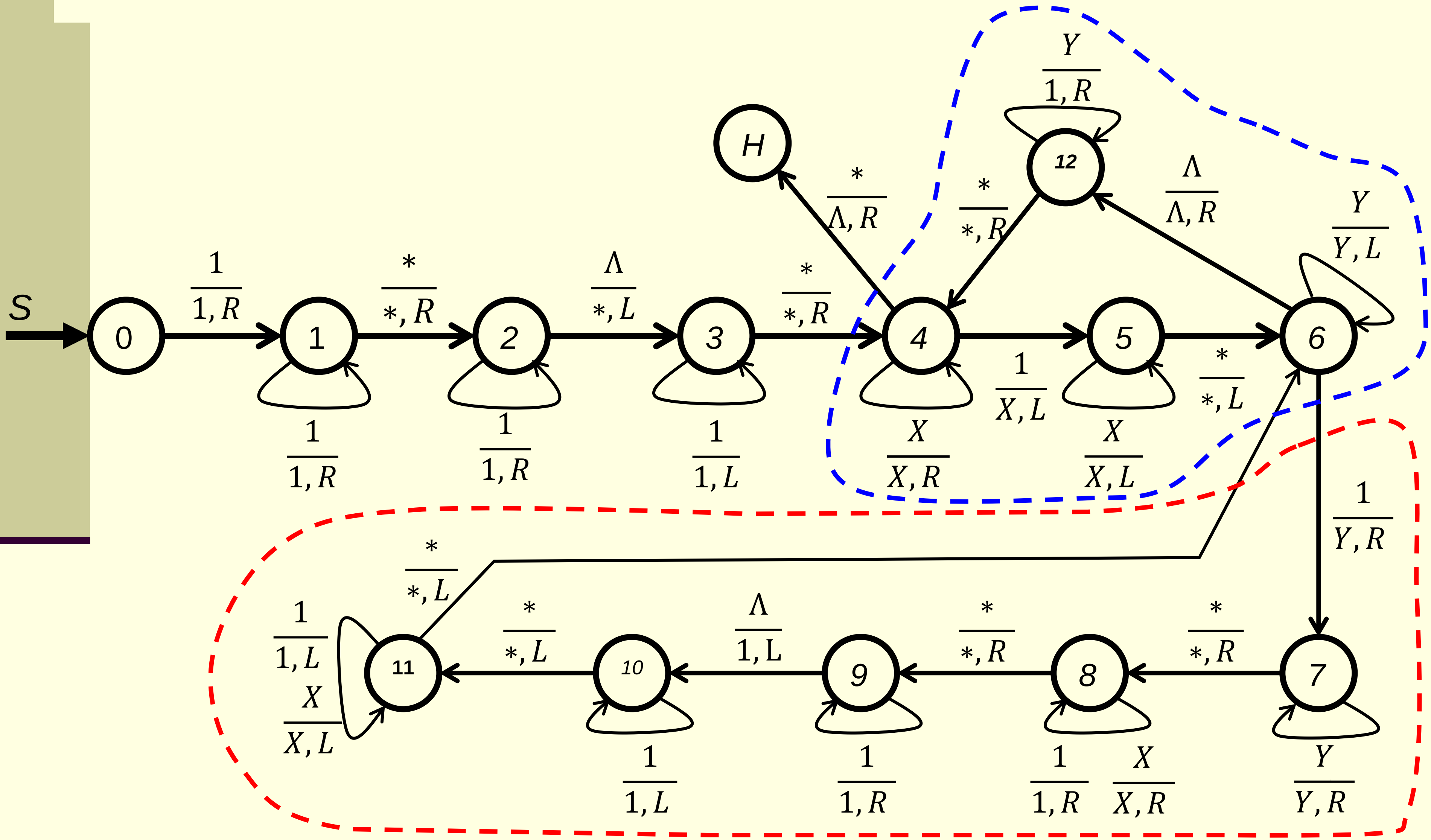
From slide 33




Λ 1 Y Y * X 1 1 1 * 1 Λ Λ Λ Λ …
Step 5.1
Λ 1 Y Y * X 1 1 1 * 1 1 Λ Λ Λ …
Step 5.2
Λ 1 Y Y * X 1 1 1 * 1 1 Λ Λ Λ …
Step 4
Λ Y Y Y * X 1 1 1 * 1 1 Λ Λ Λ …
Step 5.1
Λ Y Y Y * X 1 1 1 * 1 1 1 Λ Λ …

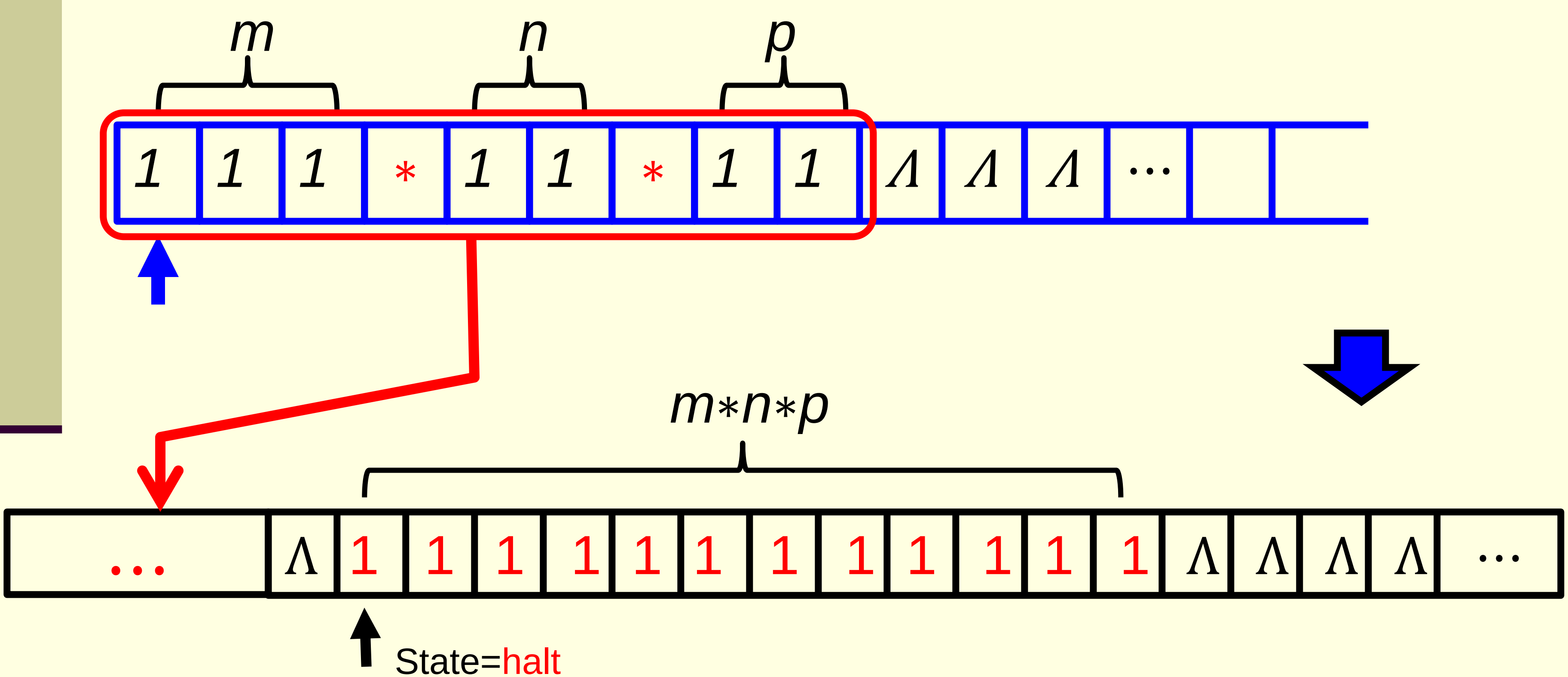

Step 5.2
Λ Y Y Y * X 1 1 1 * 1 1 1 Λ Λ ...
Step 4
Λ Y Y Y * X 1 1 1 * 1 1 1 Λ Λ ...
Step 3
Λ 1 1 1 * X 1 1 1 * 1 1 1 Λ Λ ...
Λ Y Y Y * X X 1 1 * 1 1 1 Λ Λ ...
Then make second copy of ‘111’ here

The TM will look like as follows :



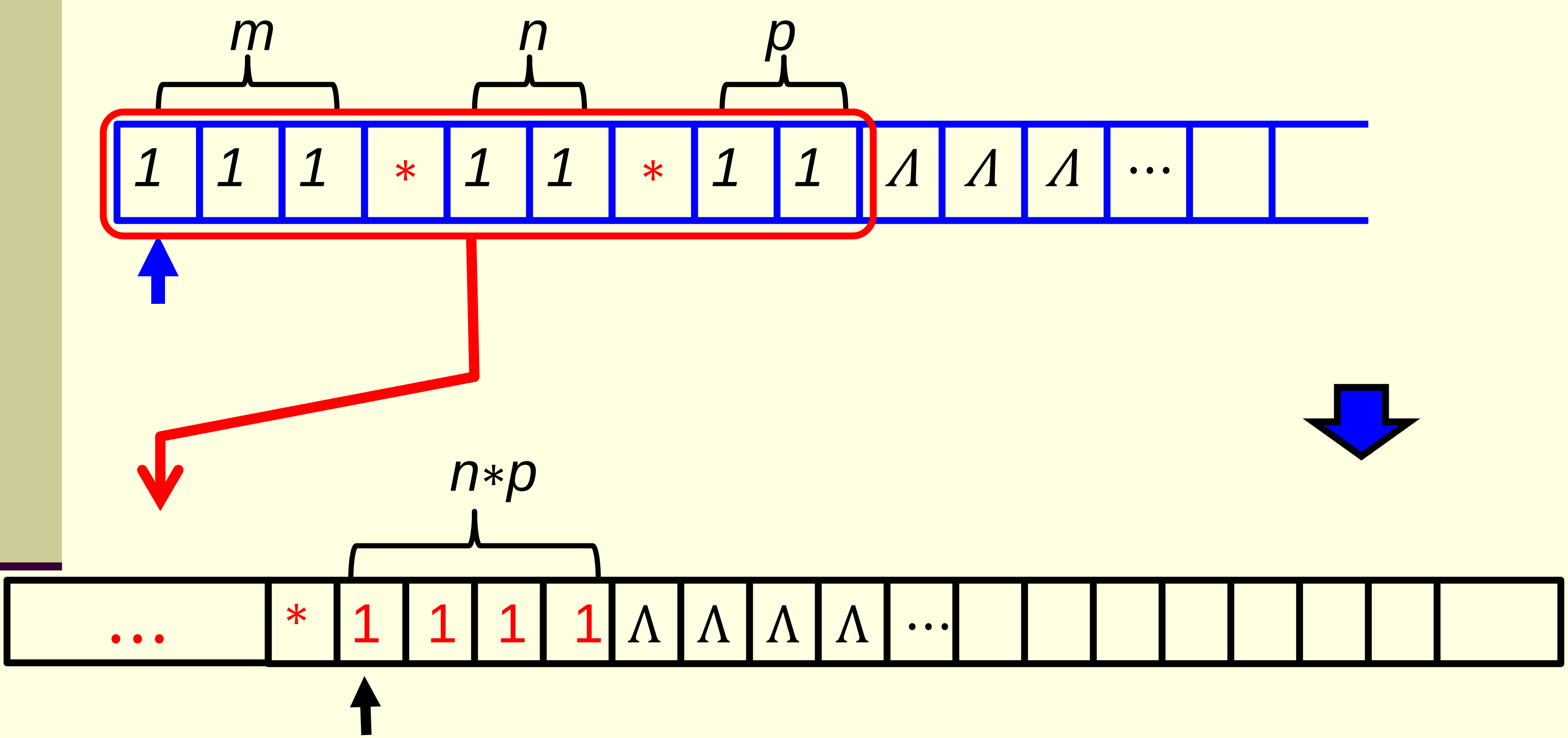
Question:

Can the above TM be modified to do multiplication of three numbers, four numbers, ..., n numbers in unary form directly?



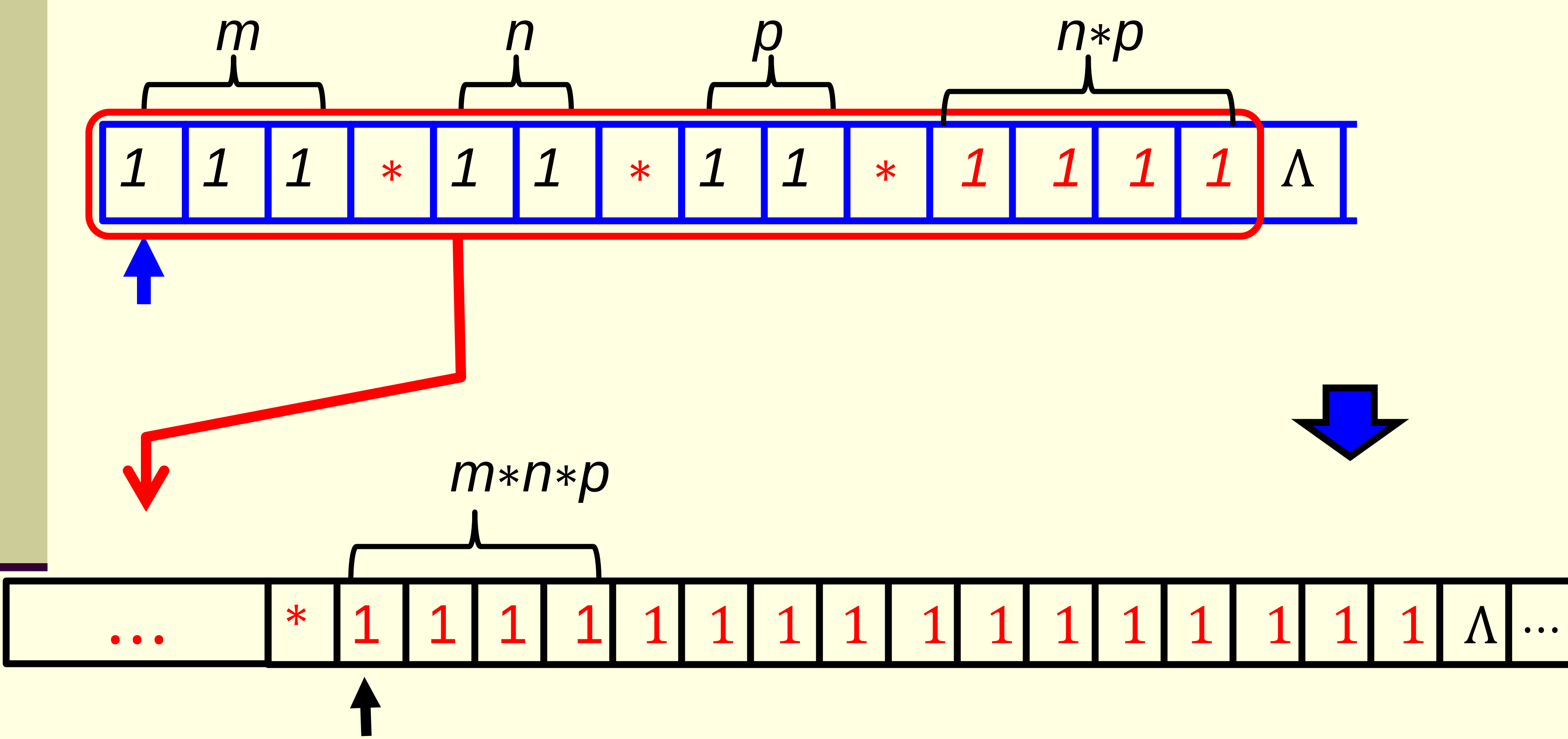
Solution:

Step 1



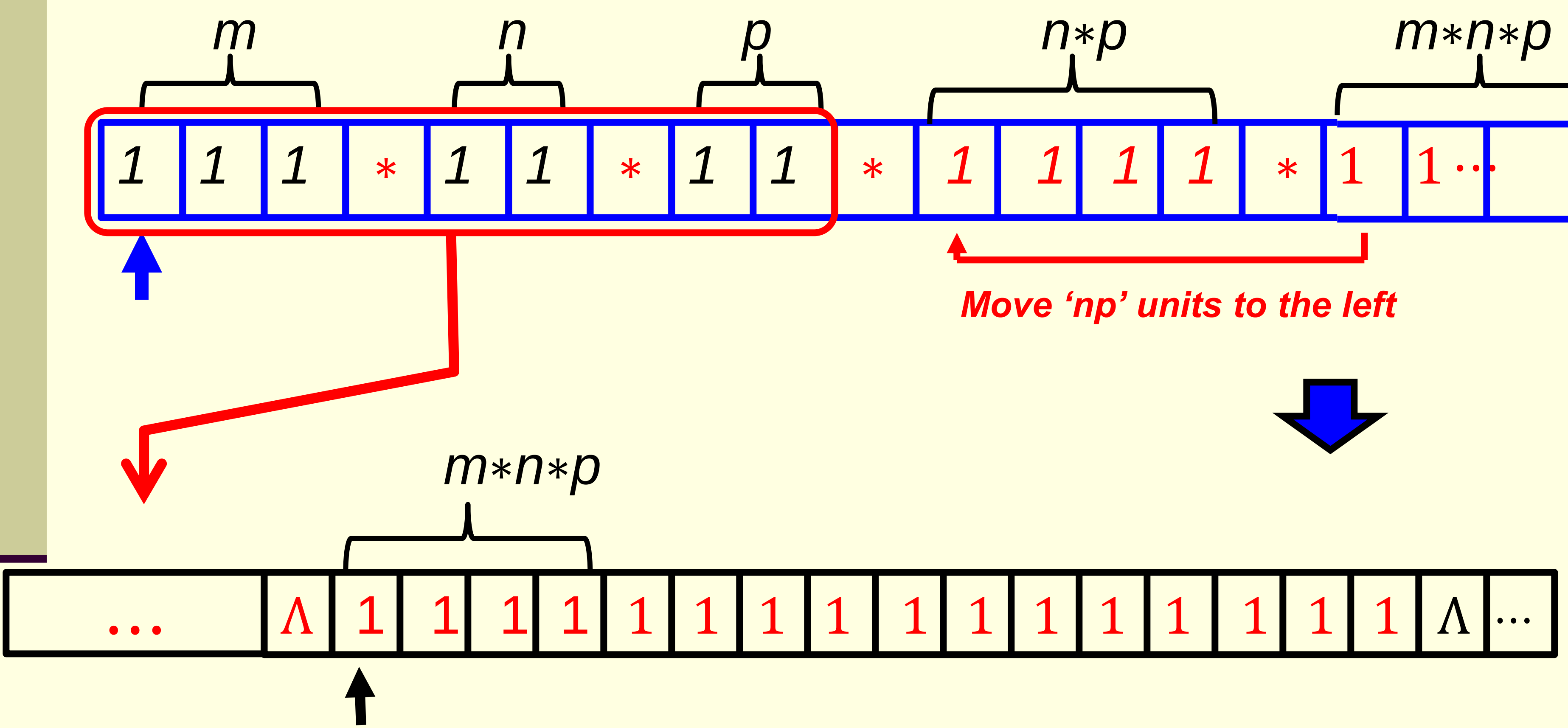
Solution:

Step 2



Solution:

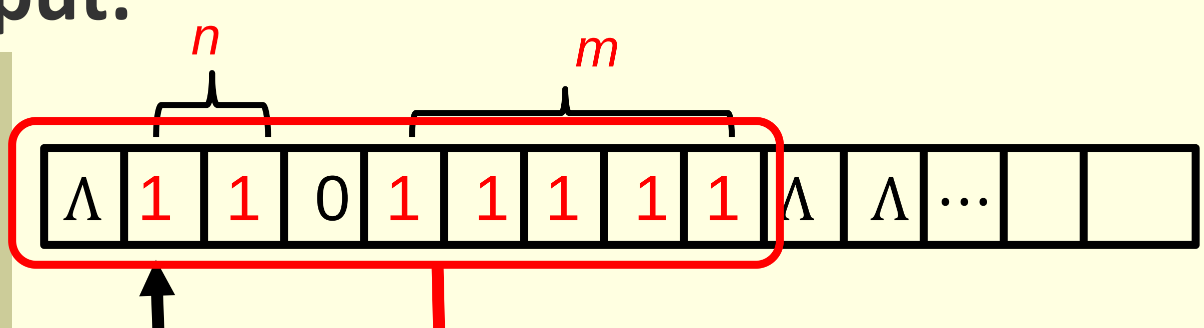
Step 3



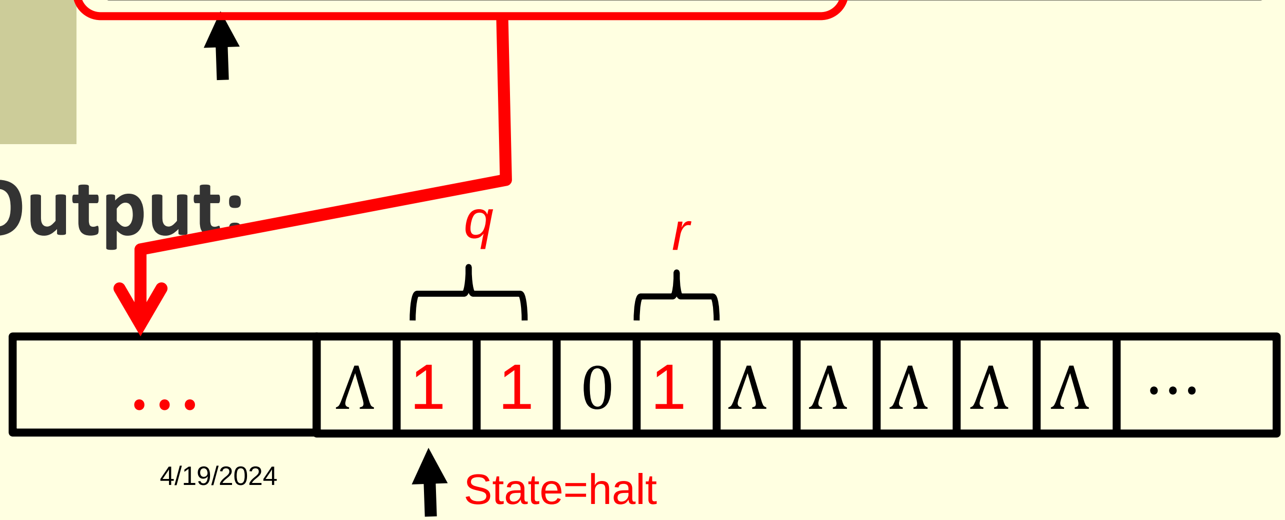
TM for the **division function** for the unary number system

Develop a TM for the division of two unary numbers m and n such that f(m/n) = q + r where q is the quotient and r is the remainder.

Input:



Output:



Basic concept: divide the unary representation of *m* into subgroups of length *n*; the number of subgroups is the quotient and the excess part is the remainder.

$$\overbrace{1\ 1\ 1\ 1}^{n} \qquad \overbrace{1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1}^{m}$$

Q = Λ

R = Λ

If *n< (current) m* do

- create a subgroup of length *n*
- add 1 to Q
- erase this subgroup from *m*

Otherwise, set the current *m* to be R

$\overbrace{1\;\;1\;\;1\;\;1}^{n}$ $\qquad$ $\overbrace{1\;\;1\;\;1\;\;1\;\;1\;\;1\;\;1\;\;1\;\;1\;\;1}^{m}$

Q = Λ

R = Λ

current m

4 < 10, so

(Increment Q by 1)

$\overbrace{1\;\;1\;\;1\;\;1}^{n}$ $\qquad$ $\overbrace{\boxed{1\;\;1\;\;1\;\;1}\;\;1\;\;1\;\;1\;\;1\;\;1\;\;1}^{m}$

(Create a subgroup of length 4)

Q = 1

R = Λ

$\overbrace{1\;\;1\;\;1\;\;1}^{n}$ $\qquad$ $\overbrace{\boxed{*\;\;*\;\;*\;\;*}\;\;1\;\;1\;\;1\;\;1\;\;1\;\;1}^{m}$

(erase this subgroup from m)

Q = 1

R = Λ

$\overbrace{1\ \ 1\ \ 1\ \ 1}^{n}$ $\qquad$ $\overbrace{\boxed{*\ \ *\ \ *\ \ *}\ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1}^{m}$

Q = 1

R = Λ

current m

$\boxed{4 < 6, \text{ so}}$

(Increment Q by 1)

$\overbrace{1\ \ 1\ \ 1\ \ 1}^{n}$ $\qquad$ $\overbrace{*\ \ *\ \ *\ \ *\ \boxed{1\ \ 1\ \ 1\ \ 1}\ 1\ \ 1}^{m}$

(Create a subgroup of length 4)

Q = 2

R = Λ

$\overbrace{1\ \ 1\ \ 1\ \ 1}^{n}$ $\qquad$ $\overbrace{*\ \ *\ \ *\ \ *\ \boxed{*\ \ *\ \ *\ \ *}\ 1\ \ 1}^{m}$

(erase this subgroup from m)

Q = 2

R = Λ

$\overbrace{1\ 1\ 1\ 1}^{n}$ $\overbrace{\boxed{*\ *\ *\ *\ \ *\ *\ *\ *}\ 1\ 1}^{m}$

Q = 2

R = Λ

current m

$\boxed{\textit{4 > 2, so}}$

$\overbrace{1\ 1\ 1\ 1}^{n}$ $\overbrace{*\ *\ *\ *\ \ *\ *\ *\ *\ \boxed{1\ 1}}^{m}$

Q = 2

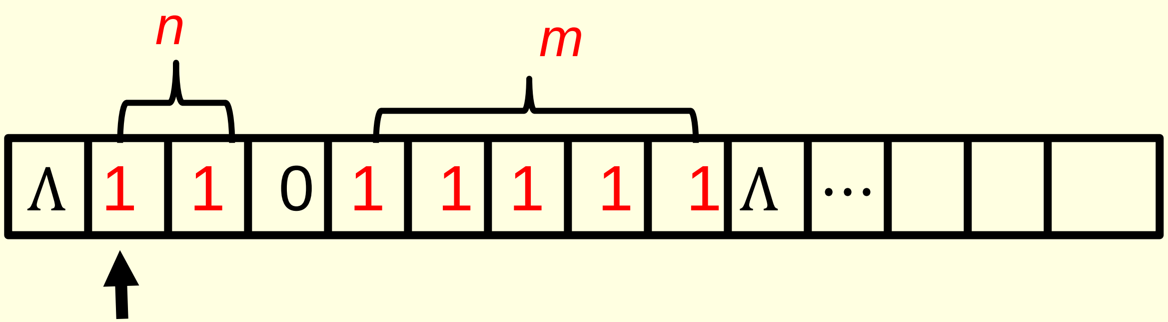
R = 2

(set this part as R)

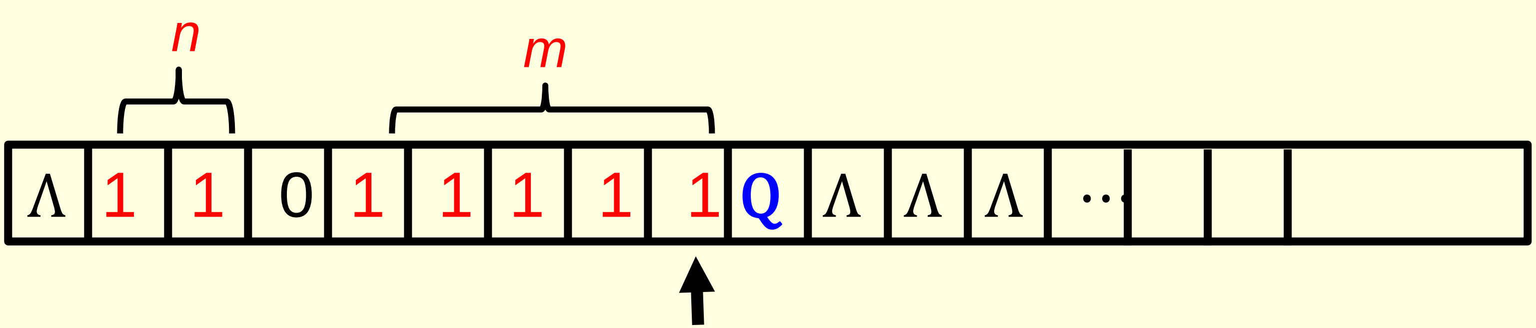
And then stop

Implementation:

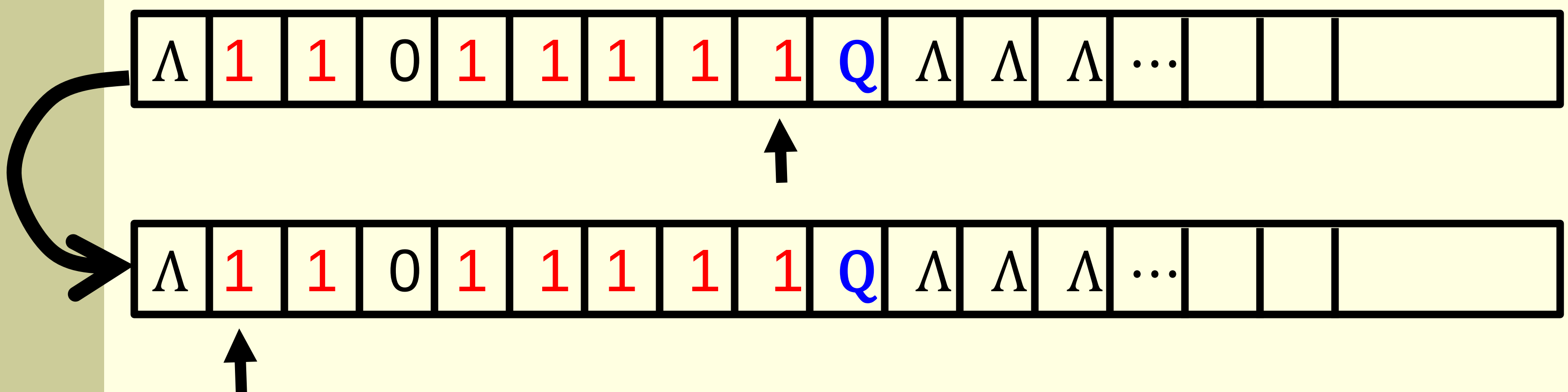
Input:



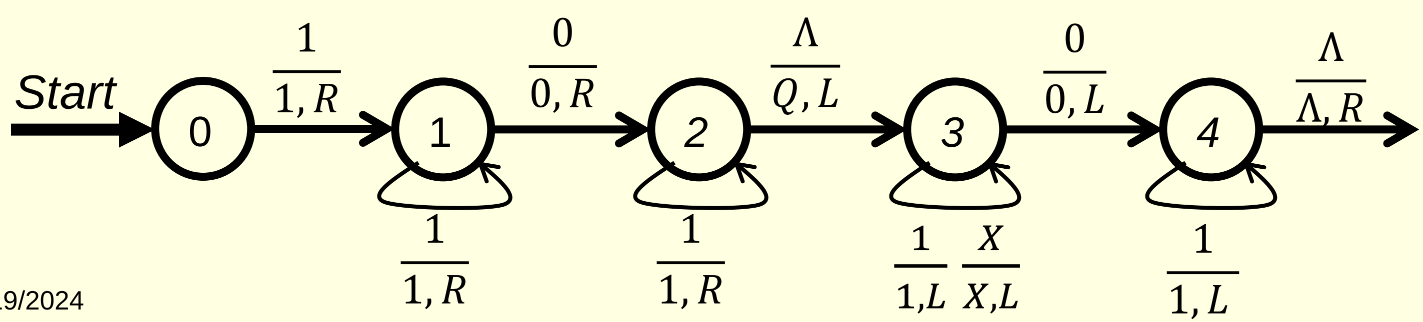
Step 1.1: move right, ignore all the 1's and 0, to find the first Λ, convert it to Q and turn left



Step 1.2: move left, ignore all the 1's, X's and 0 until a Λ is found, then turn right



The portion of the TM for Step 1.1 and Step 1.2:



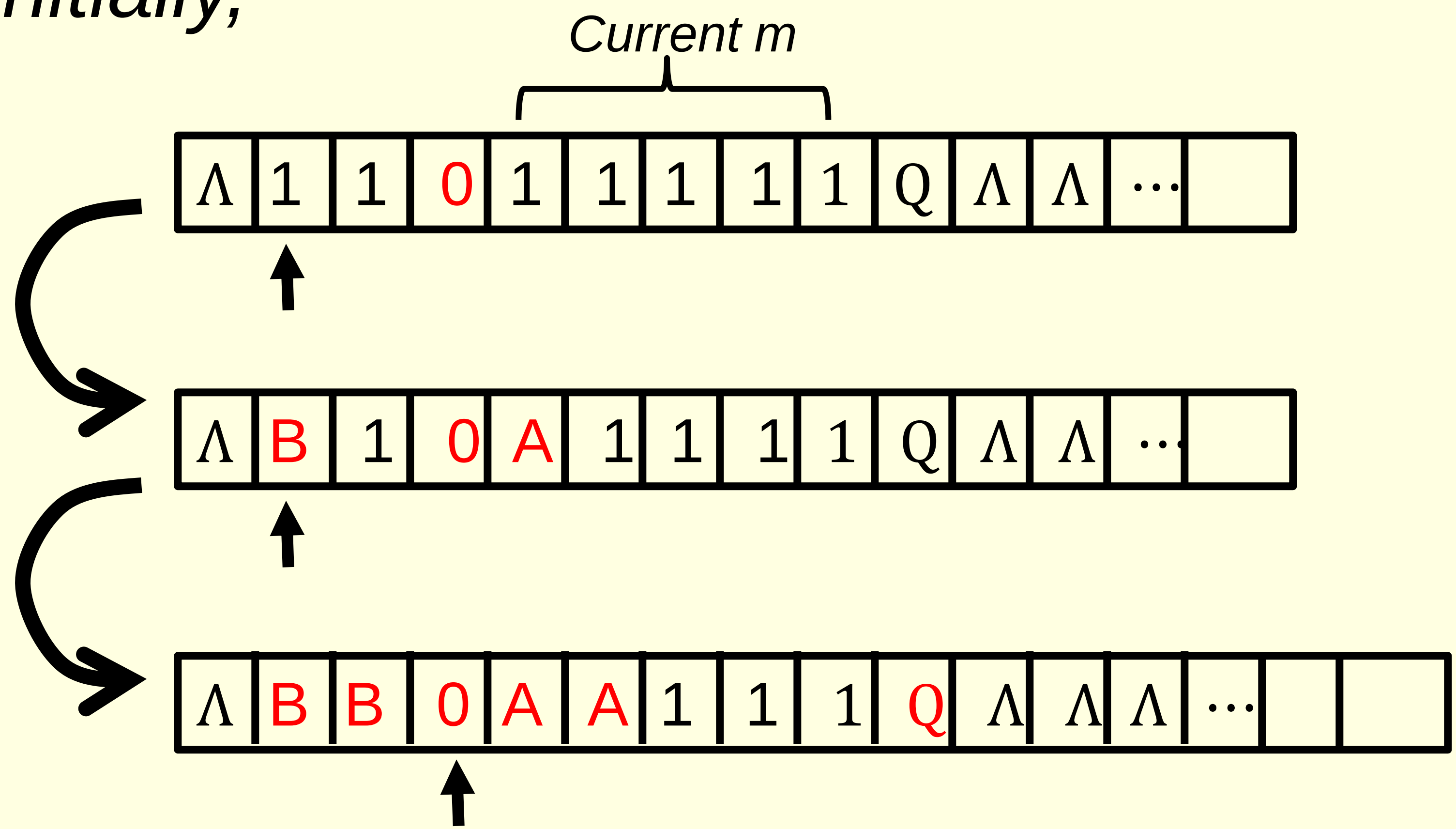
Step 2 (test if n > (current) m)

Repeat the following process: mark a 1 in n with a B then mark an available 1 in m with an A.

If all the available 1's in m are marked before all the 1's in n are marked (i.e., n > (current) m), then go to Step 4 (to format an output and then stop).

Otherwise (i.e., n <= (current) m), go to Step 3.

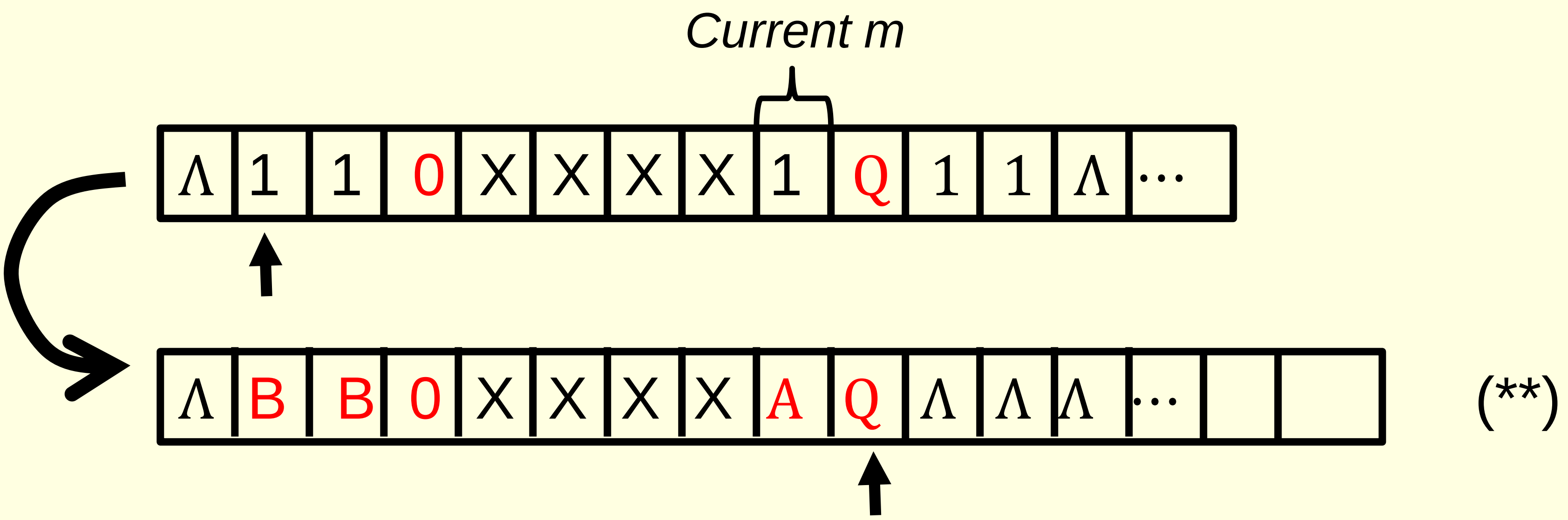
Initially,



(*)

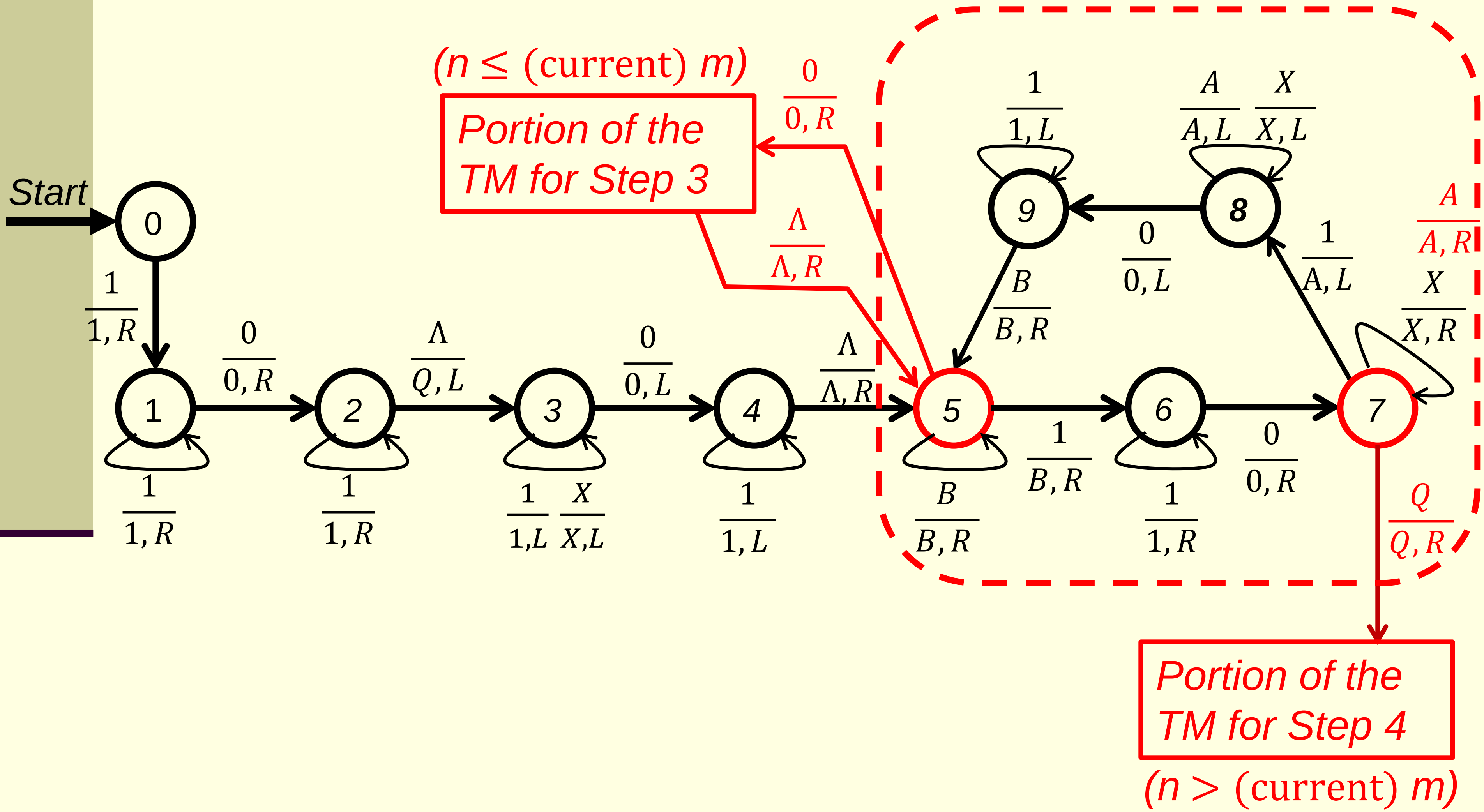
No more 1's in n to mark, but there are still some unmarked 1's in m (i.e., n <= current m), so go to Step 3

After a while,



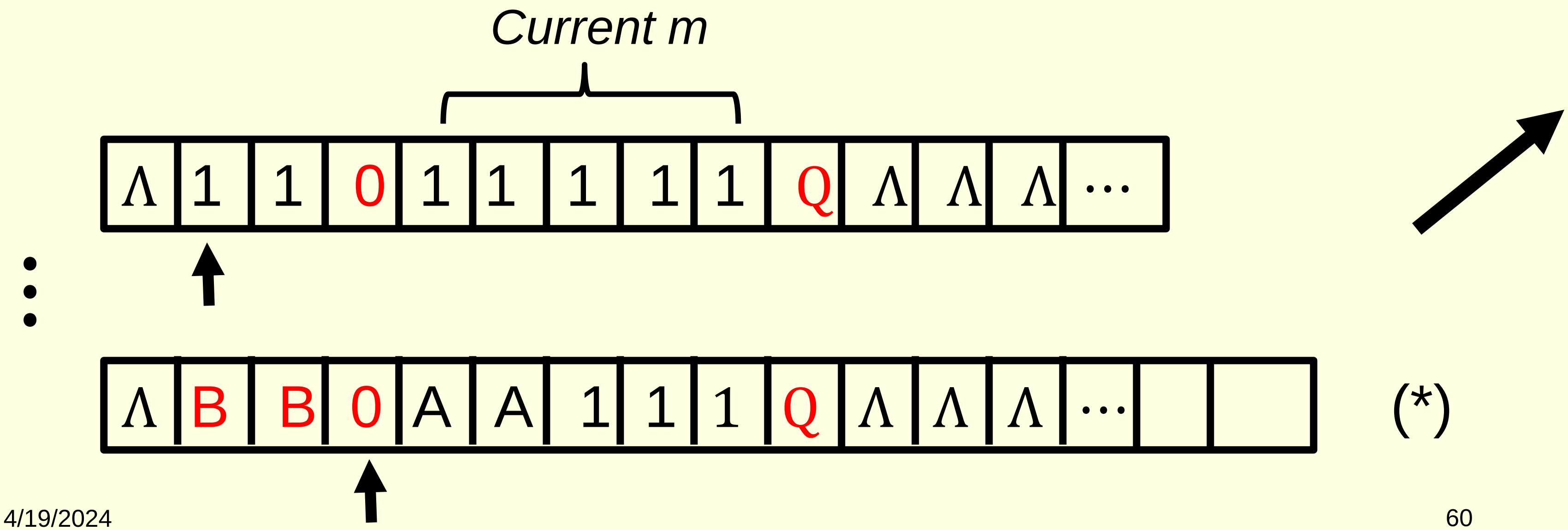
No available 1 in current m to mark/match for the 1 just marked in n (i.e., n > (current) m), so go to Step 4.

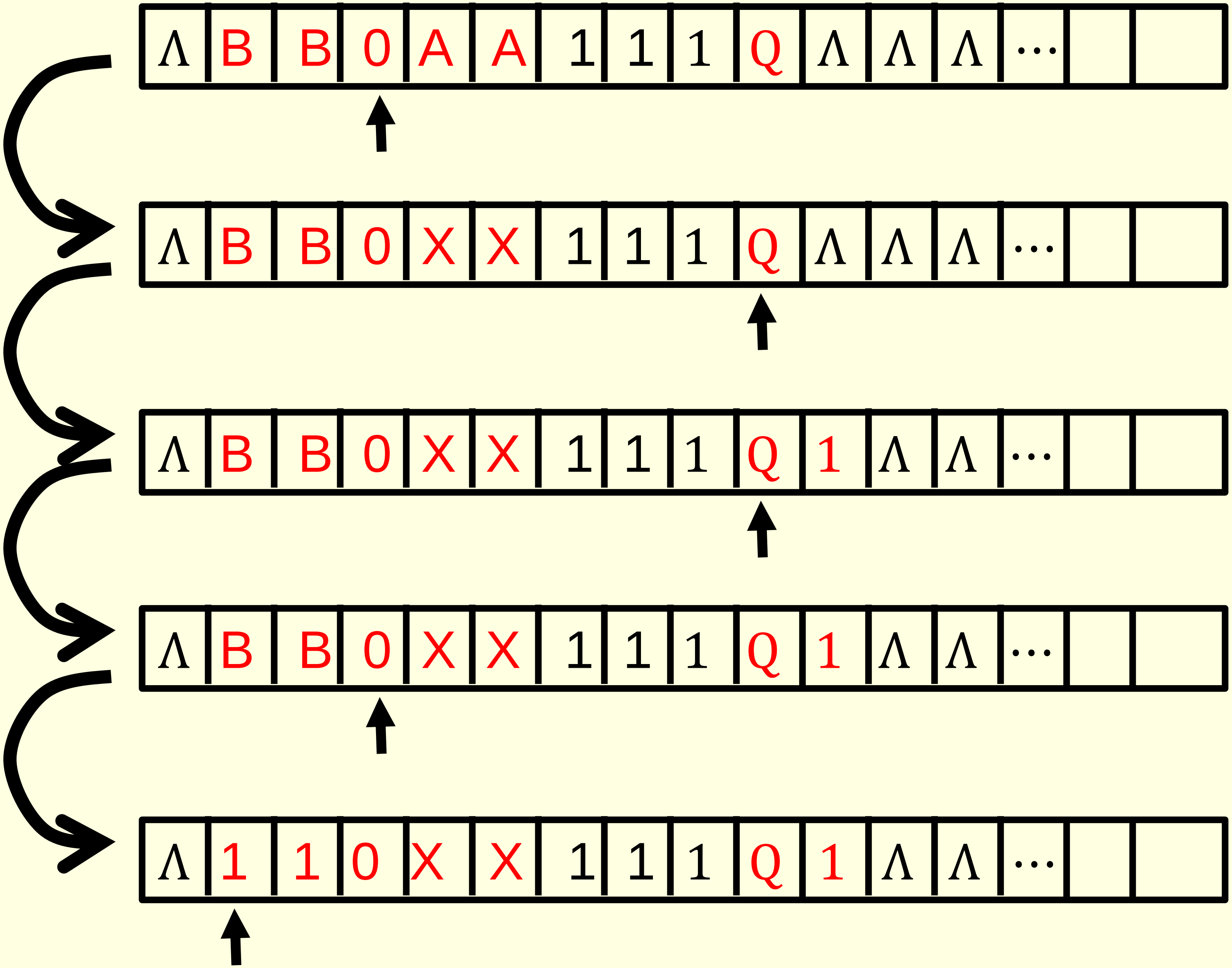
The portion of the TM for Step 1 and Step 2 is as follows :



Step 3: This step marks n 1's of m with X's (so the next 'current m' has n less available 1's than the current 'current m') and adds 1 to the quotient and then go to Step 2 (to perform the next (n > (current) m) test).

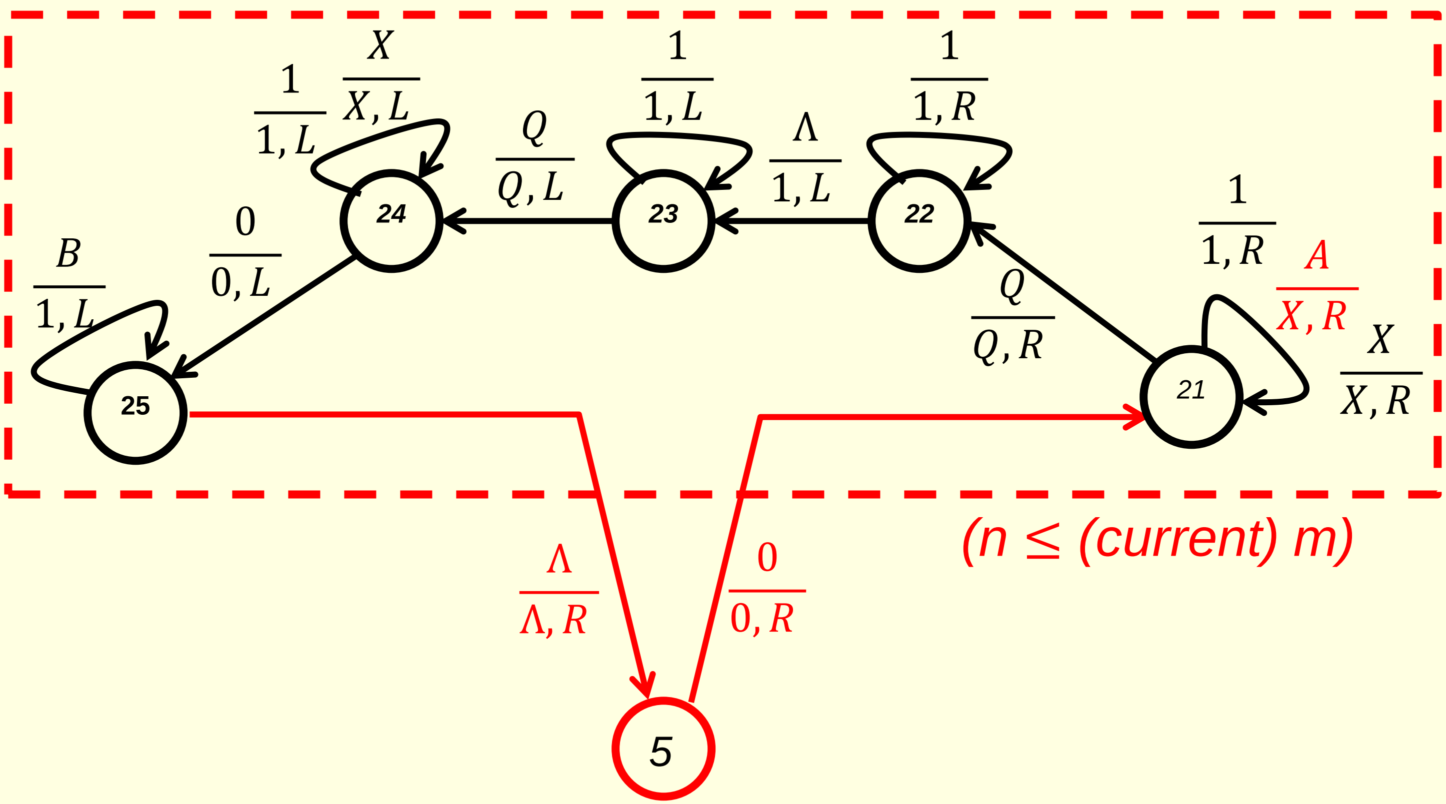
From (*) in Step 2 :





Then go to Step 2

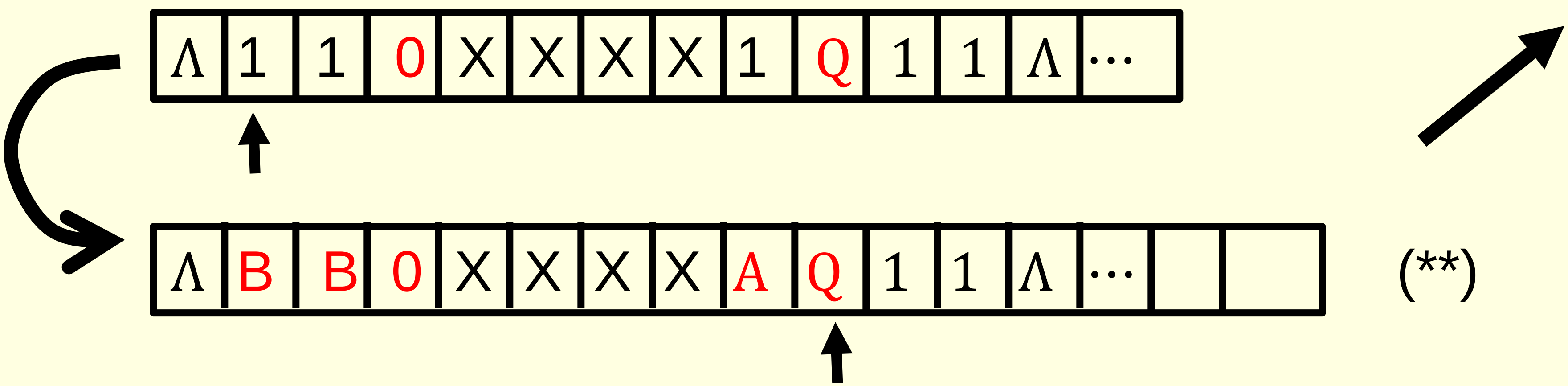
The portion of the TM that does Step 3 :



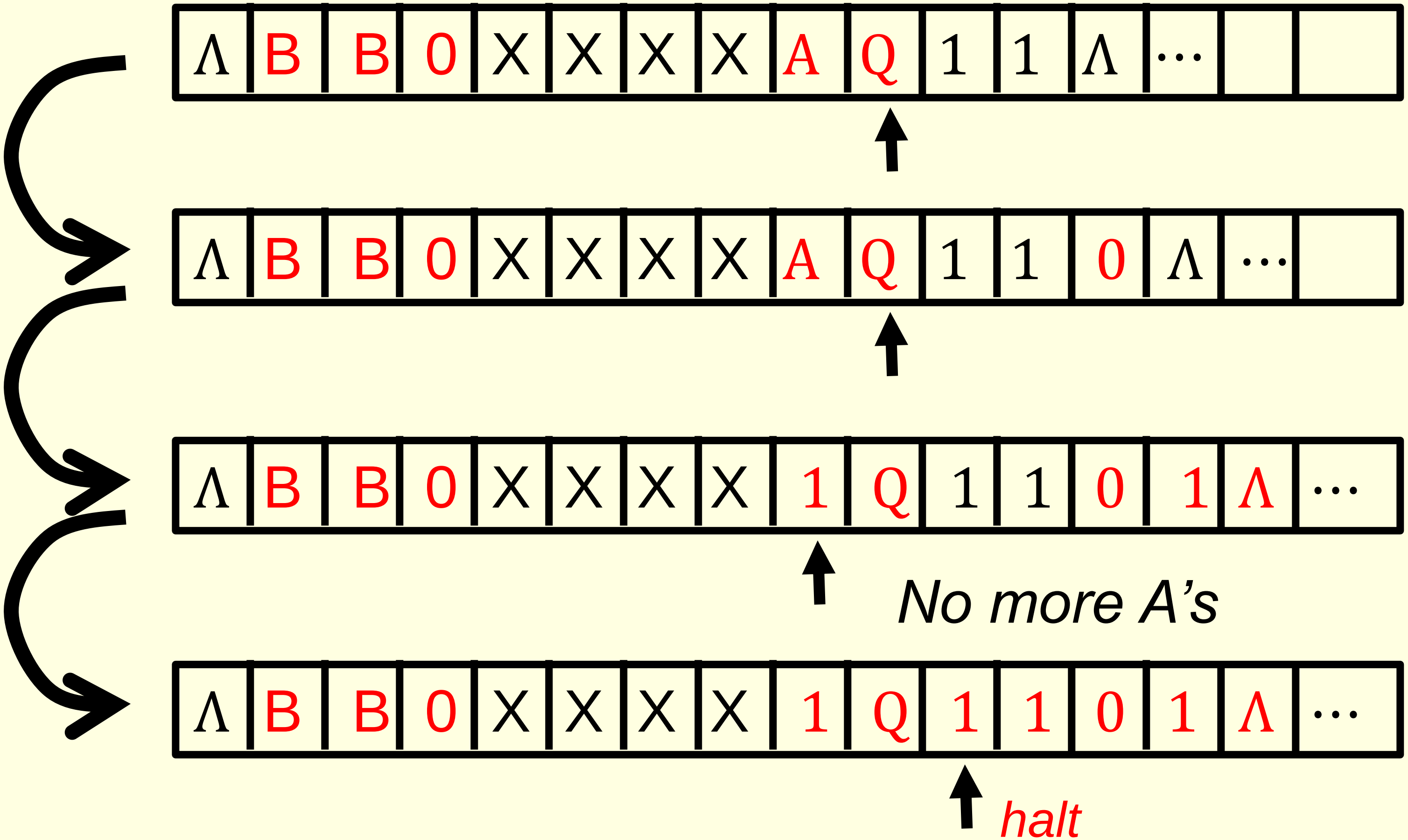
Step 4:

This step formats the output by using the number of 1’s put behind Q as the quotient and the number of available 1’s in current m as the remainder, then terminates.

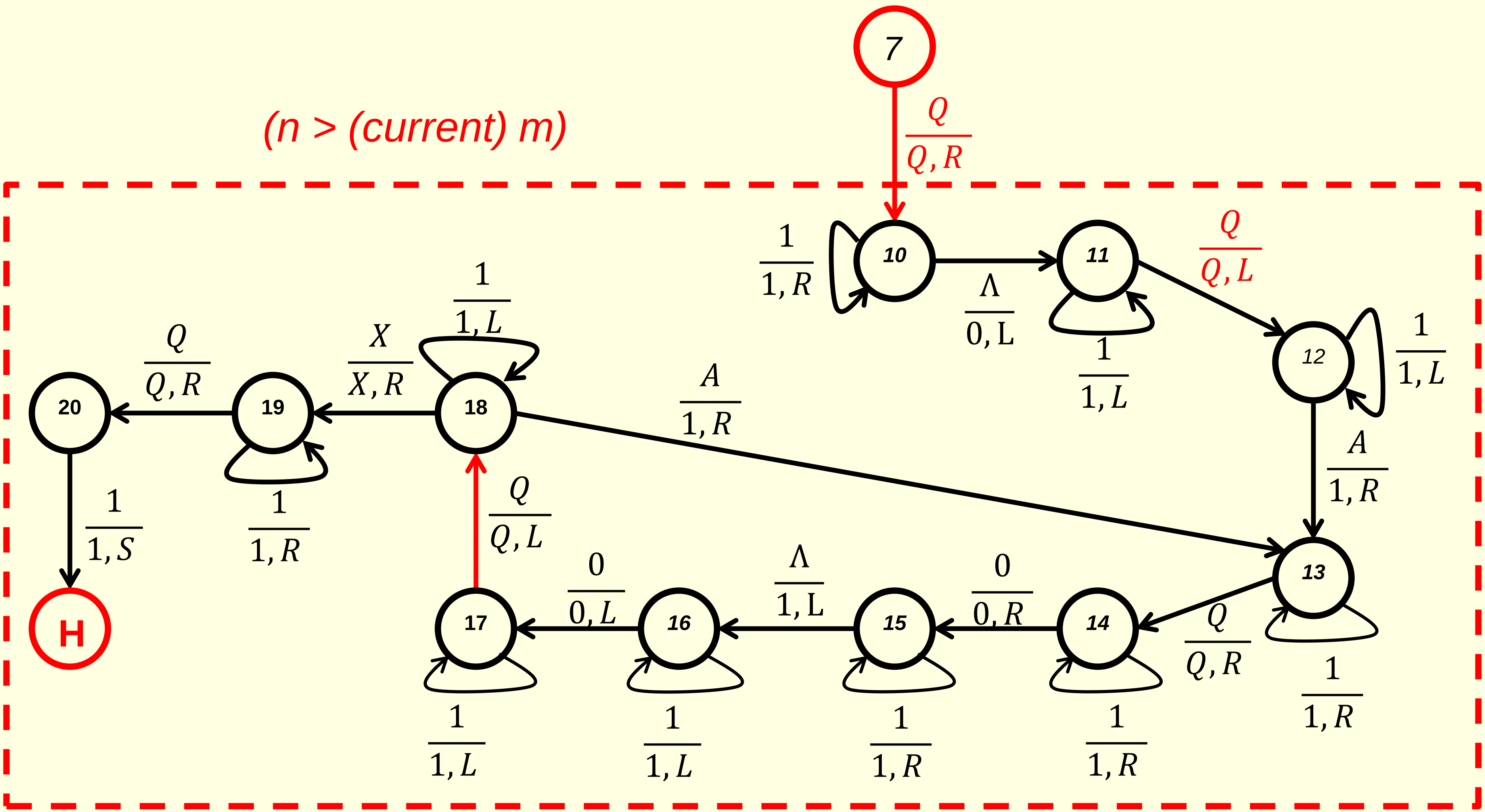
From (**) in Step 2 (slide 58):



(**)


Λ B B 0 X X X X A Q 1 1 Λ …
Λ B B 0 X X X X A Q 1 1 0 Λ …
Λ B B 0 X X X X 1 Q 1 1 0 1 Λ …
No more A's
Λ B B 0 X X X X 1 Q 1 1 0 1 Λ …
halt

The portion of the TM that performs Step 4 :



End of Turing Machines II