CS375:
Logic and Theory of Computing

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- **Weeks 13-14: Propositional Logic (Chapter 6), Predicate Logic (Chapter 7), Computational Logic (Chapter 9), Algebraic Structures (Chapter 10)**

7. Context-Free Languages & Pushdown Automat… ee Parsing

A black box assembly line

Cut open the belly

Goal:

- Declarative formalisms like CFGs, FSAs define the *legal strings of a language*--but only tell you 'this is a legal string of the language X'
- *Parsing algorithms* specify how to recognize the strings of a language and how to do syntactic analysis of a string

7. Context-Free Languages & Pushdown Automata - Context-Free Parsing

For instance,

"I like like spaghetti" could be considered a legal string

but not a legitimate sentence

structure-wise

LL(k) Grammar:

- A CFG is called an *LL(k) grammar* if a **parser** can be constructed to scan an input string from left to right and build a leftmost derivation by examining next *k* input symbols to determine the **unique** production for each derivation step.
- If a language has an LL(*k*) grammar, it is called an *LL(k) language*.

LL(k) Grammar:

Leftmost derivation

Next k input symbols → CFG production → $S \Rightarrow$ … …

Next k input symbols → CFG production → $\Rightarrow$ … …

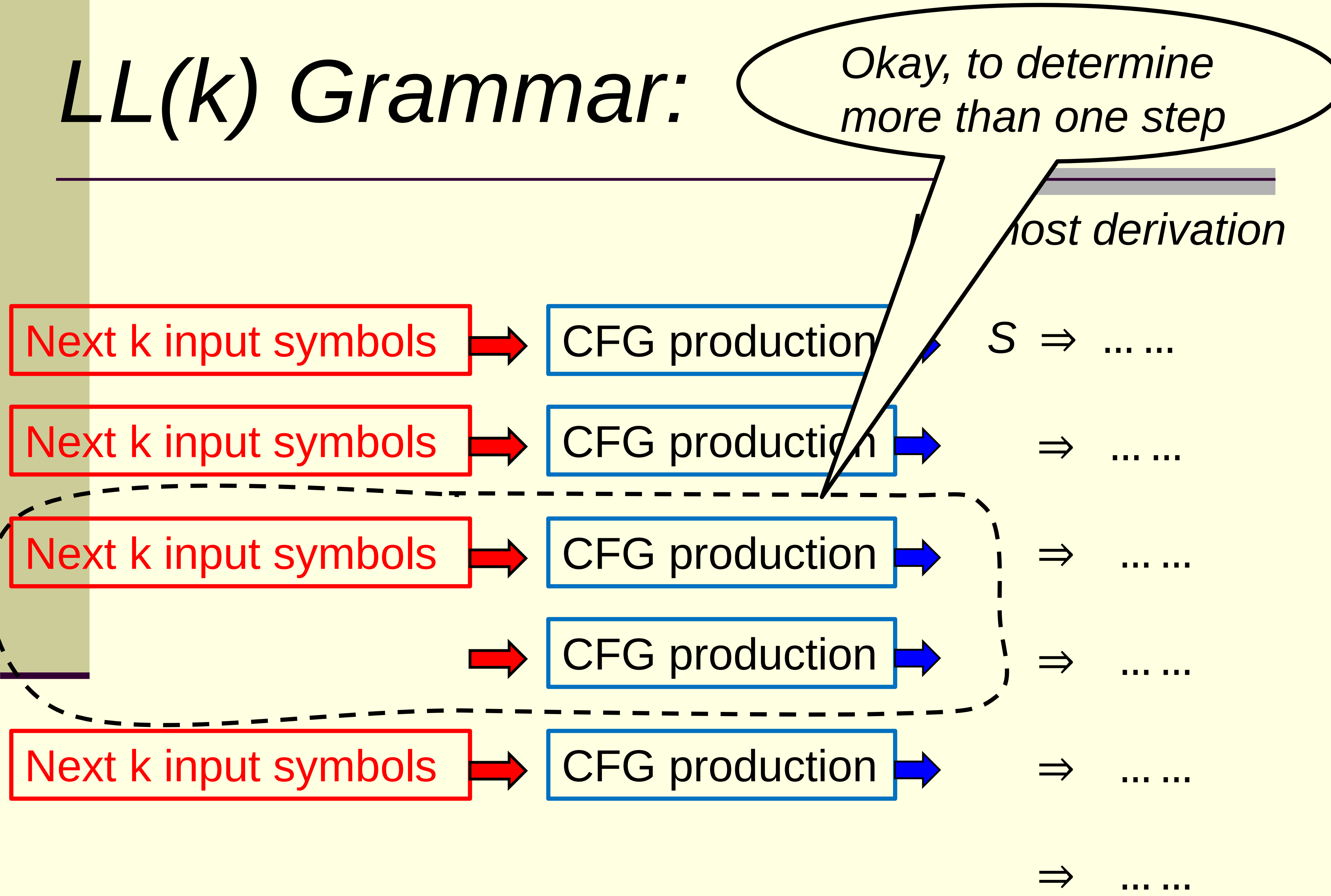
Next k input symbols → CFG production → $\Rightarrow$ … …

Next k input symbols → CFG production → $\Rightarrow$ … …

Next k input symbols → CFG production → $\Rightarrow$ … …

Next k input symbols → CFG production → $\Rightarrow$ … …

LL(*k*) Grammar:



Okay, to determine more than one step
ost derivation
Next k input symbols
CFG production
S ⇒ … …
Next k input symbols
CFG production
⇒ … …
Next k input symbols
CFG production
⇒ … …
CFG production
⇒ … …
Next k input symbols
CFG production
⇒ … …
⇒ … …

First, a few things about parse tree, parser and parsing:

$X_{02} \longrightarrow a\ A_{01}\ X_{11}$

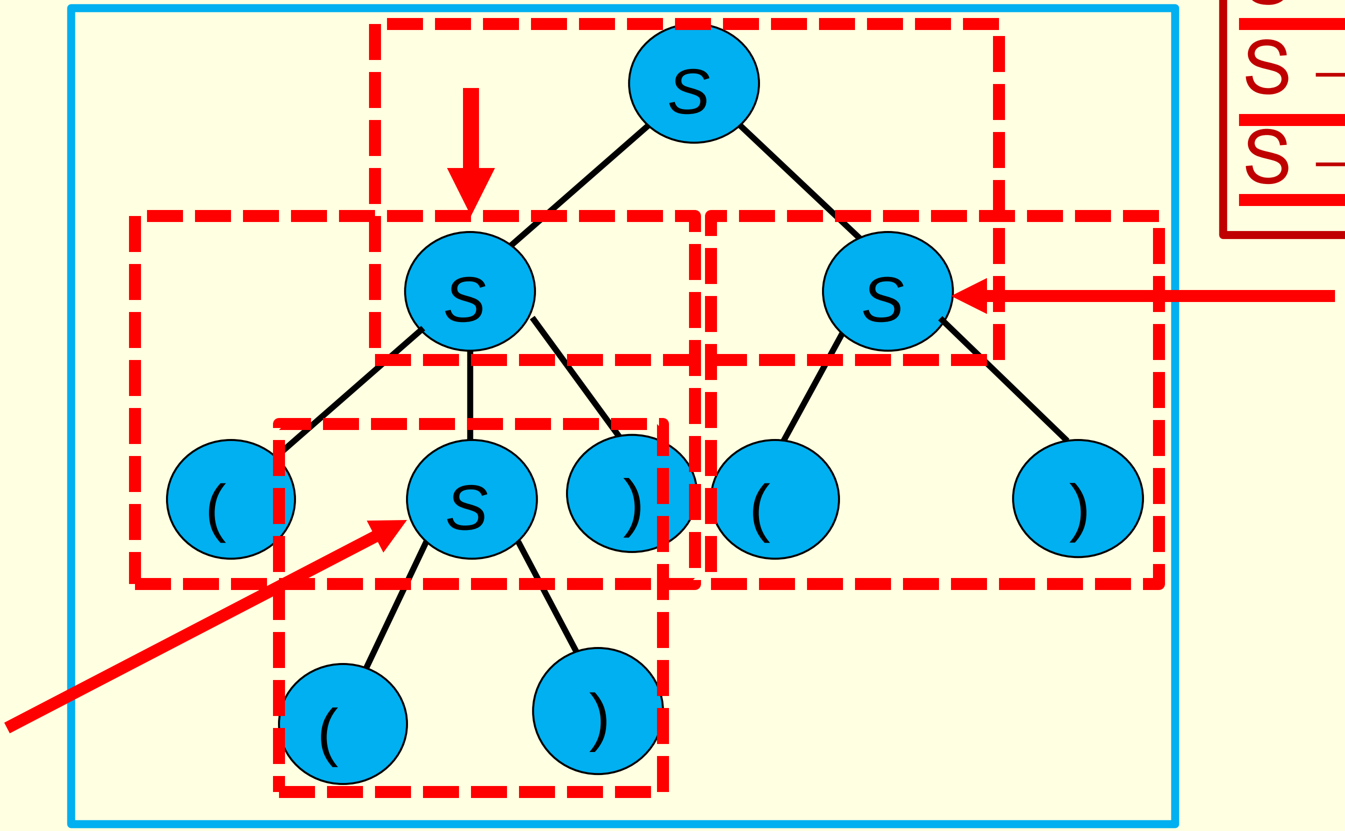
Parse Trees

- *Parse trees* are trees labeled by symbols of a particular CFG (in a particular order).
- *Leaves*: labeled by a terminal or Λ.
- *Interior nodes*: labeled by a non-terminal.
  - Children are labeled by the right side of a production for the parent.
- *Root*: must be labeled by the start symbol

Example: **Parse Tree**

S → SS | (S) | ()

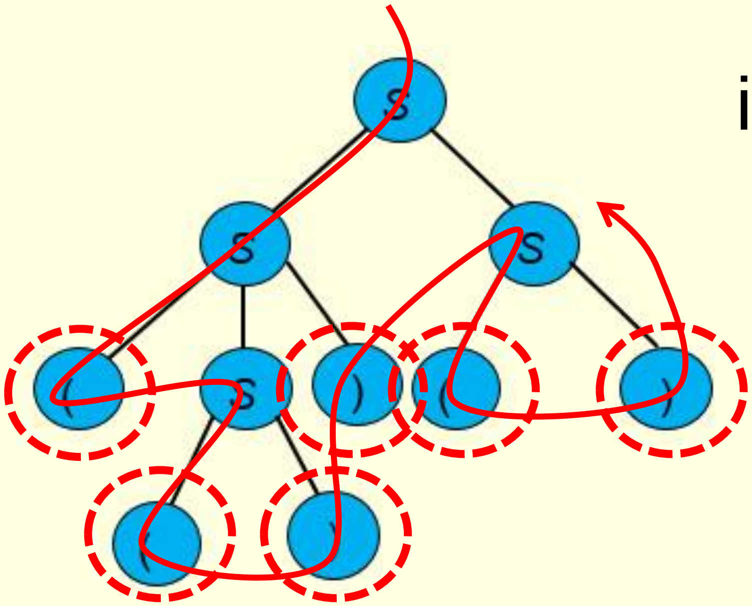
S → SS
S → (S)
S → ( )
S → ( )



Yield of a Parse Tree

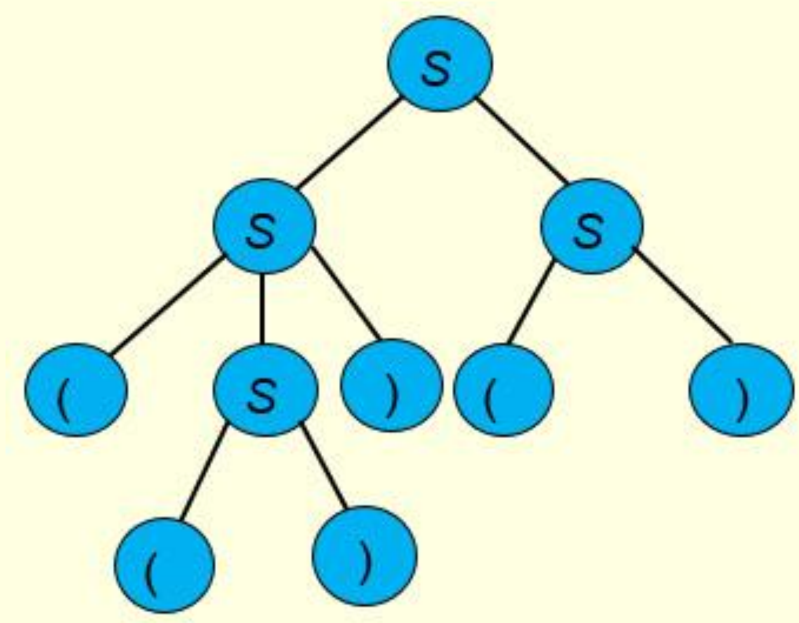
- The concatenation of the labels of the leaves in left-to-right order.
  - That is, in the order of a preorder traversal.
- is called the ***yield*** of the parse tree.

Root-L-R

- Example: yield of  is (( )) ( )

Parse Trees, Left- and Rightmost Derivations

- For every parse tree, there is a unique leftmost, and a unique rightmost derivation.



$$
\begin{aligned}
S &\Rightarrow SS \\
&\Rightarrow (S)S \\
&\Rightarrow (())S \\
&\Rightarrow (())()
\end{aligned}
$$

Derivation:

Given a grammar: **G:** $S \rightarrow SS \mid (S) \mid ()$

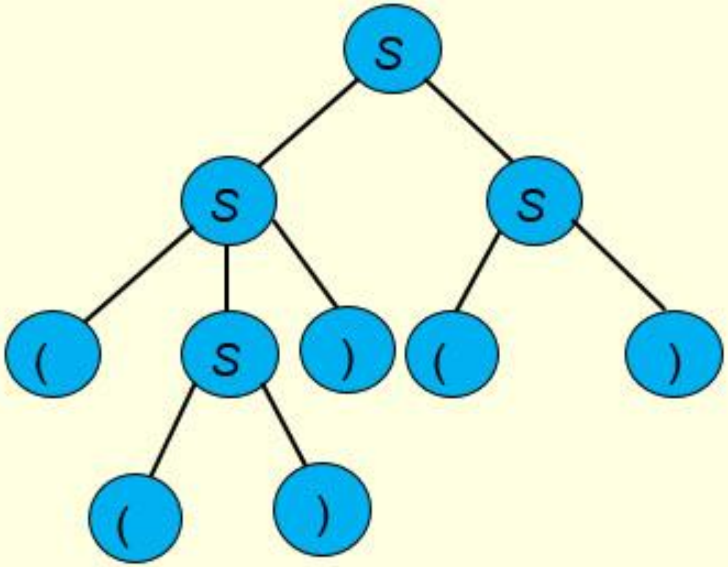
$$
\begin{aligned}
& S \\
\Rightarrow\ & SS \\
\Rightarrow\ & (S)S \\
\Rightarrow\ & (())S \\
\Rightarrow\ & (())()
\end{aligned}
$$

(leftmost) derivation

Conclusion: (())() is a legal string in **L(G)**

Parsing:

Given a parse tree

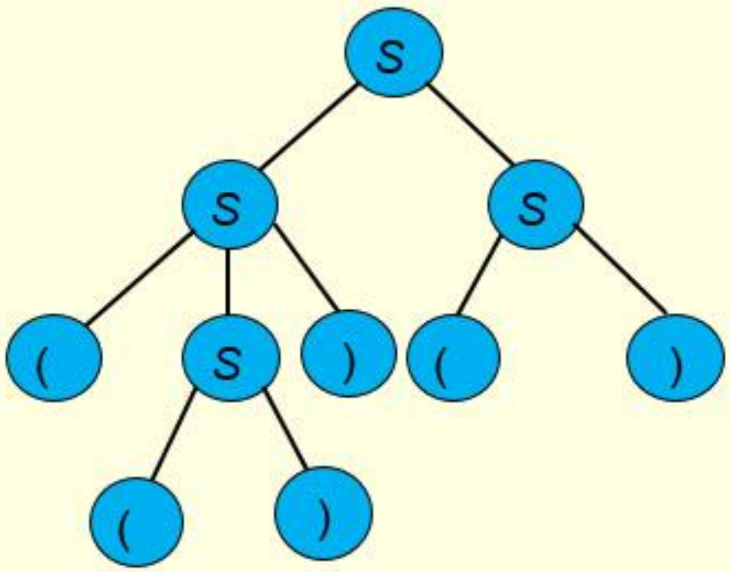


S
⇒ SS
⇒ (S)S
⇒ (())S
⇒ (())()

Conclusion: (())() is a legal string in L(G) because it has a valid structure

Parsing:

Given a parse tree



(productions)

$S \rightarrow SS$

$S \rightarrow (S)$

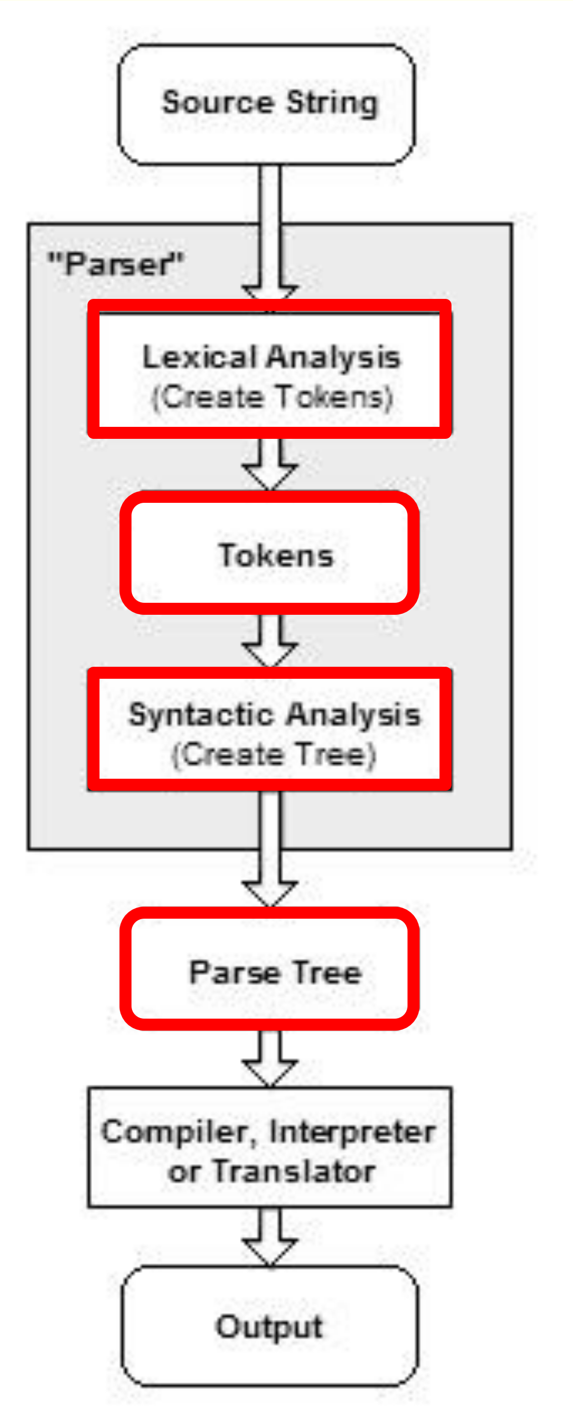
$S \rightarrow ( )$

We can even tell if a grammar is an LL(1), LL(2), … grammar

Parsing:

Actually parsing contains several steps.

But for now, a conceptual understanding is enough



Continue on LL(k) grammar …

- A CFG is called an *LL(k) grammar* if a parser can be constructed to scan an input string from left to right and build a leftmost derivation by examining next *k* input symbols to determine the unique production for each derivation step.
- If a language has an LL(*k*) grammar, it is called an *LL(k) language*.

Example. Consider the language

$$\{\ a^n b \mid n \in \mathbf{N}\}.$$

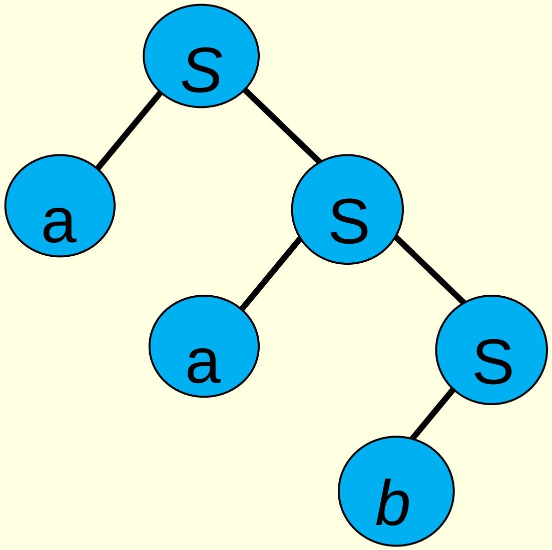
(1) It has an LL(1) grammar

$$S \rightarrow aS \mid b$$

A parser can examine one input letter to decide whether to use $S \rightarrow aS$ or $S \rightarrow b$ for the next derivation step.

Consider: $\boxed{a}\boxed{a}\boxed{b}\boxed{}$

$$S \Rightarrow \boxed{a}S$$
$$\Rightarrow a\,\boxed{a}\,S$$
$$\Rightarrow a\,a\,\boxed{b}$$



Example. Consider the language

$$\{\ a^n b \mid n \in \mathbf{N}\}.$$

(2) It has an LL(2) grammar

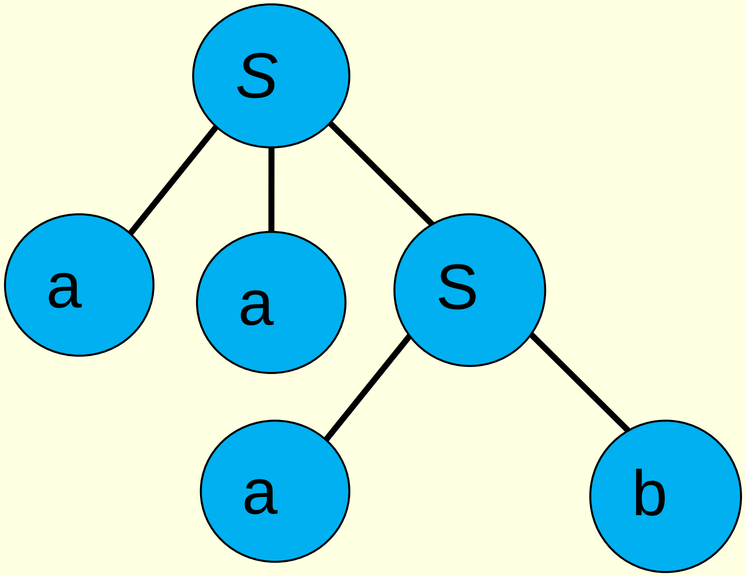
$$S \to aaS \mid ab \mid b$$

A parser can examine two input letters to decide whether to use $S \to aaS$, $S \to ab$ or $S \to b$ for the next derivation step.

Consider: $\boxed{a\ a}\boxed{a\ b}\boxed{}$

$$S \Rightarrow \boxed{a}\boxed{a}\,S$$

$$\Rightarrow a\ a\ \boxed{a}\boxed{b}$$



Example. Consider the language

$$\{\ a^n b \mid n \in \mathbf{N}\}.$$

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$$S \to aaS \mid ab \mid b$$

A parser can examine two input letters to decide whether to use $S \to aaS$, $S \to ab$ or $S \to b$ for the next derivation step.

Question 1: *Is this grammar LL(1)?*

No. *Consider* $\boxed{a}ab$

Can not determine $S \to aaS$ *or* $S \to ab$ *to use*

Example. Consider the language

$$\{\ a^n b \mid n \in \mathbf{N}\}.$$

(2) It has an LL(2) grammar

$$S \to aaS \mid ab \mid b$$

A parser can examine two input letters to decide whether to use $S \to aaS$, $S \to ab$ or $S \to b$ for the next derivation step.

Question 2: *can you find an LL(3) grammar that is not LL(2)?*

Answer: $S \to aaaS \mid aab \mid ab \mid b$

and consider the strings aaaaab, aaaab, aaab

Ambiguous Grammars

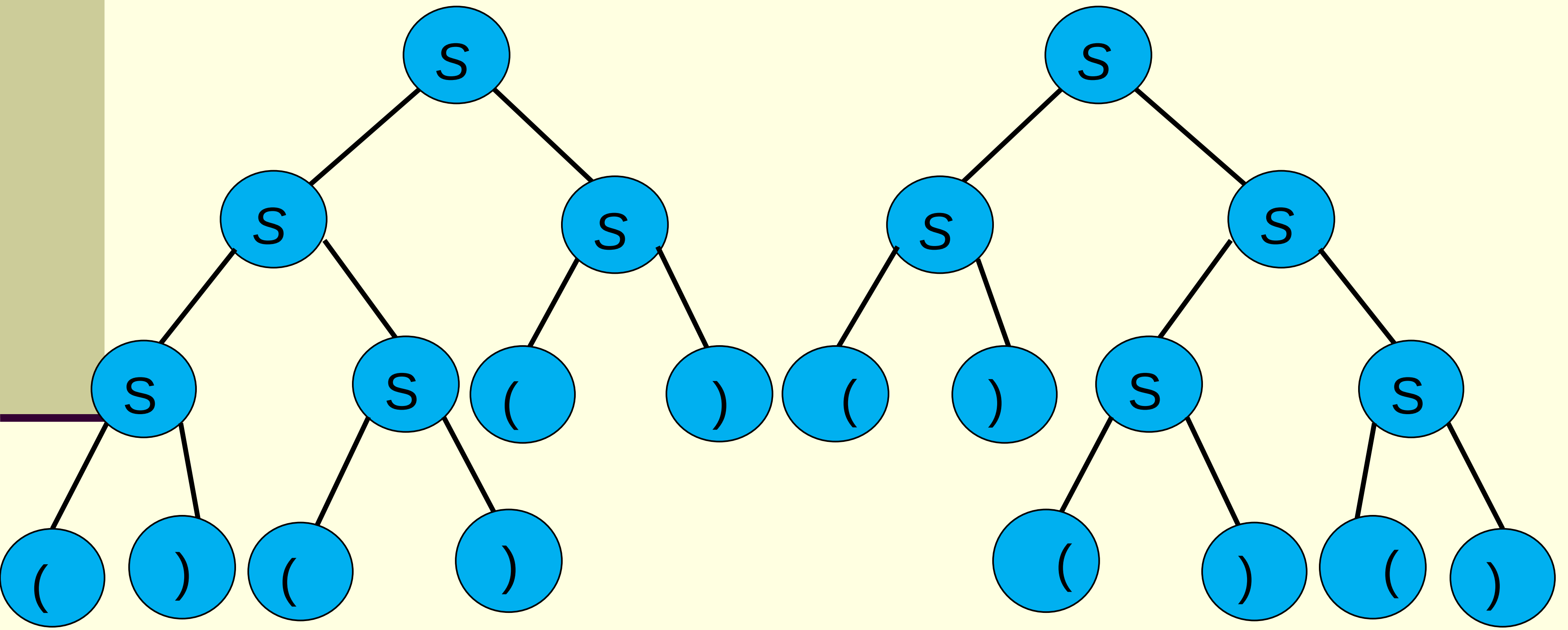
- A CFG is ambiguous if there is a string in the language that is the yield of two or more parse trees.

Not good

- Example: S -> SS | (S) | ()
- Two parse trees for ()()() on next slide.

Ambiguous Grammars

Example: ()()()



Facts:

Depending on if identifiers/symbols are included in the languages

- Most programming languages have LL(1) grammars.

  (What if a, aa, aaa are the names of three variables?)

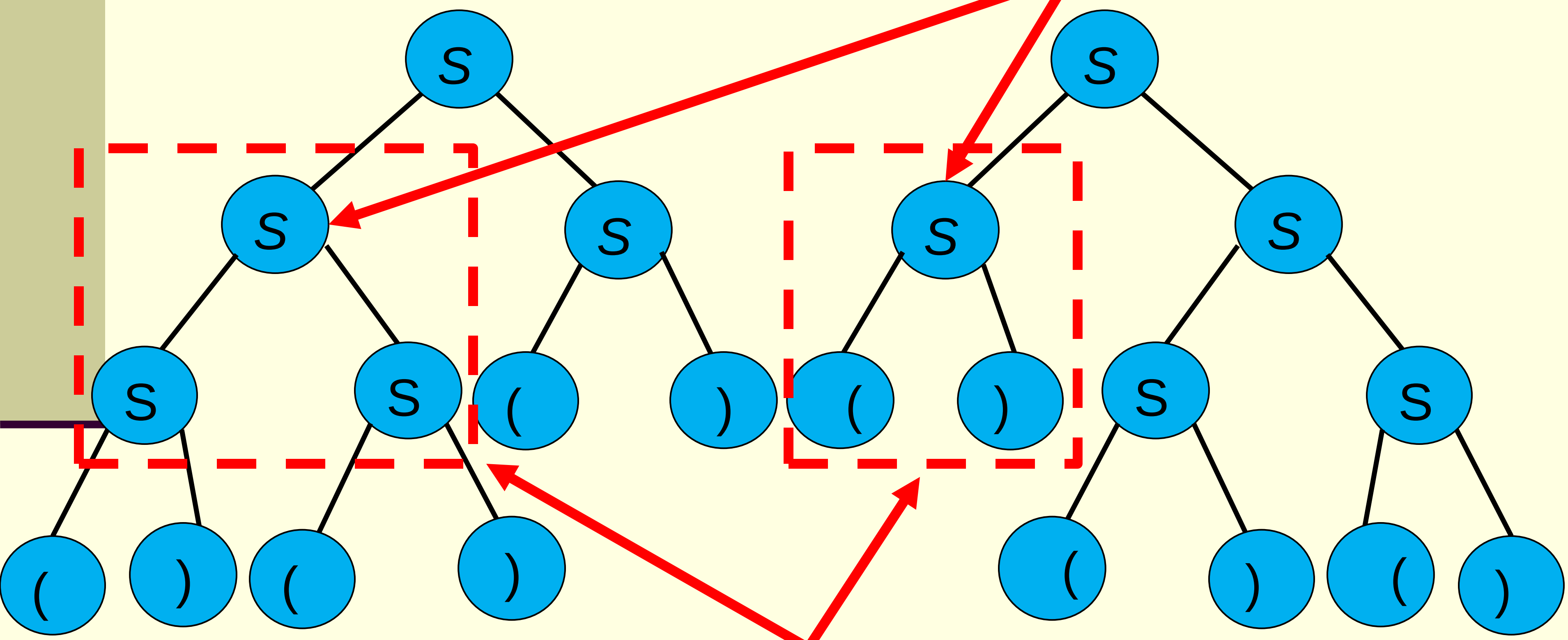
- LL(1) grammars are never ambiguous.

LL(1) grammars are
- not ambiguous and
- not left-recursive.

LL(1) grammars are not ambiguous. Why?

Example: ()()()

If the grammar is ambiguous, find the first internal node whose child nodes are different.



For that internal node, we have two different production choices for that derivation step, a contradiction.

LL(1) grammars are not left-recursive. Why?

If an LL(1) grammar is left-recursive, then there is at least a production of the form

$$S \rightarrow SA$$

But then there must be a production of the form

$$S \rightarrow B$$

to terminate the recursion. So when ‘S’ is scanned, we would have two options to choose from, a contradiction.

Questions:

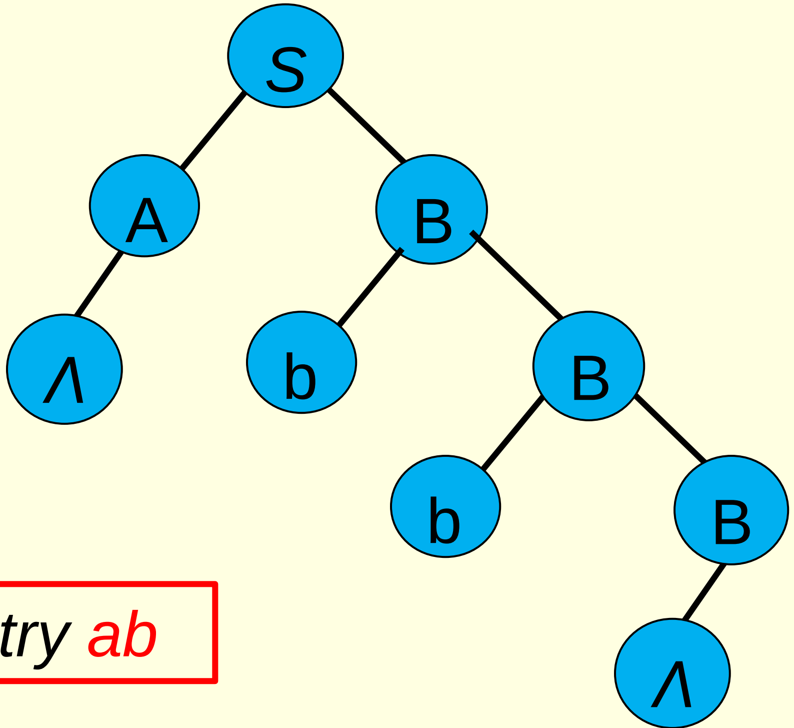
Is the grammar

$$S \to AB \;;\quad A \to aAb \mid \Lambda \;;\quad B \to bB \mid \Lambda$$

for $\{\, a^n b^{n+k} \mid n, k \in \mathbf{N} \}$ an LL(1) grammar?

Consider: $\boxed{b}\,\boxed{b}\,\boxed{}$

$$S \Rightarrow AB$$
$$\Rightarrow \Lambda\, B$$
$$\Rightarrow \Lambda\, \boxed{b}\, B$$
$$\Rightarrow \Lambda\, b\, \boxed{b} B$$
$$\Rightarrow \Lambda\, b\, b\, \Lambda$$



Then try ab

Questions:

Is the grammar

$$S \rightarrow AB\ ; \quad A \rightarrow aAb \mid \Lambda\ ; \quad B \rightarrow bB \mid \Lambda$$

for $\{\, a^n b^{n+k} \mid n, k \in \mathbf{N} \}$ an LL(1) grammar?

Consider: $\boxed{a}\boxed{b}\boxed{b}\boxed{}$

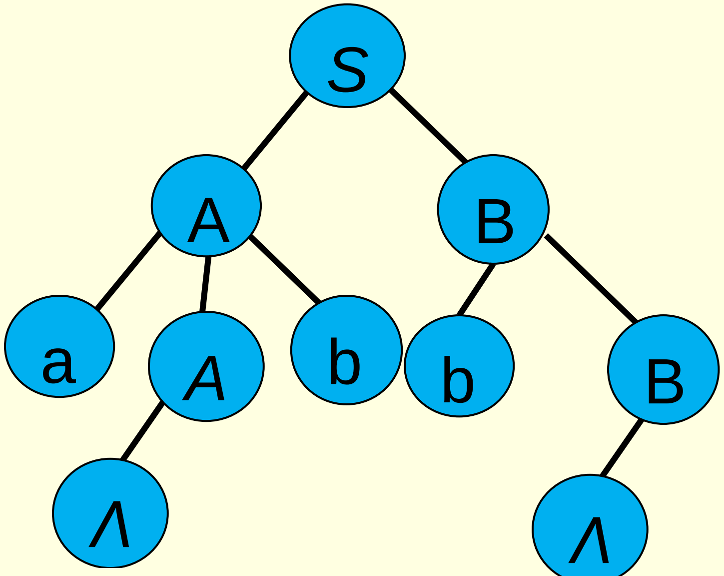
$S \Rightarrow AB$

$\Rightarrow \boxed{a}\, A\, b\, B$

$\Rightarrow a\, \Lambda\, \boxed{b}\, B$

$\Rightarrow a\, \Lambda\, b\, \boxed{b}\, B$

$\Rightarrow a\, \Lambda\, b\, b\, \Lambda$



Yes or No

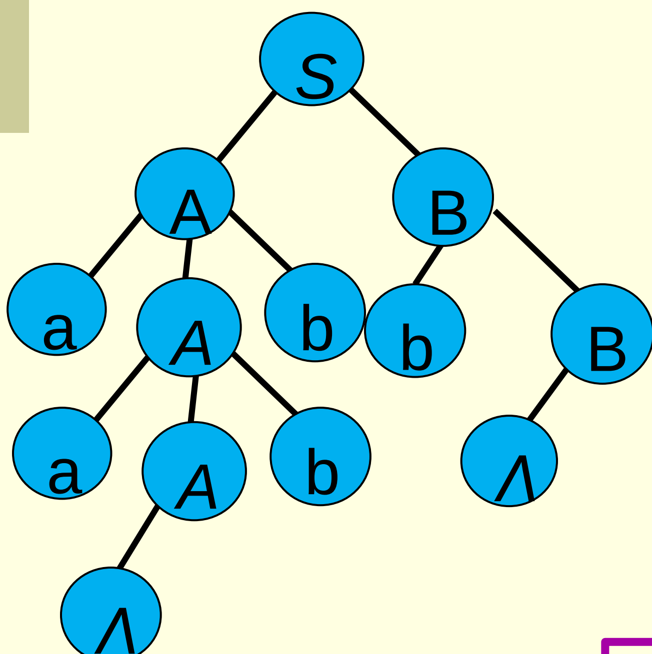
Then try aabb

Questions:

Is the grammar

$$S \to AB\ ; \quad A \to aAb \mid \Lambda\ ; \quad B \to bB \mid \Lambda$$

for $\{\, a^n b^{n+k} \mid n, k \in \mathbf{N} \}$ an LL(1) grammar?



Consider: $\boxed{a}\boxed{a}\boxed{b}\boxed{b}\boxed{b}\boxed{\ }$

$$\begin{aligned}
S &\Rightarrow AB \\
&\Rightarrow \boxed{a}\, A\, b\, B \\
&\Rightarrow a\, \boxed{a}\, A\, b\, b\, B \\
&\Rightarrow a\, a\, \boxed{\Lambda}\boxed{b}\boxed{b}\, B \\
&\Rightarrow a\, a\, \Lambda\, b\, b\, \boxed{b}\, B \\
&\Rightarrow a\, a\, \Lambda\, b\, b\, b\, \boxed{\Lambda}
\end{aligned}$$

Yes or No

Questions:

Is the grammar

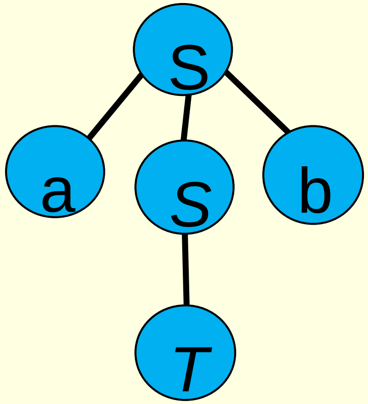
$S \to aSb \mid T\,; \quad T \to bT \mid \Lambda$

for $\{\, a^n b^{n+k} \mid n, k \in \mathbf{N}\}$ an LL(1) grammar?

Consider: $\boxed{a}\boxed{b}$

$S \Rightarrow \boxed{a}\,S\,b$

$\Rightarrow a\,\boxed{T}\,b$



It gets stuck here

Yes or No

Questions:

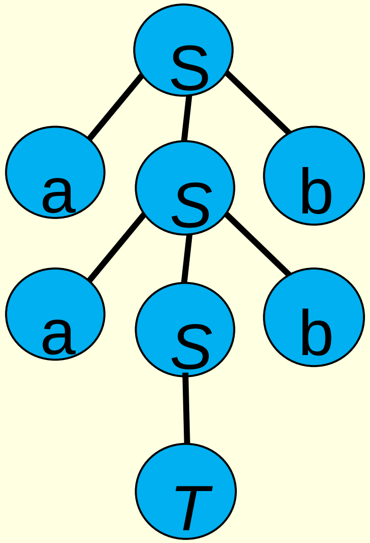
Is the grammar

$S \rightarrow aSb \mid T\ ; \quad T \rightarrow bT \mid \Lambda$

for $\{\, a^n b^{n+k} \mid n, k \in \mathbf{N} \}$ an LL(2) grammar?

Consider: $\boxed{a\ a}\boxed{b\ b}$

$$S \Rightarrow \boxed{a} S\, b$$
$$\Rightarrow a\, \boxed{a} S\, b\, b$$
$$\Rightarrow a\, a\, \boxed{T}\, b\, b$$



Yes or No

It gets stuck here

Questions:

Is the grammar

$S \rightarrow aSb \mid T\ ;\quad T \rightarrow bT \mid \Lambda$

for $\{\, a^n b^{n+k} \mid n, k \in \mathbf{N} \}$ an LL(k) grammar for k>2?

Quiz on your time

For k=3, check aaabbb

For k=4, check aaaabbbb

For k=n, check $a^n b^n$

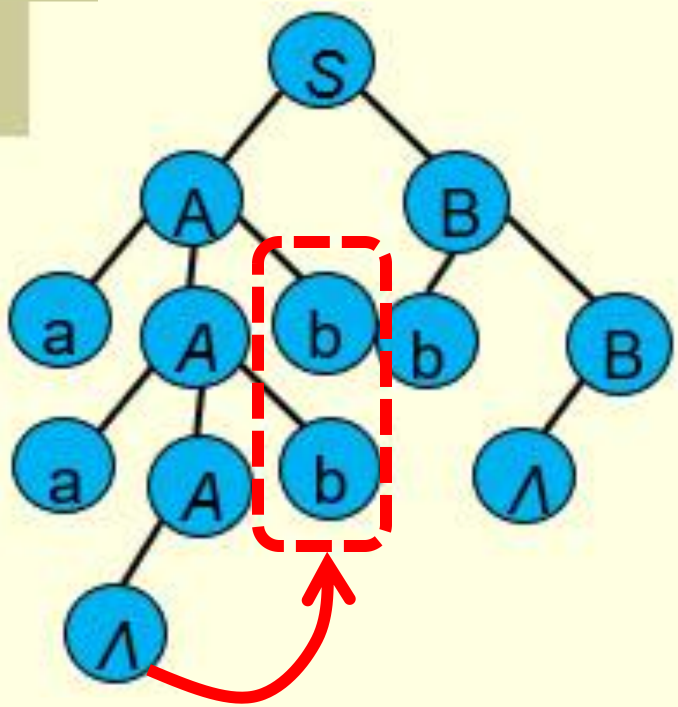
What is the difference here?

Why the grammar { S→AB; A→aAb | Λ; B→bB | Λ }

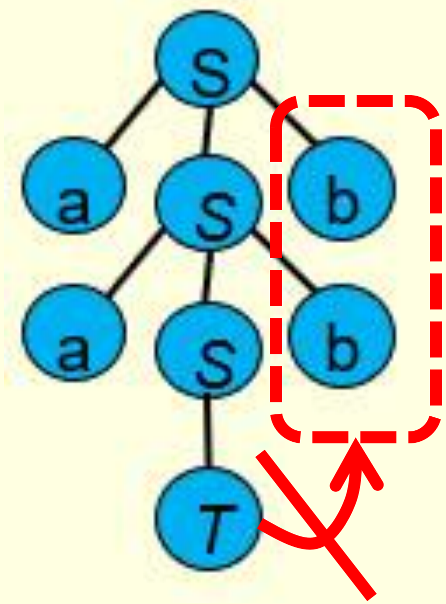
is LL(1) for { $a^n b^{n+k}$ | *n*, *k* ∈ **N**} ?

but the grammar { S → aSb | T; T → bT | Λ }

for the same language is not LL(k) for any k>0?



In the second case, we did not give the parsing process a chance to look back before it moves forward. What does this mean?



How to show a language is not an LL(k) language?

The language $\{ a^n b \mid n \in \mathbf{N}\}$ is LL(1)

$S \to aS \mid b$

The language $\{ a^n b^{n+k} \mid n, k \in \mathbf{N}\}$ is LL(1)

$S \to AB \qquad A \to aAb \mid \Lambda \qquad B \to bB \mid \Lambda$

How to show a language is not an LL(k) language?

But, even with the following grammar, the language $\{a^{n+k}b^n \mid n,k \in \mathbf{N}\}$ is not LL(k) for any k

S → AB ; A → aA | Λ ; B → aBb | Λ

Why?

Consider ab when k=1

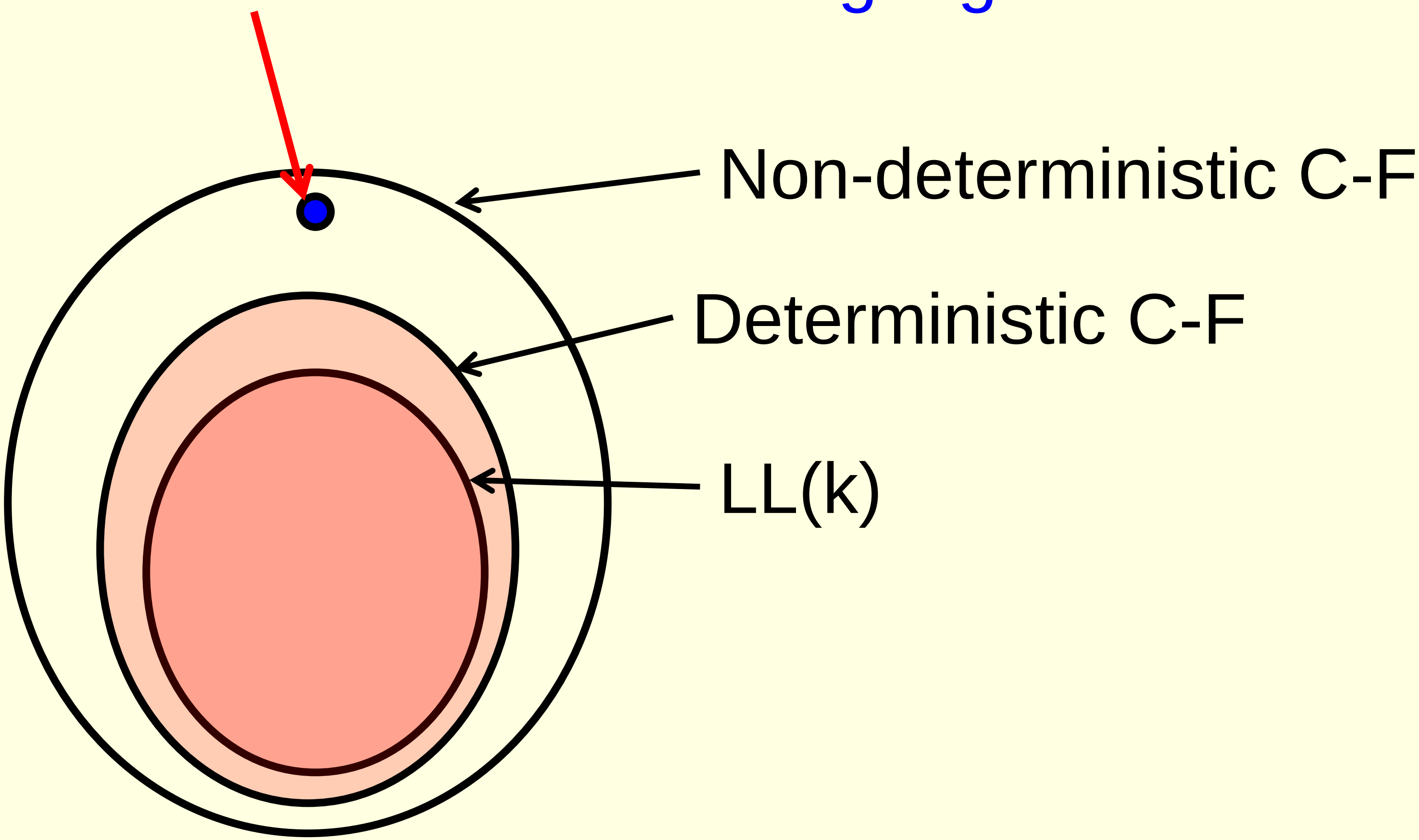
Consider aabb when k=2

Consider aaabbb when k=3

…

LL(k) languages ≠ DCF Languages

- LL(*k*) languages are deterministic CF languages,
- hence non-deterministic CF languages are not LL(k).

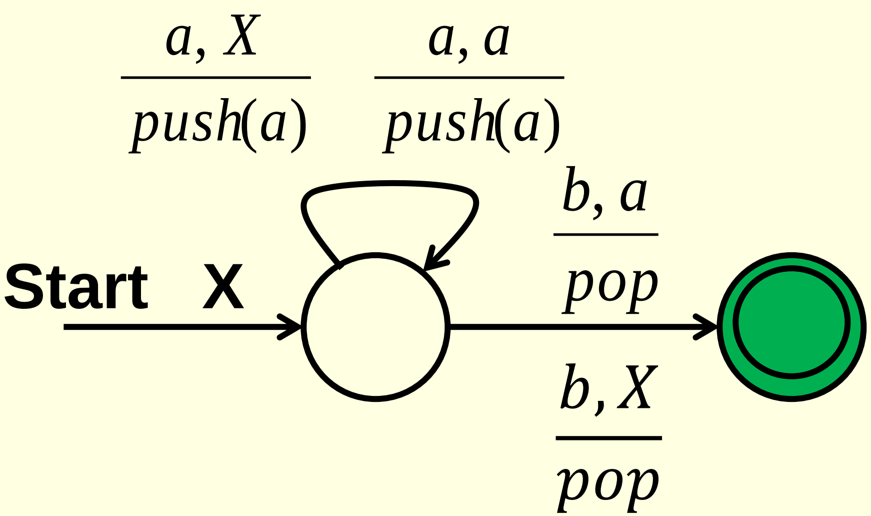


The language $\{ a^n b \mid n \in \mathbf{N}\}$ is LL(1)

$S \rightarrow aS \mid b$

Is the following PDA a PDA for this language?

Deterministic or non-deterministic?



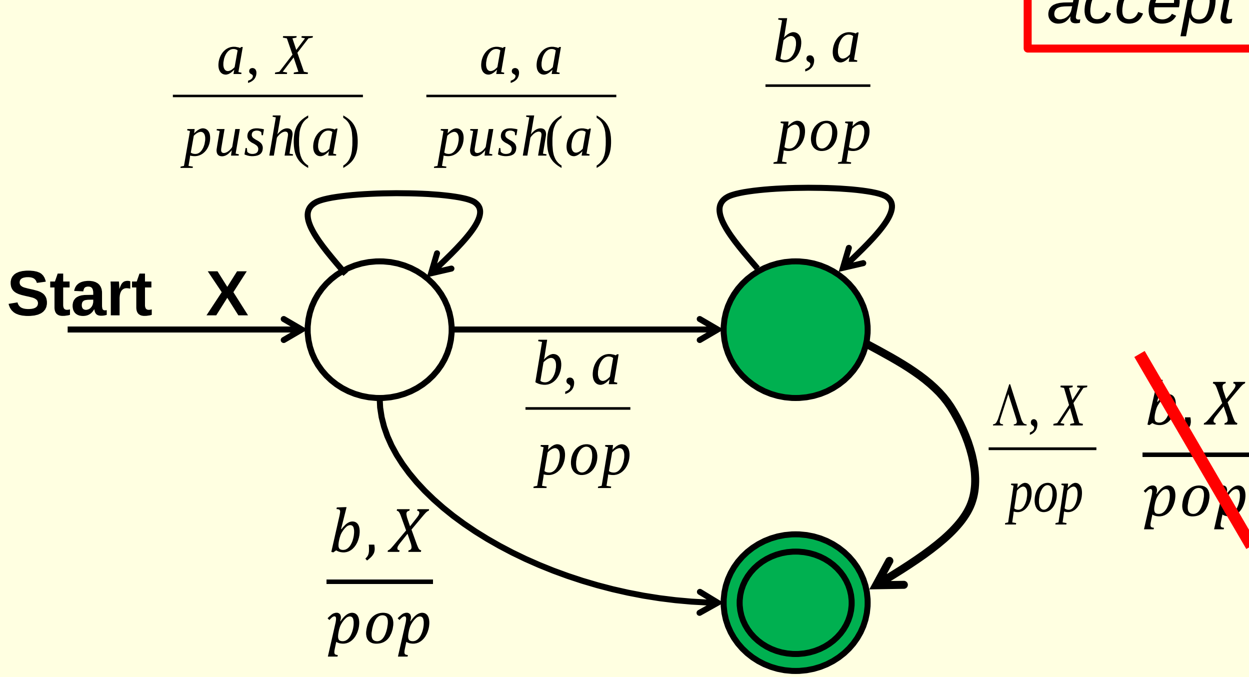
The language $\{\, a^n b^{n+k} \mid n, k \in \mathbf{N}\}$ is LL(1)

$S \to AB \qquad A \to aAb \mid \Lambda \qquad B \to bB \mid \Lambda$

Is the following PDA a PDA for this language?

Deterministic or non-deterministic?

Does this PDA accept aabbbaa?



Example:

The language $\{ a^{n+k} b^{n} \mid k, n \in \mathbf{N}\}$ is nondeterministic context-free. So it has no LL(*k*) grammar for any *k*.

Proof:

Any PDA for the language must keep a count of the *a*'s with the stack so that when the *b*'s come along the stack can be popped with each *b*.

But there might still be *a*'s on the stack (when $k > 0$), so there must be a nondeterministic state transition to a final state from the popping state. i.e., we need two instructions like,

(i, b, a, pop, i) and (i, Λ, a, pop, *final*).

Illustration:

So why is it non-deterministic?

The language $\{ a^{n+k} b^n \mid k, n \in \mathbf{N} \}$ is nondeterministic context-free. So it has no LL(k) grammar for any k.

To pop the remaining a's off the stack

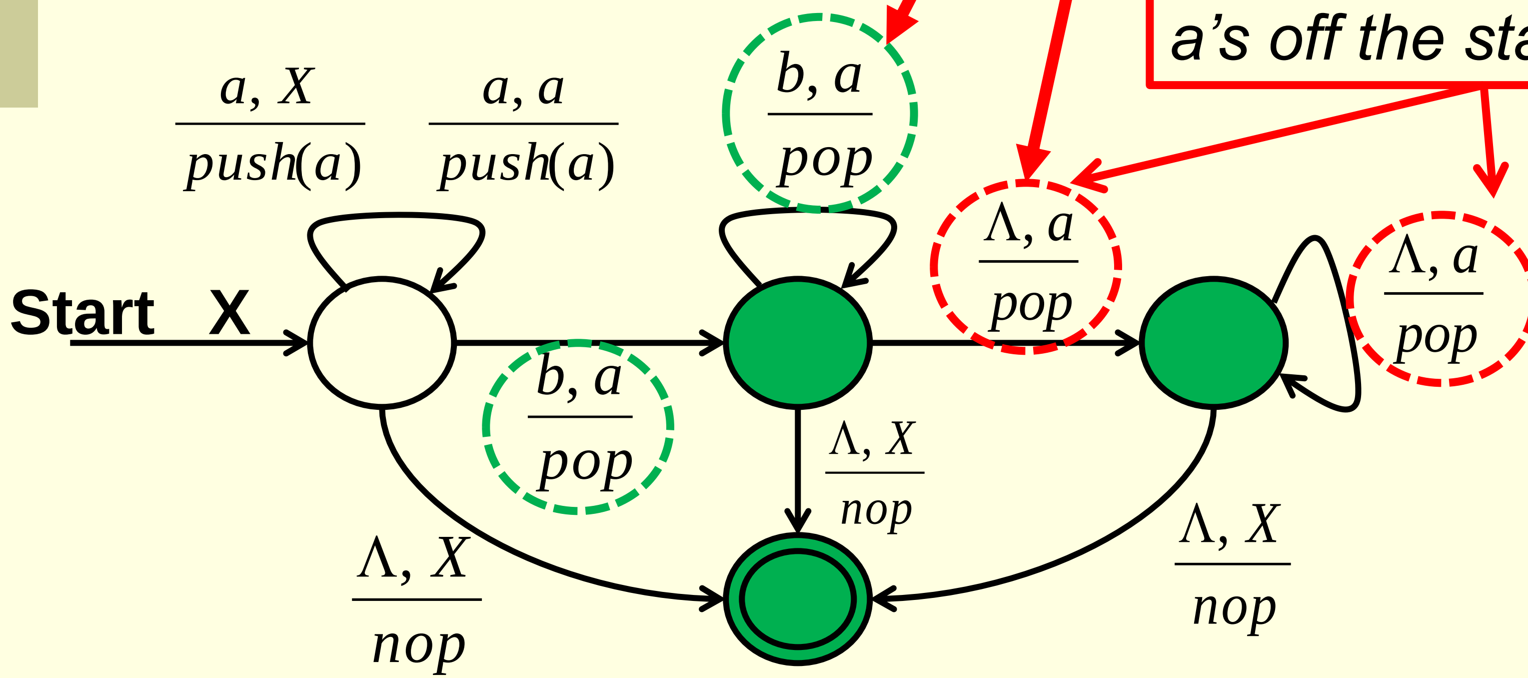
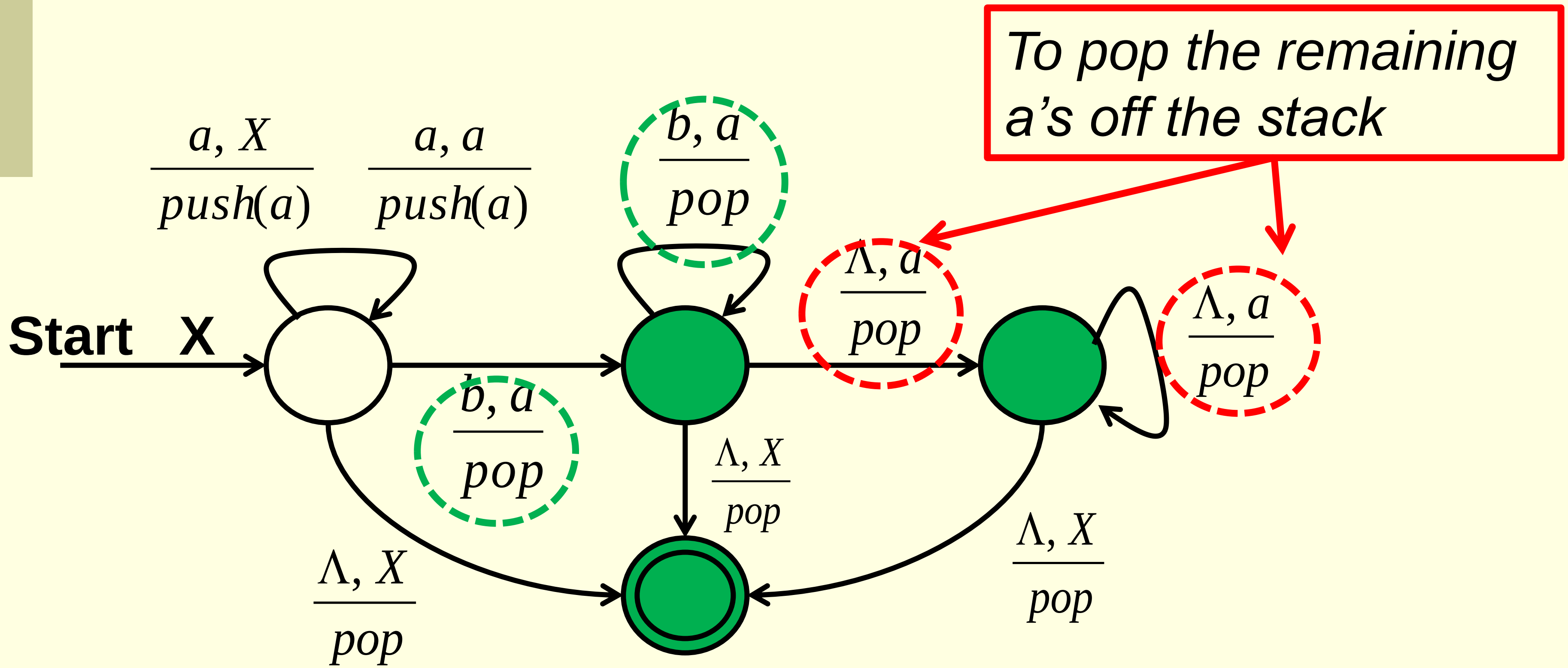


Illustration:

The language $\{\, a^{n+k} b^{n} \mid k, n \in \mathbf{N} \}$ is nondeterministic context-free. So it has no LL(k) grammar for any k.

or



Question:

The language $\{ a^{n+k} b^n \mid k, n \in \mathbf{N}\}$ is nondeterministic context-free. (So it has no LL(*k*) grammar for any *k*)

then

Is the language $\{ a^{n+k} b^n c^k \mid k, n \in \mathbf{N}\}$ nondeterministic context-free?

Keep your answer to yourself

Grammar Transformations:

"Left-factoring" an LL(k) grammar to obtain an equivalent LL(n) grammar where $n < k$.

Example.

LL(2), not LL(1)

$S \rightarrow aaS \mid ab \mid b$

factor "a" out

$S \rightarrow a(\, aS \mid b)$

$S \rightarrow aT \qquad T \rightarrow aS \mid b$

$S \rightarrow \mathrm{aT} \mid \mathrm{b} \qquad \mathrm{T} \rightarrow \mathrm{aS} \mid \mathrm{b}$

Question:

Is { S → aT | b T → aS | b } an LL(1) grammar for the language $\{ a^n b \mid n \in N \}$?

Consider: $\boxed{b}\boxed{\ }$

Consider: $\boxed{a}\boxed{b}\boxed{\ }$

Consider: $\boxed{a}\boxed{a}\boxed{b}\boxed{\ }$

$S \Rightarrow \boxed{b}$

$S \Rightarrow \boxed{a}T$

$\Rightarrow a\boxed{b}$

$S \Rightarrow \boxed{a}T$

$\Rightarrow a\boxed{a}S$

$\Rightarrow a\,a\boxed{b}$

Yes or No

Question:

{ S → aaS | ab | b } is an LL(2) grammar for the language { $a^n b$ | n ϵ N}

{ S → aT | b   T → aS | b } is an LL(1) grammar for the language { $a^n b$ | n ϵ N}

Try aaaab and get its parse trees in both cases

End of Context-Free Language and Pushdown Automata III