Gamma Distribution and Gamma Approximation

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Abstract

The Gamma distribution and related approximation properties of this distribution to certain of classes of functions are discussed. Two asymptotic estimate formulas are given.

Keywords: Gamma distribution, Gamma approximation, locally bounded functions, Lebesgue-Stieltjes integral, probabilistic methods

1 Introduction and Main Results

Probability distributions and related approximation properties have played a central role in several areas of *computer aided geometric design* (CAGD) and *numerical analysis*. For instance, the (Bernstein) basis functions of Bézier curves and surfaces are taken from a binomial distribution, and the basis functions of B-spline curves and surfaces model a simple stochastic process [1, 2]. Properties of these curves and surfaces are closely related to the approximation features of the corresponding basis functions. Therefore studying probability distributions and related approximation properties is important both in theory and applications.

The aim of this paper is to do a study in that direction. In this paper we shall consider the following Gamma random variable ξ with probability density function:

$$p_\xi(u) = \begin{cases} \frac{u^{x-1}}{\Gamma(x)}\exp(-u), & \text{if } u \geq 0 \\ 0, & \text{if } u < 0 \end{cases} \tag{1}$$

where x is a fixed number in $(0, \infty)$ and $\Gamma(x)$ is the Gamma function of x defined as follows:

$$\Gamma(x) = \int_0^\infty u^{x-1}e^{-u}du,$$

and study related approximation process based on the above Gamma distribution. We first prove the following lemma.

Lemma 1. Let $\{\xi_i\}_{i=1}^{\infty}$ be a sequence of independent random variables with the same Gamma distribution as ξ, and let $\eta_n = \sum_{i=1}^{n} \xi_i$. Then the probability distribution of the random variable η_n is

$$P(\eta_n \le y) = \frac{1}{\Gamma(nx)} \int_0^y u^{nx-1} e^{-u} du. \tag{2}$$

Proof. Let the distribution functions of the random variables ξ_i be $F_i(x),\ i = 1, 2, 3, ...,$ and let the distribution function of η_n be $F_{\eta_n}(x)$. We have

$$F_i(y) = \int_{-\infty}^{y} p_{\xi_i}(u)du = \int_0^y \frac{u^{x-1}}{\Gamma(x)} e^{-u} du$$

and

$$F_{\eta_n}(y) = (F_1 * F_2 * ... * F_n)(y),$$

where

$$(F_1 * F_2)(y) = \int_{-\infty}^{+\infty} F_2(y-u)dF_1(u),$$

and $(F_1 * F_2 * ... * F_n)(y)$ is defined recursively. By convolution of probability distributions we obtain

$$P(\eta_n \le y) = F_{\eta_n}(y) = \frac{1}{\Gamma(nx)} \int_0^y u^{nx-1} e^{-u} du.$$

Hence, Lemma 1 is proved.

In this paper we consider the following Gamma approximation process G_n^*:

$$G_n^*(f, x) = \int_{-\infty}^{+\infty} f(t/n) dF_{\eta_n}(t) = \frac{n^{nx}}{\Gamma(nx)} \int_0^{\infty} f(u) u^{nx-1} e^{-nu} du \tag{3}$$

where f belongs to the class of function Φ_B or the class of function Φ_{DB}, which are defined respectively by

$$\Phi_B = \{f |\ f \text{ is bounded on every finite subinterval of } [0, \infty),$$
$$\text{and } |f(t)| \le Me^{\beta t}, \quad (M > 0;\ \beta \ge 0;\ t \to \infty)\}.$$

and

$$\Phi_{DB} = \{f |\ f(x) - f(0) = \int_0^x h(t)dt;\ x \ge 0;\ h \text{ is bounded on every finite}$$
$$\text{subinterval of } [0, \infty), \text{ and } |f(t)| \le Me^{\beta t}, \quad (M > 0;\ \beta \ge 0;\ t \to \infty)\}.$$

For a function $f \in \Phi_B$, we introduce the following metric form:

$$\Omega_x(f, \lambda) = \sup_{t \in [x-\lambda, x+\lambda]} |f(t) - f(x)|,$$

where $x \in [0, \infty)$ is fixed, $\lambda \ge 0$.

It is clear that

$\Omega_x(f, \lambda)$ is non-decreasing with respect to λ;

$\lim_{\lambda \to 0} \Omega_x(f, \lambda) = 0$, if f is continuous at the point x;

If f is of bounded variation on $[a,b]$, and $\bigvee_a^b(f)$ denotes the total variation of f on $[a,b]$, then

$$\Omega_x(f,\lambda) \le \bigvee_{x-\lambda}^{x+\lambda}(f)$$

For further properties of $\Omega_x(f,\lambda)$, refers to Zeng and Cheng [8].

The main results of this paper are as follows:

Theorem 1. *Let $f \in \Phi_B$. If $f(x+)$ and $f(x-)$ exist at a fixed point $x \in (0,\infty)$, then for $n > 3\beta$, we have*

$$\left| G_n^*(f,x) - \frac{f(x+)+f(x-)}{2} + \frac{f(x+)-f(x-)}{3\sqrt{2\pi n x}} \right| \le \frac{x+4}{nx} \sum_{k=1}^{n} \Omega_x(g_x, x/\sqrt{k}) + O(n^{-1}), \quad (4)$$

where

$$g_x(u) = \begin{cases} f(u) - f(x+), & x < u < \infty; \\ 0, & u = x; \\ f(u) - f(x-), & 0 \le u < x. \end{cases} \quad (5)$$

We point out that Theorem 1 subsumes the case of approximation of functions of bounded variation. From Theorem 1 we get immediately

Corollary 1. *Let f be a function of bounded variation on every subinterval of $[0,\infty)$ and let $f(t) = O(e^{\beta t})$ for some $\beta \ge 0$ as $t \to \infty$. then for $x \in (0,\infty)$ and $n > 3\beta$, we have*

$$\left| G_n^*(f,x) - \frac{f(x+)+f(x-)}{2} + \frac{f(x+)-f(x-)}{3\sqrt{2\pi n x}} \right| \le \frac{x+4}{nx} \sum_{k=1}^{n} \Omega_x(g_x, x/\sqrt{k}) + O(n^{-1})$$

$$\le \frac{x+4}{nx} \sum_{k=1}^{n} \bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}} (g_x) + O(n^{-1}), \quad (6)$$

Corollary 2. *Under the conditions of Theorem 1, if $\Omega_x(g_x,\lambda) = o(\lambda)$, then*

$$G_n^*(f,x) = \frac{f(x+)+f(x-)}{2} - \frac{f(x+)-f(x-)}{3\sqrt{2\pi n x}} + o(n^{-1/2}). \quad (7)$$

Theorem 2. *Let f be a function in Φ_{DB} and let $f(t) \le Me^{\beta t}$ for some $M > 0$ and $\beta \ge 0$ as $t \to \infty$. If $h(x+)$ and $h(x-)$ exist at a fixed point $x \in (0,\infty)$, then for $n > 3\beta$ we have*

$$\left| G_n^*(f,x) - f(x) - \rho\sqrt{\frac{x}{2\pi n}} \right| \le \frac{4x+2}{n} \sum_{k=1}^{[\sqrt{n}]} \Omega_x(\psi_x, x/k) + \frac{|\rho| x^{5/2} + 7M(x+1)^{3/2} e^{3\beta x}}{x^3 n^{3/2}}, \quad (8)$$

where $\rho = h(x+) - h(x-)$, *and*

$$\psi_x(t) = \begin{cases} h(t) - h(x+), & x < t < \infty; \\ 0, & t = x; \\ h(t) - h(x-), & 0 \le t < x. \end{cases} \tag{9}$$

Remark . If f is a function with derivative of bounded variation, then $f \in \Phi_{DB}$. Thus the approximation of functions with derivatives of bounded variation is a special case of Theorem 2.

2 Auxiliary Results

To prove Theorem 1 and Theorem 2 we need some auxiliary results.

Lemma 2. *For a fixed* $x \in (0, \infty)$, *we have*

$$G_n^*(1, x) = 1,$$

$$G_n^*(u^k, x) = \frac{(nx + k - 1)(nx + k - 2)...(nx + 1)nx}{n^k}, \quad k > 1 \tag{10}$$

Proof. Direct computations give $G_n^*(1, x) = 1$, and

$$\begin{aligned} G_n^*(u^k, x) &= \frac{n^{nx}}{\Gamma(nx)} \int_0^\infty u^{nx+k-1} e^{-nu} du \\ &= \frac{n^{nx}}{n^{nx+k}\Gamma(nx)} \int_0^\infty t^{nx+k-1} e^{-t} dt \\ &= \frac{\Gamma(nx + k)}{n^k \Gamma(nx)} \\ &= \frac{(nx + k - 1)(nx + k - 2)...(nx + 1)nx}{n^k}. \end{aligned}$$

Lemma 3. *For* $x \in (0, \infty)$ *there hold*

$$G_n^*((u - x)^2, x) = \frac{x}{n}; \tag{11}$$

$$G_n^*((u - x)^4, x) \le \frac{3(x + 1)^2}{n^2}; \tag{12}$$

$$G_n^*((u - x)^6, x) \le \frac{49(x + 1)^3}{n^3}; \tag{13}$$

$$G_n^*(e^{2\beta u}, x) \le e^{6\beta x}, \qquad \text{for} \quad n > 3\beta. \tag{14}$$

Proof. By Lemma 2 and direct computations we get

$$G_n^*((t - x)^2, x) = \frac{x}{n};$$

$$G_n^*((t - x)^4, x) = \frac{3x^2}{n^2} + \frac{6x}{n^3} \le \frac{3(x + 1)^2}{n^2};$$

$$G_n^*((t - x)^6, x) = \frac{15n^2x^3 + 130nx^2 + 120x}{n^5} \le \frac{49(x + 1)^3}{n^3};$$

In addition, if $n > 3\beta$, putting $u = \dfrac{t}{n-2\beta}$, we have

$$\begin{aligned}
G_n^*(e^{2\beta u}, x) &= \frac{n^{nx}}{\Gamma(nx)} \int_0^\infty e^{2\beta u} u^{nx-1} e^{-nu} du \\
&= \frac{n^{nx}}{(n-2\beta)^{nx}\Gamma(nx)} \int_0^\infty t^{nx-1} e^{-t} dt \\
&= \left(\frac{n}{n-2\beta}\right)^{nx} \\
&\le e^{6\beta x}.
\end{aligned}$$

Lemma 4. [3, Chapter 2] *Let* $\{\xi_k\}_{k=1}^\infty$ *be a sequence of independent and identically distributed random variables with the expectation* $E\xi$*, the variance* $E(\xi - E\xi)^2 = \sigma^2 > 0$, $E(\xi - E\xi)^4 < \infty$, *and let* F_n *stand for the distribution function of* $\sum_{k=1}^{n} (\xi_k - E\xi)/\sigma\sqrt{n}$*. If* F_n *is not a lattice distribution, then the following equation holds for all* $t \in (-\infty, +\infty)$

$$F_n(t) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2} du = \frac{E(\xi - \xi)^3}{6\sigma^3\sqrt{n}} (1 - t^2) \frac{1}{\sqrt{2\pi}} e^{-t^2/2} + O(n^{-1}). \tag{15}$$

Lemma 5. Let

$$K_n(x, u) = \frac{n^{nx}}{\Gamma(nx)} \int_0^u v^{nx-1} e^{-nv} dv.$$

If $0 \le v \le u < x$, then

$$K_n(x, u) \le \frac{x}{n(x-u)^2}. \tag{16}$$

Proof. Let ξ be a Gamma random variable with probability density function defined as in (1). Let $\{\xi_i\}_{i=1}^\infty$ be a sequence of independent random variables with the same Gamma distribution as ξ. Then by direct computation we have

$$E(\xi) = x, \quad E(\xi - E\xi)^2 = \sigma^2 = x, \tag{17}$$

$$E(\xi - E\xi)^3 = 2x, \quad E(\xi - E\xi)^4 = 3x^2 + 6x < \infty. \tag{18}$$

Let $\eta_n = \sum_{i=1}^{n} \xi_i$. By the addition operation of random variables, we obtain

$$E(\eta_n) = nx, \quad E(\eta_n - E\eta_n)^2 = nx$$

through simple computation. Thus by Chebyshev inequality it follows that

$$K_n(x, u) = P(|\eta_n - nx| \ge nx - nu) \le \frac{x}{n(x-u)^2}.$$

3 Proof of Theorem 1

Let f satisfy the conditions of Theorem 1, then f can be expressed as

$$f(u)=\frac{f(x+)+f(x-)}{2}+g_x(u)+\frac{f(x+)-f(x-)}{2}\mathrm{sgn}(u-x)+\delta_x(u)\left[f(x)-\frac{f(x+)+f(x-)}{2}\right], \tag{19}$$

where $g_x(u)$ is defined in (5), $sgn(u)$ is sign function and $\delta_x(u)=\begin{cases}1, & u=x\\ 0, & u\neq x\end{cases}$.

Obviously,

$$G_n^*(\delta_x,x)=0. \tag{20}$$

Let $\{\xi_i\}_{i=1}^{\infty}$ be a sequence of independent random variables with the same Gamma distribution and their probability density functions are

$$p_{\xi_i}(u)=\begin{cases}\frac{u^{x-1}}{\Gamma(x)}\exp(-u), & \text{if } u\geq 0\\ 0, & \text{if } u<0.\end{cases}$$

where $x\in(0,\infty)$ is a parameter. Let $\eta_n=\sum\limits_{i=1}^{n}\xi_i$ and F_n^* stand for the distribution function of $\sum\limits_{i=1}^{n}(\xi_i-E\xi_i)/\sigma\sqrt{n}$. Then by Lemma 1, the probability distribution of the random variable η_n is

$$P(\eta_n\leq y)=\frac{1}{\Gamma(nx)}\int_0^y u^{nx-1}e^{-u}du.$$

Thus

$$\begin{aligned}G_n^*(\mathrm{sgn}(u-x),x)&=\frac{n^{nx}}{\Gamma(nx)}\int_x^{\infty}u^{nx-1}e^{-nu}du-\frac{n^{nx}}{\Gamma(nx)}\int_0^x u^{nx-1}e^{-nu}du\\ &=1-2P(\eta_n\leq nx)=1-2F_n^*(0).\end{aligned} \tag{21}$$

By Lemma 4, (17), (18) and straightforward computation, we have

$$1-2F_n^*(0)=-\frac{2E(\xi_1-a_1)^3}{6\sigma^3\sqrt{n}}\frac{1}{\sqrt{2\pi}}+O(n^{-1})=\frac{-2}{3\sqrt{2\pi nx}}+O(n^{-1}). \tag{22}$$

It follows from (19)–(22) that

$$\left|G_n^*(f,x)-\frac{f(x+)+f(x-)}{2}+\frac{f(x+)-f(x-)}{3\sqrt{2\pi nx}}\right|\leq|G_n^*(g_x,x)|+O(n^{-1}). \tag{23}$$

We need to estimate $|G_n^*(g_x,x)|$. Let

$$K_n(x,u)=\frac{n^{nx}}{\Gamma(nx)}\int_0^u v^{nx-1}e^{-nv}dv.$$

Then by Lebesgue-Stieltjes integral representation, we have

$$G_n^*(g_x,x)=\int_0^{\infty}g_x(u)d_uK_n(x,u) \tag{24}$$

Decompose the integral of (24) into four parts, as

$$\int_0^\infty g_x(u)d_uK_n(x,u) = \triangle_{1,n}(g_x) + \triangle_{2,n}(g_x) + \triangle_{3,n}(g_x) + \triangle_{4,n}(g_x),$$

where

$$\triangle_{1,n}(g_x) = \int_0^{x-x/\sqrt{n}} g_x(u)d_uK_n(x,u), \qquad \triangle_{2,n}(g_x) = \int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} g_x(u)d_uK_n(x,u),$$

$$\triangle_{3,n}(g_x) = \int_{x+x/\sqrt{n}}^{2x} g_x(u)d_uK_n(x,u), \qquad \triangle_{4,n}(g_x) = \int_{2x}^{\infty} g_x(u)d_uK_n(x,u).$$

We will evaluate $\triangle_{1,n}(g_x)$, $\triangle_{2,n}(g_x)$, $\triangle_{3,n}(g_x)$ and $\triangle_{4,n}(g_x)$ separately. First, for $\triangle_{2,n}(g_x)$, note that $g_x(x) = 0$. Hence

$$|\triangle_{2,n}(g_x)| \le \int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} |g_x(u) - g_x(x)|d_uK_n(x,u) \le \Omega_x(g_x, x/\sqrt{n}). \tag{25}$$

To estimate $|\triangle_{1,n}(g_x)|$, note that $\Omega_x(g_x,\lambda)$ is non-decreasing with respect to λ, thus it follows that

$$|\triangle_{1,n}(g_x)| = \left|\int_0^{x-x/\sqrt{n}} g_x(u)d_uK_n(x,u)\right| \le \int_0^{x-x/\sqrt{n}} \Omega_x(g_x, x-u)d_uK_n(x,u).$$

Using integration by parts with $y = x - x/\sqrt{n}$, we have

$$\int_0^y \Omega_x(g_x, x-u)d_uK_n(x,u) \le \Omega_x(g_x, x-y)K_n(x,y) + \int_0^y K_n(x,u)d_u(-\Omega_x(g_x, x-u)). \tag{26}$$

From (26) and Lemma 5 we get

$$|\triangle_{1,n}(g_x)| \le \Omega_x(g_x, x-y)\frac{x}{n(x-y)^2} + \int_0^y \frac{x}{n(x-u)^2}d_u(-\Omega_x(g_x, x-u)). \tag{27}$$

Using integration by parts once again, from (26), (27) it follows that

$$|\triangle_{1,n}(g_x)| \le \frac{1}{nx}\Omega_x(g_x, x) + \frac{2x}{n}\int_0^{x-x/\sqrt{n}} \frac{\Omega_x(g_x, x-u)}{(x-u)^3}dt.$$

Putting $u = x - x/\sqrt{t}$ for the last integral we get

$$\int_0^{x-x/\sqrt{n}} \frac{\Omega_x(g_x, x-u)}{(x-u)^3}du = \frac{1}{nx}\int_1^n \Omega_x(g_x, x/\sqrt{t})dt.$$

Consequently

$$|\triangle_{1,n}(g_x)| \le \frac{1}{nx}\left(\Omega_x(g_x, x) + \int_1^n \Omega_x(g_x, x/\sqrt{t})dt\right). \tag{28}$$

Using a similar method to estimate $|\triangle_{3,n}(g_x)|$, we get

$$|\triangle_{3,n}(g_x)| \le \frac{1}{nx}\left(\Omega_x(g_x, x) + \int_1^n \Omega_x(g_x, x/\sqrt{t})dt\right). \tag{29}$$

Finally, by assumption we know that $g_x(t) \le M(e^{\beta t})$ as $t \to \infty$. Using Hölder inequality and Lemma 3, we have, for $n \ge 3\beta$,

$$
\begin{array}{rcl}
|\triangle_{4,n}(g_x)| & \le & M \int_{2x}^{\infty} e^{\beta u} d_u K_n(x,u) \\
& \le & \frac{M}{x^2} \int_0^{\infty} (u-x)^2 e^{\beta u} d_u K_n(x,u) \\
& \le & \frac{M}{x^2} \left(\int_0^{\infty} (u-x)^4 d_u K_n(x,u) \right)^{1/2} \left(\int_0^{\infty} e^{2\beta u} d_u K_n(x,u) \right)^{1/2} \\
& \le & \frac{2M(x+1)e^{3\beta x}}{n}.
\end{array} \tag{30}
$$

Equations (25) and (28)–(30) lead to

$$
\begin{array}{rcl}
|G_n^*(g_x, x)| & \le & |\triangle_{1,n}(g_x)| + |\triangle_{2,n}(g_x)| + |\triangle_{3,n}(g_x)| + |\triangle_{4,n}(g_x)| \\
& \le & \Omega_x(g_x, x/\sqrt{n}) + \frac{2}{nx} \left(\Omega_x(g_x, x) + \int_1^n \Omega_x(g_x, x/\sqrt{u}) du \right) + \frac{2M(x+1)e^{3\beta x}}{n} \\
& \le & \frac{x+4}{n} \sum_{k=1}^{n} \Omega_x(g_x, x/\sqrt{k}) + \frac{2M(x+1)e^{3\beta x}}{n}.
\end{array} \tag{31}
$$

Theorem 1 now follows from (23) and (31).

4 Proof of Theorem 2

By direct computation we find that

$$
G_n^*(f,x) - f(x) = \frac{h(x+) - h(x-)}{2} G_n^*(|u-x|, x) - A_{n,x}(\psi_x) + B_{n,x}(\psi_x) + C_{n,x}(\psi_x), \tag{32}
$$

where

$$
A_{n,x}(\psi_x) = \int_0^x \left(\int_t^x \psi_x(u) du \right) d_t K_n(x,t),
$$

$$
B_{n,x}(\psi_x) = \int_x^{2x} \left(\int_x^t \psi_x(u) du \right) d_t K_n(x,t),
$$

$$
C_{n,x}(\psi_x) = \int_{2x}^{+\infty} \left(\int_x^t \psi_x(u) du \right) d_t K_n(x,t),
$$

Integration by parts derives

$$
\begin{array}{rcl}
A_{n,x}(\psi_x) & = & \int_0^x \left(\int_t^x \psi_x(u) du \right) d_t K_n(x,t) \\
& = & \int_t^x \psi_x(u) du K_n(x,t) \Big|_0^x + \int_0^x K_n(x,t) \psi_x(t) dt \\
& = & \left(\int_0^{x - x/\sqrt{n}} + \int_{x-x/\sqrt{n}}^x \right) K_n(x,v) \psi_x(v) dv
\end{array}
$$

Note that $K_n(x,v) \le 1$ and $\psi_x(x) = 0$, it follows that

$$
\left| \int_{x-x/\sqrt{n}}^x K_n(x, nv) \psi_x(v) dv \right| \le \frac{x}{\sqrt{n}} \Omega_x \left(\psi_x, \frac{x}{\sqrt{n}} \right) \le \frac{2x}{n} \sum_{k=1}^{[\sqrt{n}]} \Omega_x(\psi_x, x/k).
$$

On the other hand, by Lemma 5 and using change of variable $v = x - x/u$, we have

$$\begin{aligned}
\left|\int_0^{x-x/\sqrt{n}} K_n(x,v)\psi_x(v)dv\right| &\le \frac{x}{n}\int_0^{x-x/\sqrt{n}} \frac{\Omega_x(\psi_x, x-v)}{(x-v)^2}dv \\
&= \frac{1}{n}\int_1^{\sqrt{n}} \Omega_x(\psi_x, x/u)du \le \frac{1}{n}\sum_{k=1}^{[\sqrt{n}]}\Omega_x(\psi_x, x/k).
\end{aligned}$$

Thus, it follows that

$$|A_{n,x}(\psi_x)| \le \frac{2x+1}{n}\sum_{k=1}^{[\sqrt{n}]}\Omega_x(\psi_x, x/k). \tag{33}$$

A similar evalutation gives

$$|B_{n,x}(\psi_x)| \le \frac{2x+1}{n}\sum_{k=1}^{[\sqrt{n}]}\Omega_x(\psi_x, x/k). \tag{34}$$

Next we estimate $C_{n,x}(\psi_x)$, by the assumption that $f(t) \le Me^{\beta t}$ $(M > 0,\ \beta \ge 0)$, and using Lemma 3 we have

$$\begin{aligned}
|C_{n,x}(\psi_x)| &\le M\int_{2x}^{+\infty} e^{\beta t}d_tK_n(x,t) \\
&\le \frac{M}{x^3}\int_{2x}^{+\infty}(t-x)^3e^{\beta t}d_tK_n(x,t) \\
&\le \frac{M}{x^3}\left(\int_0^{+\infty}(t-x)^6d_tK_n(x,t)\right)^{1/2}\left(\int_0^{+\infty}e^{2\beta t}d_tK_n(x,t)\right)^{1/2} \\
&\le \frac{7M(x+1)^{3/2}e^{3\beta x}}{x^3n^{3/2}}.
\end{aligned} \tag{35}$$

Finally, we estimate the first order absolute moment of the Gamma approximation process: $G_n^*(|u-x|, x)$. By direct calculation, we have

$$\begin{aligned}
G_n^*(|u-x|,x) &= \frac{n^{nx}}{\Gamma(nx)}\int_0^{\infty}|u-x|u^{nx-1}e^{-nu}du \\
&= \frac{n^{nx}}{\Gamma(nx)}\left(\int_0^x (x-u)u^{nx-1}e^{-nu}du + \int_x^{\infty}(u-x)u^{nx-1}e^{-nu}du\right) \\
&= \frac{2n^{nx}}{\Gamma(nx)}\int_0^x (x-u)u^{nx-1}e^{-nu}du \\
&= \frac{2x}{\Gamma(nx)}\int_0^{nx} t^{nx-1}e^{-t}dt - \frac{2}{n\Gamma(nx)}\int_0^{nx} t^{nx}e^{-t}dt.
\end{aligned}$$

Note that

$$\int_0^{nx} t^{nx}e^{-t}dt = -(nx)^{nx}e^{-nx} + nx\int_0^{nx} t^{nx-1}e^{-t}dt,$$

thus

$$G_n^*(|u-x|,x) = \frac{2(nx)^{nx}}{ne^{nx}\Gamma(nx)}. \tag{36}$$

Now using Stirling's formula (cf. [4]):

$$\Gamma(z+1) = \sqrt{2\pi z}(z/e)^z e^{c_z}, \qquad (12z+1)^{-1} < c_z < (12z)^{-1},$$

from (36) we have

$$\sqrt{\frac{2x}{n\pi}} - G_n^*(|u-x|, x) = \sqrt{\frac{2x}{n\pi}}(1 - e^{-c_x}), \tag{37}$$

and a simple calculation derives

$$\frac{1}{18\sqrt{x}n^{3/2}} \le \sqrt{\frac{2x}{n\pi}}(1 - e^{-c_x}) \le \frac{1}{12\sqrt{x}n^{3/2}}. \tag{38}$$

Theorem 2 now follows from (32)–(38) combining with some simple calculations.

Acknowledgement

The first author was supported by National and Fujian provincial Science Foundation of China. The second author is supported by USA NSF under grants DMI-0422126 and DMS-0310645.

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