Subdivision Depth Computation for Extra-Ordinary Catmull-Clark Subdivision Surface Patches

Fuhua (Frank) Cheng
University of Kentucky
cheng@cs.uky.edu
http://www.cs.uky.edu/∼cheng/

Gang Chen
University of Kentucky
gchen5@cs.uky.edu
http://www.csr.uky.edu/∼gchen/

Jun-Hai Yong
Tsinghua University
yongjh@tsinghua.edu.cn
http://cgcad.thss.tsinghua.edu.cn/∼yongjh/

Abstract

A new subdivision depth computation technique for extra-ordinary Catmull-Clark subdivision surface (CCSS) patches is presented. For a given error tolerance ϵ and an extra-ordinary CCSS patch (a CCSS patch with an extraordinary vertex), the new technique determines, based on the second order forward differences of the patch's control points only, how many times the control mesh of the patch should be subdivided so that the distance between the resulting control mesh and the limit surface is smaller than ϵ. The new technique improves a previous technique by giving a subdivision depth based on the patch's curvature distribution only, instead of its dimension. Hence, with the new technique, no excessive subdivision is needed for extra-ordinary CCSS patches to meet the precision requirement and, consequently, one can make trimming, finite element mesh generation, boolean operations, and tessellation of CCSS's more efficient.

Keywords: subdivision surfaces, distance evaluation, subdivision depth computation

1 Introduction

Subdivision scheme provides a powerful method for building smooth and complex surfaces. Given a control mesh and a set of *mesh refining rules* (or, more intuitively, *corner cutting rules*), one gets a *limit surface* by recursively cutting off corners of the control mesh [2][5]. The limit surface is called a *subdivision surface* because the mesh refining process is a generalization of the uniform B-spline surface's *subdivision technique*. Subdivision surfaces can model/represent complex shape of arbitrary topology because there is no limit on the shape and topology of the control mesh of a subdivision surface. [4].

Research work for subdivision surfaces has been done in several important areas, such as surface parametrization [12][13][16][7], surface trimming [8], boolean operations [1], mesh editing [15], and *error estimate/control* [14][3]. For instance, given an error tolerance, [3] shows how many times the control mesh of a Catmull-Clark subdivision surface (CCSS) patch should be recursively subdivided so that the distance between the resulting control mesh and the *limit surface patch* would be less than the error tolerance. This error control technique, called *subdivision depth computation*, is required in all tessellation based applications of CCSS's. [3]'s subdivision depth computation technique for regular CCSS patches is optimum. However, for an extra-ordinary CCSS patch (a patch with an extra-ordinary vertex), since the subdivision depth computed by [3] depends on first order forward differences of the control points, its value could be bigger than what it actually should be and, consequently, generates excessive mesh elements for regions that are already flat enough.

In this paper we will present a new subdivision depth computation technique for extra-ordinary CCSS patches. The new technique is based on the second order forward differences of an extra-ordinary patch's control points. The computed subdivision depth reflects the patch's curvature distribution, not its dimension. Hence, with the new technique, no excessive subdivision is needed for regions that are already flat enough and, consequently, trimming, finite element mesh generation, boolean operations, and tessellation of CCSS's can be made more efficient.

The remaining part of the paper is arranged as follows. A brief review of the Catmull-Clark subdivision scheme and the subdivision depth computation technique for regular CCSS patches (to be used in the new technique) is given in Section 2. A new distance evaluation technique and a new subdivision depth computation technique for an extra-ordinary CCSS patch are given in Section 3. Examples of subdivision depth computation for extra-ordinary CCSS patches using the new techniques are presented in Section 4. Concluding remarks are given in Section 5.

2 Problem Formulation and Background

Given the control mesh of an extra-ordinary Catmull-Clark subdivision surface patch and an error tolerance ϵ, the goal here is to compute an integer d so that if the control mesh is iteratively refined (subdivided) d times, then the distance between the resulting mesh and the surface patch is smaller than ϵ. d is called the *subdivision depth* of the surface patch with respect to ϵ. Before we show the computation technique, we need to define related terms. We also need to review a distance evaluation technique and a subdivision depth computation technique for regular Catmull-Clark subdivision surface patches [3]. These techniques are needed in the new technique for extra-ordinary Catmull-Clark subdivision surface patches.

2.1 Catmull-Clark Subdivision Surfaces

Given a control mesh, the *Catmull-Clark subdivision scheme* iteratively refines (subdivides) the control mesh to form new control meshes [2]. The limit surface of the refined control meshes is called a *Catmull-Clark subdivision surface* (CCSS). The refining process consists of defining new vertices and connecting the new vertices to form new edges and faces of a new control mesh. The new vertices belong to three groups: *face points*, *edge points* and *vertex points*. For each old interior mesh face, a new *face point* is defined as the average of the vertices defining the old face. For each old interior mesh edge, a new *edge point* is defined as the average of the midpoint of the old edge and the average of the two adjacent new face points. For each old interior vertex $\mathbf{P}$, a new *vertex point* $\mathbf{Q}$ is defined as follows:

$$\mathbf{Q} = \frac{\mathbf{F}}{n} + \frac{2\mathbf{E}}{n} + \frac{(n-3)\mathbf{P}}{n}$$

where n is the number of adjacent edges of $\mathbf{P}$, $\mathbf{F}$ is the average of the new adjacent face points and $\mathbf{E}$ is the average of the midpoints of adjacent edges of $\mathbf{P}$. The new edges are formed through two connecting processes after all the new vertices are constructed:

- connecting each new face point to adjacent new edge points
- connecting each new vertex point to adjacent new edge points

New faces are then defined as those enclosed by new edges. The control mesh of a CCSS patch and the new control mesh after a refining (subdivision) process are shown in Figure 1(a) and (b), respectively. This is a conceptual drawing, the location shown for a new vertex might not be its exact physical location.

The limit surface of the iteratively refined control meshes is called a *subdivision surface* because the mesh refining (subdivision) process is a generalization of the uniform bicubic B-spline surface subdivision technique. Therefore, CCSS's include uniform B-spline surfaces and piecewise Bézier surfaces as special cases. Actually CCSS's include non-uniform B-spline surfaces and NURBS surfaces as special cases as well [11]. The Catmull-Clark mesh refining process will also be called the *Catmull-Clark subdivision*, or simply the *subdivision step* subsequently. The given control mesh will be referred to as $\mathbf{M}_0$ and the limit surface will be referred to as $\bar{\mathbf{S}}$. For each positive integer k, $\mathbf{M}_k$ refers to the control mesh obtained after applying the Catmull-Clark subdivision k times to $\mathbf{M}_0$.

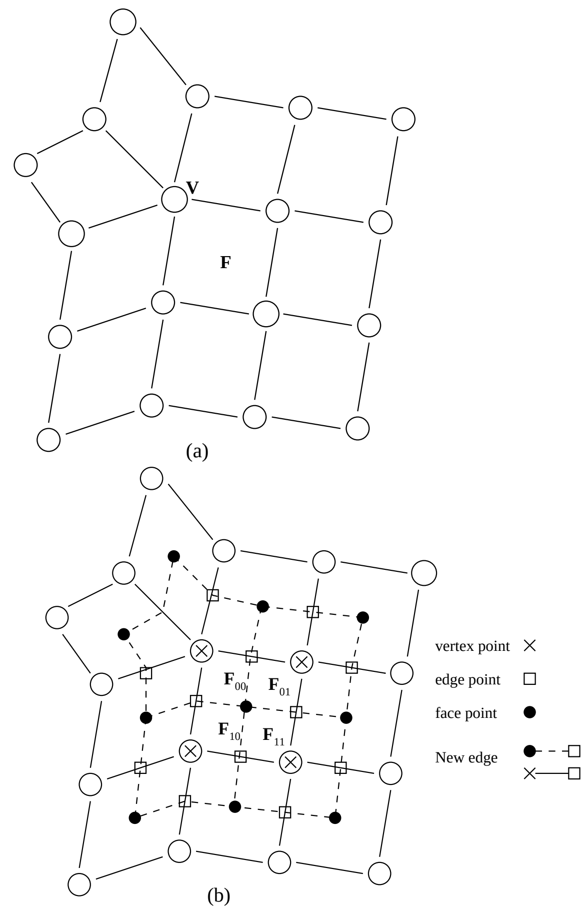

Figure 1: (a) Control mesh of an extra-ordinary patch; (b) new vertices and edges generated after a Catmull-Clark subdivision.

2.2 Regular vs. Extra-ordinary

The power of CCSS's comes from the way mesh vertices are connected. If the number of edges connected to a mesh vertex is called its *valence*, then the valence of an interior mesh vertex can be anything ≥ 3, instead of just four. Those mesh vertices whose valences are different from four are called *extra-ordinary vertices* to distinguish them from the *standard* or *regular mesh vertices*. Vertex $\mathbf{V}$ in Figure 1(a) is an extra-ordinary vertex of valence five. An interior mesh face is called an *extra-ordinary mesh face* if it has an extra-ordinary vertex. Otherwise, a *standard* or *regular mesh face*. Mesh face $\mathbf{F}$ in Figure 1(a) is an extra-ordinary mesh face. Note that after one iteration of the subdivision step, mesh faces of a CCSS are always quadrilaterals and the number of extra-ordinary vertices remains the same. After at most two iterations of the subdivision step, each mesh face has at most one extra-ordinary vertex. Therefore, without loss of generality, we shall assume all the mesh faces in $\mathbf{M}_0$ are quadrilaterals and each mesh face of $\mathbf{M}_0$ has at most one extra-ordinary vertex.

For each interior face $\mathbf{F}$ of $\mathbf{M}_k$, $k \geq 0$, there is a corresponding patch $\mathbf{S}$ in the limit surface $\bar{\mathbf{S}}$. $\mathbf{F}$ and $\mathbf{S}$ can be parametrized on the same parameter space $\Omega = [0,\ 1] \times [0,\ 1]$ [12]. $\mathbf{F}$ is a bilinear *rule surface*. $\mathbf{S}$ is a uniform bicubic B-spline surface patch if $\mathbf{F}$ is a regular face. However, if $\mathbf{F}$ is an extra-ordinary face then $\mathbf{S}$, defined by $2n+8$ control points where n is the valence of $\mathbf{F}$'s extra-ordinary vertex, can not be parametrized as a uniform B-spline patch. In such a case, $\mathbf{S}$ is called an *extra-ordinary patch*. Otherwise, a *regular patch* or *standard patch*. The control mesh shown in Figure 1(a) is the control mesh of an extra-ordinary patch whose extra-ordinary vertex is of valence five.

2.3 Distance and Subdivision Depth

For a given interior mesh face $\mathbf{F}$, let $\mathbf{S}$ be the corresponding patch in the limit surface $\bar{\mathbf{S}}$. The control mesh of $\mathbf{S}$ contains $\mathbf{F}$ as the center face. If we perform a subdivision step on the control mesh, we get four new mesh faces in the place of $\mathbf{F}$. This is the case no matter $\mathbf{F}$ is a regular face or an extra-ordinary face (see Figure 1(b) for the four new faces $\mathbf{F}_{00}$, $\mathbf{F}_{10}$, $\mathbf{F}_{01}$ and $\mathbf{F}_{11}$ obtained in the place of the extra-ordinary face $\mathbf{F}$ shown in Figure 1(a)). Since each of these new faces corresponds to a quarter subpatch of $\mathbf{S}$, we shall call these new faces *subfaces* of $\mathbf{F}$ even though they are not pyhsically subsets of $\mathbf{F}$. Therefore, each subdivision step generates four new subfaces for the center face $\mathbf{F}$ of the control mesh. Because the correspondence between $\mathbf{F}$ and $\mathbf{S}$ is one-to-one, sometime, instead of saying performing a subdivision step on $\mathbf{S}$, we shall simply say performing a subdivision step on $\mathbf{F}$.

The *distance* between an interior mesh face $\mathbf{F}$ and the corresponding patch $\mathbf{S}$ is defined as the maximum of $\|\mathbf{F}(u,v) - \mathbf{S}(u,v)\|$:

$$D_{\mathbf{F}} = max_{(u,v)\in\Omega} \|\mathbf{F}(u,v) - \mathbf{S}(u,v)\| \qquad (1)$$

where Ω is the unit square parameter space of $\mathbf{F}$ and $\mathbf{S}$. $D_{\mathbf{F}}$ is also called the distance between $\mathbf{S}$ and its control mesh. For a given $\epsilon > 0$, the *subdivision depth* of $\mathbf{F}$ with respect to ϵ is a positive integer d such that if $\mathbf{F}$ is recursively subdivided d times, the distance between each of the resulting subfaces and the corresponding subpatch is smaller than zero.

2.4 Subdivision Depth Computation for Regular Patches

A regular patch is a standard uniform bicubic B-spline surface patch. Therefore, the computation process for a regular patch is the same as the computation process for a standard uniform B-spline surface patch. We review the evaluation of the distance between a B-spline patch and its control mesh first.

2.4.1 Distance Evaluation

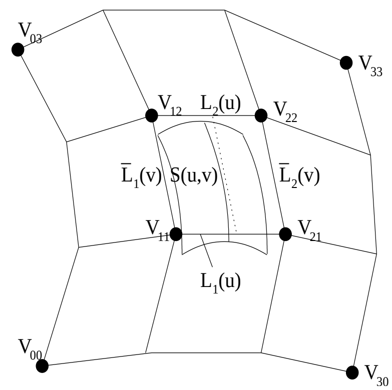

Figure 2: Definition of $\mathbf{L}(u,v) = (1-v)\mathbf{L}_1(u) + v\mathbf{L}_2(u) = (1-u)\bar{\mathbf{L}}_1(v) + u\bar{\mathbf{L}}_2(v)$.

Let $\mathbf{S}(u,v)$ be a uniform bicubic B-spline surface patch defined on the unit square $\Omega = [0,1] \times [0,1]$ with control points $\mathbf{V}_{i,j}$, $0 \leq i,j \leq 3$,

$$\mathbf{S}(u,v) = \sum_{i=0}^{3} N_{i,3}(u) \sum_{j=0}^{3} N_{j,3}(v)\mathbf{V}_{i,j}, \quad 0 \leq u,v \leq 1 \quad (2)$$

where $N_{k,3}(t)$ are standard B-spline basis functions of degree three, and let $\mathbf{L}(u,v)$ be the bilinear parametrization of the center mesh face $\{\mathbf{V}_{1,1}, \mathbf{V}_{2,1}, \mathbf{V}_{2,2}, \mathbf{V}_{1,2}\}$ (see Figure 2):

$$\begin{aligned}\mathbf{L}(u,v) \quad = \quad & (1-v)[(1-u)\mathbf{V}_{1,1} + u\mathbf{V}_{2,1}] \\ & +v[(1-u)\mathbf{V}_{1,2} + u\mathbf{V}_{2,2}], \quad 0 \leq u,v \leq 1.\end{aligned}$$

The distance between $\mathbf{S}(u,v)$ and $\mathbf{L}(u,v)$, i.e., the maximum of $\|\mathbf{L}(u,v) - \mathbf{S}(u,v)\|$, satisfies the inequality of the following lemma [3].

Lemma 1: The distance between $\mathbf{L}(u,v)$ and $\mathbf{S}(u,v)$ satisfies the following inequality

$$\max_{0\leq u,v\leq 1} \|\mathbf{L}(u,v) - \mathbf{S}(u,v)\| \leq \frac{1}{3}M$$

where M is the second order norm of $\mathbf{S}(u,v)$ defined as follows

$$\begin{aligned} M \quad = \quad & \max_{i,j}\{\ \|2\mathbf{V}_{i,j} - \mathbf{V}_{i-1,j} - \mathbf{V}_{i+1,j}\|\ , \\ & \|2\mathbf{V}_{i,j} - \mathbf{V}_{i,j-1} - \mathbf{V}_{i,j+1}\|\ \} \end{aligned} \qquad (3)$$

2.4.2 Recurrence Formula for Second Order Norm

Let $\mathbf{V}_{i,j}$, $0 \le i,j \le 3$, be the control points of a uniform bicubic B-spline surface patch $\mathbf{S}(u,v)$. We use $\mathbf{V}^k_{i,j}$ to represent the new control points of the surface patch after k levels of recursive subdivision. The indexing of the new control points follows the convention that $\mathbf{V}^k_{0,0}$ is always the *face point* of the mesh face $\{\mathbf{V}^{k-1}_{0,0}, \mathbf{V}^{k-1}_{1,0}, \mathbf{V}^{k-1}_{1,1}, \mathbf{V}^{k-1}_{0,1}\}$. The new control points $\mathbf{V}^k_{ij}$ are called the *level-k control points* of $\mathbf{S}(u,v)$ and the new control mesh will be called the *level-k control mesh* of $\mathbf{S}(u,v)$.

If we divide the parameter space of the surface patch, Ω, into 4^k regions as follows:

$$\Omega^k_{mn} = [\frac{m}{2^k}, \frac{m+1}{2^k}] \times [\frac{n}{2^k}, \frac{n+1}{2^k}], \qquad 0 \le m,n \le 2^k - 1$$

and denote the corresponding subpatches $\mathbf{S}^k_{mn}(u,v)$, then each $\mathbf{S}^k_{mn}(u,v)$ is a uniform bicubic B-spline surface patch defined by the level-k control point set $\{\mathbf{V}^k_{pq} \mid m \le p \le m+3, n \le q \le n+3\}$. $\mathbf{S}^k_{mn}(u,v)$ is called a *level-k subpatch* of $\mathbf{S}(u,v)$. Let $\mathbf{L}^k_{mn}(u,v)$ be the bilinear parametrization of the center face of $\mathbf{S}^k_{mn}$'s control mesh, $\{\mathbf{V}^k_{pq} \mid p = m+1, m+2; q = n+1, n+2\}$. We say the *distance between* $\mathbf{S}(u,v)$ *and the level-k control mesh is smaller than* ϵ if the distance between each level-k subpatch $\mathbf{S}^k_{mn}(u,v)$ and the corresponding level-k bilinear plane $\mathbf{L}^k_{mn}(u,v)$, $0 \le m,n \le 2^k - 1$, is smaller than ϵ. A technique to compute a subdivision depth k for a given ϵ so that the distance between $\mathbf{S}(u,v)$ and the level-k control mesh is smaller than ϵ is presented in [3]. The following lemma is needed in the derivation of the computation process. If we use M^k_{mn} to represent the second order norm of $\mathbf{S}^k_{mn}(u,v)$, i.e., the maximum norm of the second order forward differences of the control points of $\mathbf{S}^k_{mn}(u,v)$, then the lemma shows the second order norm of $\mathbf{S}^k_{mn}(u,v)$ converges at a rate of 1/4 of the level-$(k-1)$ second order norm [3].

Lemma 2 If M^k_{mn} is the second order norm of $\mathbf{S}^k_{mn}(u,v)$ then we have

$$M^k_{mn} \le \left(\frac{1}{4}\right)^k M \tag{4}$$

where M is the second order norm of $\mathbf{S}(u,v)$ defined in (3).

2.4.3 Subdivision Depth Computation

With Lemmas 1 and 2, it is easy to see that, for any $0 \le m,n \le 2^{k-1}$, we have

$$\begin{aligned} &\max_{0\le u,v\le 1} \|\mathbf{L}^k_{mn}(u,v) - \mathbf{S}^k_{mn}(u,v)\| \\ &\le \tfrac{1}{3} M^k_{mn} \le \tfrac{1}{3}\left(\tfrac{1}{4}\right)^k M \end{aligned} \tag{5}$$

where M^k_{mn} and M are the second order norms of $\mathbf{S}^k_{mn}(u,v)$ and $\mathbf{S}(u,v)$, respectively. Hence, if k is large enough to make the right side of the above inequality smaller than ϵ, we have

$$\max_{0\le u,v\le 1} \|\mathbf{L}^k_{mn}(u,v) - \mathbf{S}^k_{mn}(u,v)\| \le \epsilon$$

for every $0 \le m,n \le 2^{k-1}$. This leads to the following subdivision depth computation process for a regular CCSS patch [3].

Theorem 3 Let $\mathbf{V}_{ij}$, $0 \le i,j \le 3$, be the control points of a uniform bicubic B-spline surface patch $\mathbf{S}(u,v)$. For any given $\epsilon > 0$, if

$$k \ge \lceil \log_4(\frac{M}{3\epsilon}) \rceil$$

levels of recursive subdivision are performed on the control points of $\mathbf{S}(u,v)$ then the distance between $\mathbf{S}(u,v)$ and the level-k control mesh is smaller than ϵ where M is the second order norm of $\mathbf{S}(u,v)$ defined in (3).

3 Subdivision Depth Computation for Extra-Ordinary Patches

The main idea of the new technique is the same, i.e., developing a distance evaluation mechanism that has a recursive nature so that results from different subdivision levels can be related through a recurrence formula. The distance evaluation mechanism will utilize second order norm, instead of first order norm, as a measurement scheme because of its capability in measuring both length and height, but the pattern of second order forward differences used in the distance evaluation process will be different. In the following, we will define second order forward difference pattern to be used for an extra-ordinary patch and derive a recurrence formula for the corresponding second order norm, like the one used for regular patch in Section 2.

3.1 Second Order Norm and Recurrence Formula

Let $\mathbf{V}_i$, $i = 1, 2, ..., 2n+8$, be the control points of an extra-ordinary patch $\mathbf{S}(u,v) = \mathbf{S}^0_0(u,v)$, with $\mathbf{V}_1$ being an extra-ordinary vertex of valence n. The control points are ordered following J. Stam's fashion [12] (Figure 3(a)). For convenience of subsequent reference, we shall call the control mesh of $\mathbf{S}(u,v)$ $\Pi = \Pi^0_0$. By performing a subdividion step on Π, one gets $2n+17$ new vertices $\mathbf{V}^1_i$, $i = 1, ..., 2n+17$ (see Figure 3(b)). These control points form four control point sets Π^1_0, Π^1_1, Π^1_2 and Π^1_3, representing control meshes of the subpatches $\mathbf{S}^1_0(u,v)$, $\mathbf{S}^1_1(u,v)$, $\mathbf{S}^1_2(u,v)$ and $\mathbf{S}^1_3(u,v)$, respectively (see Figure 3(b)) where

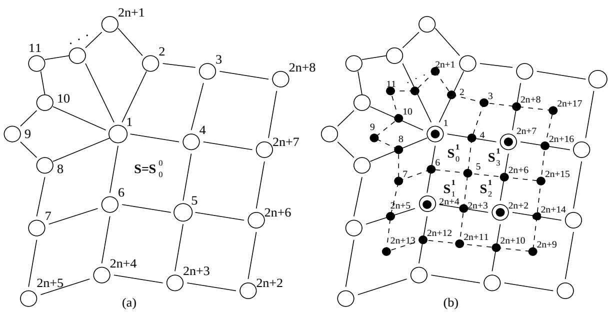

Figure 3: (a) Ordering of control points of an extra-ordinary patch. (b) Ordering of new control points (solid dots) after a Catmull-Clark subdivision.

$\Pi_0^1 = \{ \mathbf{V}_i^1 \mid 1 \le i \le 2n+8 \}$, and the other three control point sets Π_1^1, Π_2^1 and Π_3^1 are shown in Figure 4. $\mathbf{S}_0^1(u,v)$ is an extra-ordinary patch but $\mathbf{S}_1^1(u,v)$, $\mathbf{S}_2^1(u,v)$ and $\mathbf{S}_3^1(u,v)$ are regular patches. Therefore, second order norm similar to (3) can be defined for $\mathbf{S}_1^1$, $\mathbf{S}_2^1$ and $\mathbf{S}_3^1$.

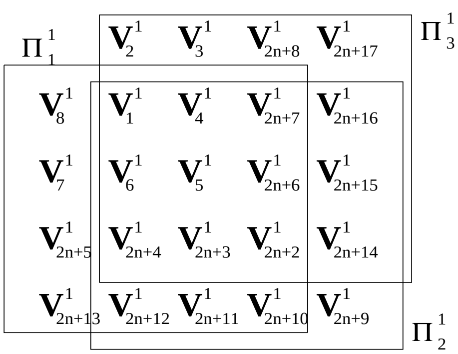

Figure 4: Control vertices of subpatches $\mathbf{S}_1^1$, $\mathbf{S}_2^1$ and $\mathbf{S}_3^1$.

To define a second order norm for $\mathbf{S}$, one needs to choose appropriate second order forward differences from Π. For the second order norm to be recursively defined, second order forward differences that are required in the child control meshes should also appear in the parent control mesh. For instance, $2\mathbf{V}_1 - \mathbf{V}_4 - \mathbf{V}_8$ and $2\mathbf{V}_1 - \mathbf{V}_2 - \mathbf{V}_6$ should be chosen for Π because these patterns are required for Π_1^1 and Π_3^1, respectively. On the other hand, for a recurrence formula to hold effectively, second order forward differences that are not required in the child control meshes should not be used in the parent control mesh either. For instance, one should not choose $2\mathbf{V}_1 - \mathbf{V}_2 - \mathbf{V}_8$ for Π because this pattern is not required in any of Π_1^1, Π_2^1 or Π_3^1. Therefore, for those cases that involves the extra-ordinary point $\mathbf{V}_1$ as the center point, one should only consider

$$2\mathbf{V}_1 - \mathbf{V}_{2i} - \mathbf{V}_{2(i\%n+2)}, \qquad 1 \le i \le n. \tag{6}$$

To ensure the boundary of the vicinity of the extra-ordinary point is covered (Figure 5(a)), one should consider

$$2\mathbf{V}_{2(i\%n+1)} - \mathbf{V}_{2i+1} - \mathbf{V}_{2(i\%n+1)+1}, \qquad 1 \le i \le n. \tag{7}$$

One also has to consider second order forward differences

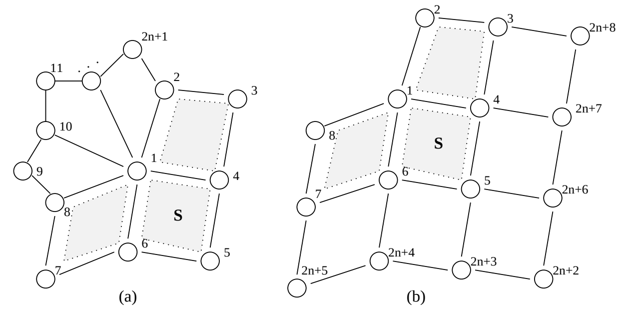

Figure 5: (a) Vicinity of the extra-ordinary point. (b) The extended remaining part.

that cover the extended remaining part (Figure 5(b)). There are ten of them (actually twelve, but two of them have been used in (7)). So, totally, $2n+10$ ($n+10$ when $n=3$) second order forward differences should be considered for Π and the *second order norm* of $\mathbf{S}$, $M = M_0$, is defined as the maximum norm of these $2n+10$ second order forward differences:

$$
\begin{aligned}
M = max\{ & \{ \|2\mathbf{V}_1 - \mathbf{V}_{2i} - \mathbf{V}_{2(i\%n+2)}\| \mid 1 \le i \le n \} \cup \\
& \{ \|2\mathbf{V}_{2(i\%n+1)} - \mathbf{V}_{2i+1} - \mathbf{V}_{2(i\%n+1)+1}\| \mid 1 \le i \le n \} \cup \\
& \{ \| 2V_3 - V_2 - V_{2n+8} \|, \| 2V_4 - V_1 - V_{2n+7} \|, \\
& \| 2V_5 - V_6 - V_{2n+6} \|, \| 2V_5 - V_4 - V_{2n+3} \|, \\
& \| 2V_6 - V_1 - V_{2n+4} \|, \| 2V_7 - V_8 - V_{2n+5} \|, \\
& \| 2V_{2n+7} - V_{2n+6} - V_{2n+8} \|, \\
& \| 2V_{2n+6} - V_{2n+2} - V_{2n+7} \|, \\
& \| 2V_{2n+3} - V_{2n+2} - V_{2n+4} \|, \\
& \| 2V_{2n+4} - V_{2n+3} - V_{2n+5} \| \} \}
\end{aligned}
\tag{8}
$$

Following this definition, one can define a similar second order norm, M_1, for the control mesh of $\mathbf{S}_0^1$. In general, if $\mathbf{S}_0^k$ is an extra-ordinary patch with control mesh Π_0^k after k Catmull-Clark subdivision steps, $k \ge 1$, then by performing a Catmull-Clark subdivision step on Π_0^k, we get four subpatches $\mathbf{S}_0^{k+1}$, $\mathbf{S}_1^{k+1}$, $\mathbf{S}_2^{k+1}$ and $\mathbf{S}_3^{k+1}$ with control points Π_0^{k+1}, Π_1^{k+1}, Π_2^{k+1} and Π_3^{k+1}, respectively. These control point sets are defined similar to Π_i^1, $0 \le i \le 3$ (simply replacing the sup-index '1' with '$k+1$' of points in Π_i^1). $\mathbf{S}_0^{k+1}$ is again an extra-ordinary patch and $\mathbf{S}_1^{k+1}$, $\mathbf{S}_2^{k+1}$ and $\mathbf{S}_3^{k+1}$ are regular patches. Therefore, we can define second order norm similar to (8) for both $\mathbf{S}_0^k$ and $\mathbf{S}_0^{k+1}$. The second order norms of $\mathbf{S}_0^k$ and $\mathbf{S}_0^{k+1}$

are denoted M_k and M_{k+1}, respectively. We have the following lemma for M_k and M_{k+1}. The proof of Lemma 4 is given in Appendix A.

Lemma 4: For any $k \geq 0$, if M_k represents the second order norm of the extra-ordinary sub-patch $\mathbf{S}_0^k$ after k Catmull-Clark subdivision steps, then M_k satisfies the following inequality

$$M_{k+1} \leq \begin{cases} \frac{2}{3}M_k, & n=3 \\ 0.72M_k, & n=5 \\ (\frac{3}{4}+\frac{8n-46}{4n^2})M_k, & n>5 \end{cases}$$

Actually, the lemma works in a more general sense, i.e., if M_k stands for the second order norm of the control mesh $\mathbf{M}_k$, instead of Π_0^k, the lemma still works. The second order norm of $\mathbf{M}_k$ is defined as follows: for regions not involving the extra-ordinary point, use standard second order forward differences; for the vicinity of the extra-ordinary point, use second order forward differences defined in (8). The proof is essentially the same.

3.2 Distance Evaluation

To compute the distance between the extra-ordinary patch $\mathbf{S}(u,v)$ and the center face of its control mesh, $\mathbf{L}(u,v)$, we need to parameterize the patch $\mathbf{S}(u,v)$ first.

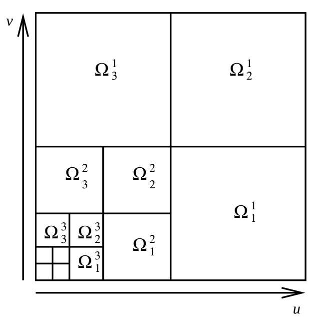

Figure 6: Ω-partition of the unit square.

Note that by iteratively performing Catmull-Clark subdivision on $\mathbf{S}(u,v)$, we get a sequence of regular patches $\{\ \mathbf{S}_b^m\ \}$, $m \geq 1$, $b = 1,2,3$, and a sequence of extra-ordinary patches $\{\ \mathbf{S}_0^m\ \}$, $m \geq 1$. The extra-ordinary patches converge to a limit point which is the value of $\mathbf{S}$ at $(0,0)$ [6]. This limit point and the regular patches $\{\ \mathbf{S}_b^m\ \}$, $m \geq 1$, $b = 1,2,3$, form a partition of $\mathbf{S}$. If we use Ω_b^m to represent the parameter space corresponding to $\mathbf{S}_b^m$ then $\{\ \Omega_b^m\ \}$, $m \geq 1$, $b = 1,2,3$, form a partition of the unit square $\Omega = [0,1]\times[0,1]$ (see Figure 6) with

$$\begin{aligned} \Omega_1^m &= [\tfrac{1}{2^m}, \tfrac{1}{2^{m-1}}] \times [0, \tfrac{1}{2^m}], \\ \Omega_2^m &= [\tfrac{1}{2^m}, \tfrac{1}{2^{m-1}}] \times [\tfrac{1}{2^m}, \tfrac{1}{2^{m-1}}], \\ \Omega_3^m &= [0, \tfrac{1}{2^m}] \times [\tfrac{1}{2^m}, \tfrac{1}{2^{m-1}}]. \end{aligned} \tag{9}$$

The parametrization of $\mathbf{S}(u,v)$ is done as follows. For any $(u,v) \in \Omega$ but $(u,v) \neq (0,0)$, first find the Ω_b^m that contains (u,v). m and b can be computed as follows.

$$m(u,v) = \min\{\lceil log_{\frac{1}{2}} u \rceil,\ \lceil log_{\frac{1}{2}} v \rceil\}$$

$$b(u,v) = \begin{cases} 1, & if\ 2^m u \geq 1\ and\ 2^m v \leq 1 \\ 2, & if\ 2^m u \geq 1\ and\ 2^m v \geq 1 \\ 3, & if\ 2^m u \leq 1\ and\ 2^m v \geq 1 \end{cases} \tag{10}$$

Then map this Ω_b^m to the unit square with the following mapping

$$(u,v) \rightarrow\ (u_m, v_m)$$

where

$$t_m = (2^m t)\%1 = \begin{cases} 2^m t, & if\ \ 2^m t \leq 1 \\ 2^m t - 1, & if\ \ 2^m t > 1 \end{cases}. \tag{11}$$

The value of $\mathbf{S}(u,v)$ is equal to the value of $\mathbf{S}_b^m$ at (u_m, v_m), i.e.,

$$\mathbf{S}(u,v) = \mathbf{S}_b^m(u_m, v_m).$$

Let $\mathbf{L}_b^m(u,v)$ be the bilinear parametrization of the center face of $\mathbf{S}_b^m$'s control mesh. Since $\mathbf{S}_b^m$ is a regular patch, following Lemma 1, we have

$$\|\mathbf{L}_b^m(u,v) - \mathbf{S}_b^m(u,v)\| \leq \frac{1}{3}M_b^m$$

where M_b^m is the second order norm of the contol mesh of $\mathbf{S}_b^m$. But the second order norm of $\mathbf{S}_b^m$ is smaller than the second order norm of $\mathbf{M}_m$, M_m. Hence, the above inequality can be written as

$$\|\mathbf{L}_b^m(u,v) - \mathbf{S}_b^m(u,v)\| \leq \frac{1}{3}M_m. \tag{12}$$

So the maximum distance between the original extra-ordinary mesh $\mathbf{L}(u,v)$ and the patch $\mathbf{S}(u,v)$ can be written as

$$\begin{aligned} &\|\ \mathbf{L}(u,v) - \mathbf{S}(u,v)\ \| \\ &= \|\ \mathbf{L}(u,v) - \mathbf{L}_b^m(u_m,v_m) + \mathbf{L}_b^m(u_m,v_m) - \mathbf{S}(u,v)\ \| \\ &\leq \|\ \mathbf{L}(u,v) - \mathbf{L}_b^m(u_m,v_m)\ \| \\ &\quad + \|\ \mathbf{L}_b^m(u_m,v_m) - \mathbf{S}_b^m(u_m,v_m)\ \| \end{aligned} \tag{13}$$

where $0 \leq u,v \leq 1$ and u_m and v_m are defined in (11). Since the second term on the right hand side can be estimated using (12), the only thing we need to work with is $\|\mathbf{L}(u,v) - \mathbf{L}_b^m(u_m,v_m)\ \|$.

It is easy to see that if $(u,v) \in \Omega_b^m$ then $(u,v) \in \Omega_0^k$ for any $0 \leq k < m$ where

$$\Omega_0^k = [0,\ \frac{1}{2^k}] \times [0,\ \frac{1}{2^k}]. \tag{14}$$

Ω_0^k corresponds to the subpatch $\mathbf{S}_0^k$. This means that $(2^k u, 2^k v)$ is within the parameter space of $\mathbf{S}_0^k$ for $0 \leq$

$k < m$, i.e., $(2^k u, 2^k v) = (u_k, v_k)$ where u_k and v_k are defined in (11). Consequently, we can consider $\mathbf{L}_0^k(u_k, v_k)$ for $0 \le k < m$ where $\mathbf{L}_0^k$ is the bilinear parametrization of the center face of the control mesh of $\mathbf{S}_0^k$ (with the understanding that $\mathbf{L}_0^0 = \mathbf{L}$). What we want to do here is to write the first term on the right hand side of (13) as

$$\begin{aligned}
&\mathbf{L}(u,v) - \mathbf{L}_b^m(u_m, v_m) = \\
&\mathbf{L}_0^0(u,v) - \mathbf{L}_0^1(u_1,v_1) + \mathbf{L}_0^1(u_1,v_1) - \mathbf{L}_0^2(u_2,v_2) \\
&\quad + \mathbf{L}_0^2(u_2,v_2) - \mathbf{L}_0^3(u_3,v_3) + \mathbf{L}_0^3(u_3,v_3) - \mathbf{L}_0^4(u_4,v_4) \\
&\quad + \cdots + \mathbf{L}_0^{m-1}(u_{m-1},v_{m-1}) - \mathbf{L}_b^m(u_m,v_m)
\end{aligned} \tag{15}$$

and get an estimate for its norm by estimating the norm of each consecutive pair on the right hand side. We have the following two lemmas. The proofs of these lemmas are shown in Appendice B and C, respectively.

Lemma 5: If $(u, v) \in \Omega_b^m$ where b and m are defined in (10) then for any $0 \le k < m - 1$ we have

$$\| \mathbf{L}_0^k(u_k, v_k) - \mathbf{L}_0^{k+1}(u_{k+1}, v_{k+1}) \| \le \frac{1}{\min\{\, n, 8 \,\}} M_k$$

where M_k is the second order norm of $\mathbf{M}_k$ and $\mathbf{L}_0^0 = \mathbf{L}$.

Lemma 6: If $(u, v) \in \Omega_b^m$ where b and m are defined in (10) then we have

$$\begin{aligned}
&\| \mathbf{L}_0^{m-1}(u_{m-1}, v_{m-1}) - \mathbf{L}_b^m(u_m, v_m) \| \\
&\le \begin{cases} \frac{1}{4} M_{m-1}, & if \quad b = 2 \\ \frac{1}{8} M_{m-1}, & if \quad b = 1 \text{ or } 3 \end{cases}
\end{aligned}$$

where M_{m-1} is the second order norm of $\mathbf{M}_{m-1}$.

By applying Lemmas 5 and 6 on (15) and then using (12) on (13), we have the following lemma. Proof of this lemma is shown in Appendix D.

Lemma 7: The maximum of $\| \mathbf{L}(u,v) - \mathbf{S}(u,v) \|$ satisfies the following inequality

$$\| \mathbf{L}(u,v) - \mathbf{S}(u,v) \| \le \begin{cases} M_0, & n = 3 \\ \frac{5}{7} M_0, & n = 5 \\ \frac{4n}{n^2-8n+46} M_0, & 5 < n \le 8 \\ \frac{n^2}{4(n^2-8n+46)} M_0, & n > 8 \end{cases} \tag{16}$$

where $M = M_0$ is the second order norm of the extra-ordinary patch $\mathbf{S}(u, v)$.

Since the coefficient in the third case $(4n/(n^2-8n+46))$ is smaller than the coefficient in the second case $(5/7)$, we can combine these two cases into one case $(5 \le n \le 8)$ to make the above expression (16) simpler.

3.3 Subdivision Depth Computation

Lemma 7 is important because it not only provides us with a second order norm based simple mechanism to estimate the distance between an extra-ordinary surface patch and its control mesh, it also allows us to estimate the distance between a level-k control mesh and the surface patch for any $k > 0$. This is because the distance between a level-k control mesh and the surface patch is dominated by the distance between the level-k extra-ordinary subpatch and the corresponding control mesh which, accoriding to Lemma 7, is

$$\| \mathbf{L}_k(u,v) - \mathbf{S}(u,v) \| \le \begin{cases} M_k, & n = 3 \\ 0.72 M_k, & 5 \le n \le 8 \\ \frac{n^2}{4(n^2-8n+46)} M_k, & n > 8 \end{cases}$$

where M_k is the second order norm of $\mathbf{S}(u, v)$'s level-k control mesh, $\mathbf{M}_k$ (see the remark at the end of Section 3.1 for the definition of M_k). By combining the above result with Lemma 4, we have the following subdivision depth computation theorem for extra-ordinary surface patches.

Theorem 8: Given an extra-ordinary surface patch $\mathbf{S}(u, v)$ and an error tolerance ϵ, if k levels of subdivisions are iteratively performed on the control mesh of $\mathbf{S}(u, v)$, where

$$k = \left\lceil log_w \frac{M}{z\epsilon} \right\rceil$$

with M being the second order norm of $\mathbf{S}(u, v)$ defined in (8),

$$w = \begin{cases} \frac{3}{2}, & n = 3 \\ \frac{25}{18}, & n = 5 \\ \frac{4n^2}{3n^2+8n-46}, & n > 5 \end{cases}$$

and

$$z = \begin{cases} 1, & n = 3 \\ \frac{25}{18}, & 5 \le n \le 8 \\ \frac{2(n^2-8n+46)}{n^2}, & n > 8 \end{cases}$$

then the distance between $\mathbf{S}(u, v)$ and the level-k control mesh is smaller than ϵ.

4 Examples

Some examples of the presented distance evaluation and subdivision depth computation techniques are given in this section. In Figures 7(a) and 7(c), the distances between the blue mesh faces of the control meshes and the corresponding limit surface patches are 0.16 and 0.81, respectively. For the blue mesh face shown in Figure 7(a), the subdivision depths for the error tolerances 0.1, 0.01,

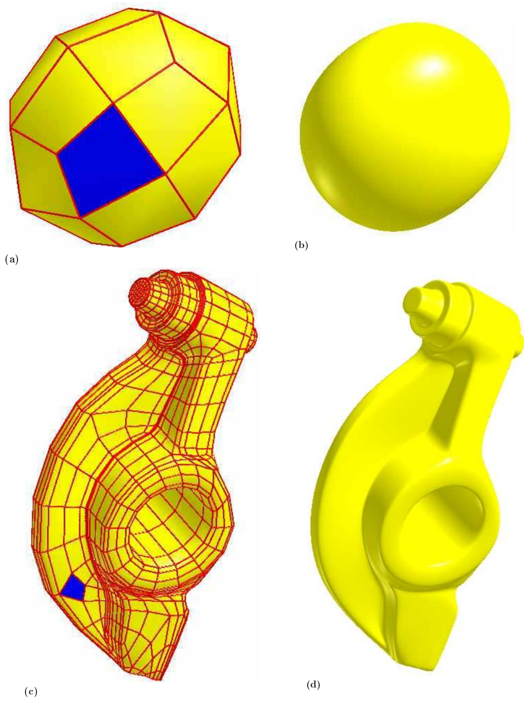

Figure 7: Examples: **(a)** an extra-ordinary CCSS mesh face of valence 3, **(b)** limit surface of the control mesh shown in (a), **(c)** an extra-ordinary CCSS mesh face of valence 5, **(d)** limit surface of the control mesh shown in (c).

0.001, and 0.0001 are 2, 7, 13, and 19, respectively. For the blue mesh face shown in Figure 7(c), the subdivision depths for the error tolerances 0.1, 0.01, 0.001, and 0.0001 are 7, 14, 21, and 28, respectively. Note that in the previous approach [3], the subdivision depths for these error tolerances are 9, 24, 40, and 56, respectively. Hence, the new approach presented in this paper indeed improves the previous, first order norm based approach.

5 Conclusions

A new subdivision depth computation technique for extra-ordinary CCSS patches is presented. The new technique computes the subdivision depth based on norms of the second order forward differences, not the first order forward differences, of the patch's control points. Hence, the computed subdivision depth reflects the curvature distribution of the extra-ordinary patch, not its dimension. Our result also points out that as long as the design objective can be achieved, one should try to use extra-ordinary vertices with smaller valence because, according to Theorem 8, smaller valence gives higher convergence rate and, consequently, smaller subdivision depth for the same precision.

Although the new technique improves the previous approach [3], it is not clear if the new approach is optimum for extra-ordinary CCSS patches. This will be a study direction in the future.

6 Appendix A: Proof of Lemma 4

For convenience of subsequent reference, we introduce two notations here:

$$\left\{\begin{array}{lll} \mathbf{A} & = & \sum_{i=1}^{n}(2\mathbf{V}_1^k - \mathbf{V}_{2i}^k - \mathbf{V}_{2[(i+2)\%n)]}^k) \\ & = & 2n\mathbf{V}_1^k - 2\sum_{i=1}^{n}\mathbf{V}_{2i}^k \\ \mathbf{B} & = & \sum_{i=1}^{n}(\mathbf{V}_{2[i\%n+1]}^k - \mathbf{V}_{2i+1}^k - \mathbf{V}_{2[i\%n+1]+1}^k) \\ & = & 2\sum_{i=1}^{n}\mathbf{V}_{2i}^k - 2\sum_{i=1}^{n}\mathbf{V}_{2i+1}^k \end{array}\right.$$

Note that $\|\mathbf{A}\| \le nM_k$ and $\|\mathbf{B}\| \le nM_k$. Then $\mathbf{V}_1^{k+1}$ can be expressed as

$$\begin{array}{lll} \mathbf{V}_1^{k+1} & = & \frac{\mathbf{F}}{n} + \frac{2\mathbf{E}}{n} + \frac{(n-3)\mathbf{V}_1^k}{n} \\ & = & \mathbf{V}_1^k \frac{6\sum_1^n \mathbf{V}_{2i}^k + \sum_1^n \mathbf{V}_{2i+1}^k - 7n\mathbf{V}_1^k}{4n^2} \\ & = & \mathbf{V}_1^k - \frac{1}{8n^2}(7\mathbf{A} + \mathbf{B}) \end{array} \quad (17)$$

where $\mathbf{V}_1^k$ is the extra-ordinary vertex with subdivision depth k.

Second order forward differences involved in M_{k+1} can be classified into four categories: F-E-F, E-F-E, E-V-E and V-E-V, where F indicates a new face point, E indicates a new edge point and V indicates a new vertex point. For some categories, we also need to consider if an extra-ordinary mesh face is involved in the second order forward difference.

Category 1 (F-E-F):
The proof is similar to the same category in the proof of Lemma 3 in [3] and the result is the same, i.e., the norm of each second order forward difference of this type is $\le M_k/4$ where M_k is the second order norm of the extra-ordinary patch after k subdivisions. Hence the proof is omitted.

Category 2 (E-F-E):
Like the above category, the proof of this category is similar to the same category in the proof of Lemma 3 in [3] and the result is the same as well (i.e., $\le M_k/4$). Hence the proof is omitted.

Category 3 (E-V-E):
Case 1: When the extra-ordinary vertex point V_1^{k+1} is involved. Consider $2V_1^{k+1} - V_2^{k+1} - V_6^{k+1}$ only. The proof of other cases is similar.

Note that from (17) and definition of edge points we have

$$\begin{array}{l} \mathbf{V}_1^{k+1} = \mathbf{V}_1^k - \frac{1}{8n^2}(7\mathbf{A} + \mathbf{B}) \\ \mathbf{V}_2^{k+1} = \frac{3}{8}(\mathbf{V}_1^k + \mathbf{V}_2^k) + \frac{1}{16}(\mathbf{V}_3^k + \mathbf{V}_4^k + \mathbf{V}_{2n}^k + \mathbf{V}_{2n+1}^k) \\ \mathbf{V}_6^{k+1} = \frac{3}{8}(\mathbf{V}_1^k + \mathbf{V}_6^k) + \frac{1}{16}(\mathbf{V}_4^k + \mathbf{V}_5^k + \mathbf{V}_7^k + \mathbf{V}_8^k) \end{array}$$

Hence,

$$
\begin{aligned}
&\|2\mathbf{V}_1^{k+1}-\mathbf{V}_2^{k+1}-\mathbf{V}_6^{k+1}\| \\
&=\|2[\mathbf{V}_1^k-\tfrac{1}{8n^2}(7\mathbf{A}+\mathbf{B})]-\tfrac{3}{8}\mathbf{V}_1^k-\tfrac{3}{8}\mathbf{V}_2^k-\tfrac{1}{16}\mathbf{V}_{2n}^k \\
&\quad-\tfrac{1}{16}\mathbf{V}_{2n+1}^k-\tfrac{1}{16}\mathbf{V}_3^k-\tfrac{1}{16}\mathbf{V}_4^k-\tfrac{3}{8}\mathbf{V}_1^k-\tfrac{3}{8}\mathbf{V}_6^k-\tfrac{1}{16}\mathbf{V}_4^k \\
&\quad-\tfrac{1}{16}\mathbf{V}_5^k-\tfrac{1}{16}\mathbf{V}_7^k-\tfrac{1}{16}\mathbf{V}_8^k\| \\
&=\|\tfrac{5}{4}\mathbf{V}_1^k-\tfrac{3}{8}\mathbf{V}_2^k-\tfrac{3}{8}\mathbf{V}_6^k-\tfrac{1}{16}\mathbf{V}_{2n}^k-\tfrac{1}{16}\mathbf{V}_{2n+1}^k-\tfrac{1}{16}\mathbf{V}_3^k \\
&\quad-\tfrac{1}{8}\mathbf{V}_4^k-\tfrac{1}{16}\mathbf{V}_5^k-\tfrac{1}{16}\mathbf{V}_7^k-\tfrac{1}{16}\mathbf{V}_8^k-\tfrac{1}{4n^2}(7\mathbf{A}+\mathbf{B})\| \\
&=\|\tfrac{1}{2}(2\mathbf{V}_1^k-\mathbf{V}_2^k-\mathbf{V}_6^k)+\tfrac{1}{16}(2\mathbf{V}_1^k-\mathbf{V}_{2n}^k-\mathbf{V}_4^k) \\
&\quad+\tfrac{1}{16}(2\mathbf{V}_1^k-\mathbf{V}_4^k-\mathbf{V}_8^k)+\tfrac{1}{16}(2\mathbf{V}_2^k-\mathbf{V}_{2n+1}^k-\mathbf{V}_3^k) \\
&\quad+\tfrac{1}{16}(2\mathbf{V}_6^k-\mathbf{V}_5^k-\mathbf{V}_7^k)-\tfrac{7}{4n^2}\mathbf{A}-\tfrac{1}{4n^2}\mathbf{B}\|
\end{aligned}
$$

Let

$$
\begin{aligned}
\mathbf{D_A}&=\mathbf{A}-(2\mathbf{V}_1^k-\mathbf{V}_2^k-\mathbf{V}_6^k)-(2\mathbf{V}_1^k-\mathbf{V}_4^k-\mathbf{V}_8^k) \\
&\quad-(2\mathbf{V}_1^k-\mathbf{V}_{2n}^k-\mathbf{V}_4^k) \\
\mathbf{D_B}&=\mathbf{B}-(2\mathbf{V}_2^k-\mathbf{V}_{2n+1}^k-\mathbf{V}_3^k)-(2\mathbf{V}_6^k-\mathbf{V}_5^k-\mathbf{V}_7^k)
\end{aligned}
$$

We have $\|\mathbf{D_A}\| \le (n-3)M_k$ and $\|\mathbf{D_B}\| \le (n-2)M_k$. Hence,

$$
\begin{aligned}
&\|2\mathbf{V}_1^{k+1}-\mathbf{V}_2^{k+1}-\mathbf{V}_6^{k+1}\| \\
&=\|(\tfrac{1}{2}-\tfrac{7}{4n^2})(2\mathbf{V}_1^k-\mathbf{V}_2^k-\mathbf{V}_6^k) \\
&\quad+(\tfrac{1}{16}-\tfrac{7}{4n^2})(2\mathbf{V}_1^k-\mathbf{V}_{2n}^k-\mathbf{V}_4^k) \\
&\quad+(\tfrac{1}{16}-\tfrac{7}{4n^2})(2\mathbf{V}_1^k-\mathbf{V}_4^k-\mathbf{V}_8^k) \\
&\quad+(\tfrac{1}{16}-\tfrac{1}{4n^2})(2\mathbf{V}_2^k-\mathbf{V}_{2n+1}^k-\mathbf{V}_3^k) \\
&\quad+(\tfrac{1}{16}-\tfrac{1}{4n^2})(2\mathbf{V}_6^k-\mathbf{V}_5^k-\mathbf{V}_7^k)-\tfrac{7}{4n^2}\mathbf{D_A}-\tfrac{1}{4n^2}\mathbf{D_B}\|.
\end{aligned}
$$

In the above equation, $\mathbf{V}_8^k$ should be replaced with $\mathbf{V}_2^k$ if $n=3$.

For the case n=3,

$$
\begin{aligned}
&\|2\mathbf{V}_1^{k+1}-\mathbf{V}_2^{k+1}-\mathbf{V}_6^{k+1}\| \\
&\le[(\tfrac{1}{2}-\tfrac{7}{4n^2})+(\tfrac{7}{4n^2}-\tfrac{1}{16})+(\tfrac{7}{4n^2}-\tfrac{1}{16}) \\
&\quad+(\tfrac{1}{16}-\tfrac{1}{4n^2})+(\tfrac{1}{16}-\tfrac{1}{4n^2})+\tfrac{7(n-3)}{4n^2}+\tfrac{n-2}{4n^2}]M_k \\
&=(\tfrac{1}{2}+\tfrac{8n-18}{4n^2})M_k=(\tfrac{1}{2}+\tfrac{6}{36})M_k=\tfrac{2}{3}M_k
\end{aligned}
$$

For the case n=5,

$$
\begin{aligned}
&\|2\mathbf{V}_1^{k+1}-\mathbf{V}_2^{k+1}-\mathbf{V}_6^{k+1}\| \\
&\le[(\tfrac{1}{2}-\tfrac{7}{4n^2})+(\tfrac{7}{4n^2}-\tfrac{1}{16})+(\tfrac{7}{4n^2}-\tfrac{1}{16}) \\
&\quad+(\tfrac{1}{16}-\tfrac{1}{4n^2})+(\tfrac{1}{16}-\tfrac{1}{4n^2})+\tfrac{7(n-3)}{4n^2}+\tfrac{n-2}{4n^2}]M_k \\
&=(\tfrac{1}{2}+\tfrac{8n-18}{4n^2})M_k=(\tfrac{1}{2}+\tfrac{22}{100})M_k=0.72M_k
\end{aligned}
$$

For the case n>5,

$$
\begin{aligned}
&\|2\mathbf{V}_1^{k+1}-\mathbf{V}_2^{k+1}-\mathbf{V}_6^{k+1}\| \\
&\le[(\tfrac{1}{2}-\tfrac{7}{4n^2})+(\tfrac{1}{16}-\tfrac{7}{4n^2})+(\tfrac{1}{16}-\tfrac{7}{4n^2}) \\
&\quad+(\tfrac{1}{16}-\tfrac{1}{4n^2})+(\tfrac{1}{16}-\tfrac{1}{4n^2})+\tfrac{7(n-3)}{4n^2}+\tfrac{n-2}{4n^2}]M_k \\
&=(\tfrac{1}{2}+\tfrac{4}{16}+\tfrac{-7-7-7-1-1+7n-21+n-2}{4n^2})M_k \\
&=(\tfrac{3}{4}+\tfrac{8n-46}{4n^2})M_k
\end{aligned}
$$

Case 2: When the extra-ordinary point is not involved. The proof of this case is similar to the same category in the proof of Lemma 3 in [3] and the result is the same as well (i.e., $\le M_k/4$). Hence the proof is omitted.

Category 4 (V-E-V):
Case 1: When the extra-ordinary vertex point V_1^k+1 is involved. Consider $2\mathbf{V}_4^{k+1}-\mathbf{V}_1^{k+1}-\mathbf{V}_{2n+7}^{k+1}$ only. The proof of other cases is similar:

$$
\begin{aligned}
&\|2\mathbf{V}_4^{k+1}-\mathbf{V}_1^{k+1}-\mathbf{V}_{2n+7}^{k+1}\| \\
&=\|\tfrac{3}{4}\mathbf{V}_1^k+\tfrac{3}{4}\mathbf{V}_4^k+\tfrac{1}{8}\mathbf{V}_2^k+\tfrac{1}{8}\mathbf{V}_5^k+\tfrac{1}{8}\mathbf{V}_6^k-\mathbf{V}_1^k \\
&\quad+\tfrac{1}{8n^2}(7\mathbf{A}+\mathbf{B})-\tfrac{9}{16}\mathbf{V}_4^k-\tfrac{1}{64}(\mathbf{V}_2^k+\mathbf{V}_6^k+\mathbf{V}_{2n+6}^k \\
&\quad+\mathbf{V}_{2n+8}^k)-\tfrac{3}{32}(\mathbf{V}_1^k+\mathbf{V}_3^k+\mathbf{V}_5^k+\mathbf{V}_{2n+7}^k)\| \\
&=\|\tfrac{1}{8}(\mathbf{V}_4^k-\mathbf{V}_1^k-\mathbf{V}_{2n+7}^k)+\tfrac{1}{64}(2\mathbf{V}_{2n+7}^k-\mathbf{V}_{2n+6}^k \\
&\quad-\mathbf{V}_{2n+8}^k)-\tfrac{7}{64}(2\mathbf{V}_1^k-\mathbf{V}_2^k-\mathbf{V}_6^k) \\
&\quad-\tfrac{1}{32}(2\mathbf{V}_4^k-\mathbf{V}_3^k-\mathbf{V}_5^k)+\tfrac{1}{8n^2}(7\mathbf{A}+\mathbf{B})\| \\
&=\|\tfrac{1}{8}(2\mathbf{V}_4^k-\mathbf{V}_1^k-\mathbf{V}_{2n+7}^k)+\tfrac{1}{64}(2\mathbf{V}_{2n+7}^k-\mathbf{V}_{2n+6}^k \\
&\quad-\mathbf{V}_{2n+8}^k)-(\tfrac{7}{64}-\tfrac{7}{8n^2})(2\mathbf{V}_1^k-\mathbf{V}_2^k-\mathbf{V}_6^k) \\
&\quad-(\tfrac{1}{32}-\tfrac{1}{8n^2})(2\mathbf{V}_4^k-\mathbf{V}_3^k-\mathbf{V}_5^k)+\tfrac{7}{8n^2}[\mathbf{A}-(2\mathbf{V}_1^k \\
&\quad-\mathbf{V}_2^k-\mathbf{V}_6^k)]+\tfrac{1}{8n^2}[\mathbf{B}-(2\mathbf{V}_4^k-\mathbf{V}_3^k-\mathbf{V}_5^k)]\| \\
&\le[\tfrac{1}{8}+\tfrac{1}{64}+(\tfrac{7}{64}-\tfrac{7}{8n^2})+(\tfrac{1}{32}-\tfrac{1}{8n^2})+\tfrac{7(n-1)}{8n^2}+\tfrac{n-1}{8n^2}]M_k \\
&=(\tfrac{9}{32}-\tfrac{1}{n^2}+\tfrac{n-1}{n^2})M_k=(\tfrac{9}{32}+\tfrac{n-2}{n^2})M_k.
\end{aligned}
$$

Case 2: When the extra-ordinary vertex point V_1^{k+1} is not involved. The proof of this case is similar to the same category in the proof of Lemma 3 in [3] and the result is the same as well (i.e., $\le M_k/4$). Hence the proof is omitted.

The lemma now follows from combining the results from the above four categories.

7 Appendix B: Proof of Lemma 5

The center faces of $\mathbf{S}_0^k$ and $\mathbf{S}_0^{k+1}$ are $\{\ \mathbf{V}_1^k, \mathbf{V}_6^k, \mathbf{V}_5^k, \mathbf{V}_4^k\ \}$ and $\{\ \mathbf{V}_1^{k+1}, \mathbf{V}_6^{k+1}, \mathbf{V}_5^{k+1}, \mathbf{V}_4^{k+1}\ \}$, respectively. By definition and the fact that $u_{k+1}=2u_k$ and $v_{k+1}=2v_k$, we have

$$
\begin{aligned}
\mathbf{L}_0^k(u_k,v_k) &= (1-v_k)[(1-u_k)V_1^k+u_kV_6^k] \\
&\quad+v_k[(1-u_k)V_4^k+u_kV_5^k]
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbf{L}_0^{k+1}(u_{k+1},v_{k+1}) \\
&=(1-v_{k+1})[(1-u_{k+1})\mathbf{V}_1^{k+1}+u_{k+1}\mathbf{V}_6^{k+1}] \\
&\quad+v_{k+1}[(1-u_{k+1})\mathbf{V}_4^{k+1}+u_{k+1}\mathbf{V}_5^{k+1}] \\
&=(1-2v_k)[(1-2u_k)\mathbf{V}_1^{k+1}+2u_k\mathbf{V}_6^{k+1}] \\
&\quad+2v_k[(1-2u_k)\mathbf{V}_4^{k+1}+2u_k\mathbf{V}_5^{k+1}].
\end{aligned}
$$

Using the expression of (17), we get

$$\begin{aligned}
&\| \mathbf{L}_0^k(u_k,v_k) - \mathbf{L}_0^{k+1}(u_{k+1},v_{k+1}) \| \\
&= \| (1-v_k)[(1-u_k)\mathbf{V}_1^k + u_k\mathbf{V}_6^k] + v_k[(1-u_k)\mathbf{V}_4^k \\
&\quad + u_k\mathbf{V}_5^k] - (1-2v_k)[(1-2u_k)(\mathbf{V}_1^k - \tfrac{1}{8n^2}(7\mathbf{A}+\mathbf{B})) \\
&\quad + 2u_k(\tfrac{3}{8}\mathbf{V}_1^k + \tfrac{3}{8}\mathbf{V}_6^k + \tfrac{1}{16}\mathbf{V}_4^k + \tfrac{1}{16}\mathbf{V}_5^k + \tfrac{1}{16}\mathbf{V}_7^k + \tfrac{1}{16}\mathbf{V}_8^k)] \\
&\quad - 2v_k[(1-2u_k)(\tfrac{3}{8}\mathbf{V}_1^k + \tfrac{3}{8}\mathbf{V}_4^k + \tfrac{1}{16}\mathbf{V}_2^k + \tfrac{1}{16}\mathbf{V}_3^k \\
&\quad + \tfrac{1}{16}\mathbf{V}_5^k + \tfrac{1}{16}\mathbf{V}_6^k) + \tfrac{u_k}{2}(\mathbf{V}_1^k + \mathbf{V}_4^k + \mathbf{V}_5^k + \mathbf{V}_6^k)] \| \\
&= \| \tfrac{1}{8}[u_k(2\mathbf{V}_1^k - \mathbf{V}_4^k - \mathbf{V}_8^k + 2\mathbf{V}_6^k - \mathbf{V}_5^k - \mathbf{V}_7^k) \\
&\quad + v_k(2\mathbf{V}_1^k - \mathbf{V}_2^k - \mathbf{V}_6^k + 2\mathbf{V}_4^k - \mathbf{V}_3^k - \mathbf{V}_5^k) + u_kv_k\cdot \\
&\quad (-8\mathbf{V}_1^k + 2\mathbf{V}_2^k + 2\mathbf{V}_3^k - 2\mathbf{V}_4^k + 4\mathbf{V}_5^k - 2\mathbf{V}_6^k + 2\mathbf{V}_7^k \\
&\quad + 2\mathbf{V}_8^k)] + \tfrac{1}{8n^2}(1-2u_k)(1-2v_k)(7\mathbf{A}+\mathbf{B}) \| \\
&= \| \tfrac{1}{8}[u_k(2\mathbf{V}_1^k - \mathbf{V}_4^k - \mathbf{V}_8^k) + u_k(2\mathbf{V}_6^k - \mathbf{V}_5^k - \mathbf{V}_7^k) \\
&\quad + v_k(2\mathbf{V}_1^k - \mathbf{V}_2^k - \mathbf{V}_6^k) + v_k(2\mathbf{V}_4^k - \mathbf{V}_3^k - \mathbf{V}_5^k) \\
&\quad - 2u_kv_k(2\mathbf{V}_1^k - \mathbf{V}_2^k - \mathbf{V}_6^k) - 2u_kv_k(2\mathbf{V}_1^k - \mathbf{V}_4^k - \mathbf{V}_8^k) \\
&\quad - 2u_kv_k(2\mathbf{V}_6^k - \mathbf{V}_5^k - \mathbf{V}_7^k) - 2u_kv_k(2\mathbf{V}_4^k - \mathbf{V}_3^k - \mathbf{V}_5^k)] \\
&\quad + \tfrac{1}{8n^2}(1-2u_k)(1-2v_k)(7\mathbf{A}+\mathbf{B}) \| \\
&= \| \tfrac{1}{8}[(u_k - 2u_kv_k)(2\mathbf{V}_1^k - \mathbf{V}_4^k - \mathbf{V}_8^k) + (u_k - 2u_kv_k)\cdot \\
&\quad (2\mathbf{V}_6^k - \mathbf{V}_5^k - \mathbf{V}_7^k) + (v_k - 2u_kv_k)(2\mathbf{V}_1^k - \mathbf{V}_2^k - \mathbf{V}_6^k) \\
&\quad + (v_k - 2u_kv_k)(2\mathbf{V}_4^k - \mathbf{V}_3^k - \mathbf{V}_5^k)] \\
&\quad + \tfrac{1}{8n^2}(1-2u_k)(1-2v_k)(7\mathbf{A}+\mathbf{B}) \| \\
&\le \tfrac{1}{8}[2(u_k - 2u_kv_k) + 2(v_k - 2u_kv_k)]M_k \\
&\quad + \tfrac{1}{n}(1-2u_k)(1-2v_k)M_k \\
&= [\tfrac{1}{4}(u_k+v_k) - u_kv_k + \tfrac{1}{n}(1-2u_k)(1-2v_k)]M_k
\end{aligned}$$

To find the maximum value of the above inequality, we set

$$f(u,v) = \frac{1}{4}(u+v) - uv + \frac{1}{n}(1-2u)(1-2v), \quad 0 \le u,v \le \frac{1}{2}.$$

The partial derivatives of f with respect to u and v are

$$f_u(u,v) = \frac{1}{4} - v - \frac{2}{n}(1-2v)$$

and

$$f_v(u,v) = \frac{1}{4} - u - \frac{2}{n}(1-2u),$$

respectively. By setting f_u and f_v to zero and solving for u and v, we get

$$u = \frac{n-8}{4n-16}, \qquad v = \frac{n-8}{4n-16}.$$

On the other hand, it is easy to see that f is a linear function on the boundary of its domain $\Omega_0^1 = [0,1/2] \times [0,1/2]$. Hence, the maximum of $f(u,v)$ on Ω_0^1 must occur at one of the corners of Ω_0^1 or at $(\frac{n-8}{4n-16}, \frac{n-8}{4n-16})$. Note that

$$f(0,0) = \frac{1}{n}, \quad f(\frac{1}{2},0) = \frac{1}{8}, \quad f(0,\frac{1}{2}) = \frac{1}{8}, \quad f(\frac{1}{2},\frac{1}{2}) = 0$$

and

$$f(\frac{n-8}{4n-16}, \frac{n-8}{4n-16}) = \frac{n}{16(n-4)}.$$

Therefore the largest value occurs at $(0,0)$ when $n \le 8$, which is $\frac{1}{n}$, and the largest value occurs at at $(\frac{1}{2},0)$ or $(0,\frac{1}{2})$ when $n > 8$, which is $\frac{1}{8}$. Hence, by combining these cases, we get

$$\| \mathbf{L}_0^k(u_k,v_k) - \mathbf{L}_0^{k+1}(u_{k+1}v_{k+1} \| \le \frac{1}{\min\{n,8\}} M_k.$$

8 Appendix C: Proof of Lemma 6

We consider the case $b = 2$ first, i.e., $\frac{1}{2^m} \le u,v < \frac{1}{2^{m-1}}$. Since $2^{m-1}u$ and $2^{m-1}v$ are both smaller than 1 and $2^m u$ and $2^m v$ are both bigger than 1, according to definition (11), we have

$$u_{m-1} = 2^{m-1}u, \qquad v_{m-1} = 2^{m-1}v$$

and

$$u_m = 2^m u - 1 = 2u_{m-1} - 1, \quad v_m = 2^m v - 1 = 2v_{m-1} - 1.$$

The center face of $\mathbf{S}_0^{m-1}$ is $\{ \mathbf{V}_1^{m-1}, \mathbf{V}_6^{m-1}, \mathbf{V}_5^{m-1}, \mathbf{V}_4^{m-1} \}$ and the center face of $\mathbf{S}_2^m$ is $\{ \mathbf{V}_5^m, \mathbf{V}_{2n+3}^m, \mathbf{V}_{2n+2}^m, \mathbf{V}_{2n+6}^m \}$. Hence, we have

$$\begin{aligned}
&\mathbf{L}_0^{m-1}(u_{m-1},v_{m-1}) \\
&= (1-v_{m-1})[(1-u_{m-1})\mathbf{V}_1^{m-1} + u_{m-1}\mathbf{V}_6^{m-1}] \\
&\quad + v_{m-1}[(1-u_{m-1})\mathbf{V}_4^{m-1} + u_{m-1}\mathbf{V}_5^{m-1}]
\end{aligned}$$

and

$$\begin{aligned}
&\mathbf{L}_2^m(u_m,v_m) \\
&= (1-v_m)[(1-u_m)\mathbf{V}_5^m + u_m\mathbf{V}_{2n+3}^m] \\
&\quad + v_m[(1-u_m)\mathbf{V}_{2n+6}^m + u_m\mathbf{V}_{2n+2}^m] \\
&= 2(1-v_{m-1})[2(1-u_{m-1})\mathbf{V}_5^m + (2u_{m-1}-1)\mathbf{V}_{2n+3}^m] \\
&\quad + (2v_{m-1}-1)[2(1-u_{m-1})\mathbf{V}_{2n+6}^m \\
&\quad + (2u_{m-1}-1)\mathbf{V}_{2n+2}^m]
\end{aligned}$$

If we define $\mathbf{A}_5$ and $\mathbf{B}_5$ as follows:

$$\begin{aligned}
\mathbf{A}_5 &= 2[(2\mathbf{V}_5^{m-1} - \mathbf{V}_4^{m-1} - \mathbf{V}_{2n+3}^{m-1}) \\
&\quad + (2\mathbf{V}_5^{m-1} - \mathbf{V}_6^{m-1} - \mathbf{V}_{2n+6}^{m-1})] \\
\mathbf{B}_5 &= (2\mathbf{V}_4^{m-1} - \mathbf{V}_1^{m-1} - \mathbf{V}_{2n+7}^{m-1}) \\
&\quad + (2\mathbf{V}_6^{m-1} - \mathbf{V}_1^{m-1} - \mathbf{V}_{2n+4}^{m-1}) \\
&\quad + (2\mathbf{V}_{2n+3}^{m-1} - \mathbf{V}_{2n+2}^{m-1} - \mathbf{V}_{2n+4}^{m-1}) \\
&\quad + (2\mathbf{V}_{2n+6}^{m-1} - \mathbf{V}_{2n+2}^{m-1} - \mathbf{V}_{2n+7}^{m-1})
\end{aligned}$$

then $\mathbf{V}_{2n+2}^m$ can be expressed as

$$\mathbf{V}_{2n+2}^m = \mathbf{V}_5^{m-1} - \frac{1}{128}(7\mathbf{A}_5 + \mathbf{B}_5).$$

Note that $\| \mathbf{A}_5 \| \leq 4M_{m-1}$ and $\| \mathbf{B}_5 \| \leq 4M_{m-1}$. Therefore,

$$\begin{aligned}
&\| \mathbf{L}_0^{m-1}(u_{m-1}, v_{m-1}) - \mathbf{L}_2^m(u_m, v_m) \| \\
&= \| (1 - v_{m-1})[(1 - u_{m-1})\mathbf{V}_1^{m-1} + u_{m-1}\mathbf{V}_6^{m-1}] \\
&\quad + v_{m-1}[(1 - u_{m-1})\mathbf{V}_4^{m-1} + u_{m-1}\mathbf{V}_5^{m-1}] \\
&\quad - 2(1 - v_{m-1})[2(1 - u_{m-1})\tfrac{\mathbf{V}_1^{m-1}+\mathbf{V}_4^{m-1}+\mathbf{V}_5^{m-1}+\mathbf{V}_6^{m-1}}{4} \\
&\quad + (2u_{m-1} - 1)(\tfrac{3}{8}\mathbf{V}_5^{m-1} + \tfrac{3}{8}\mathbf{V}_6^{m-1} + \tfrac{1}{16}\mathbf{V}_1^{m-1} \\
&\quad + \tfrac{1}{16}\mathbf{V}_4^{m-1} + \tfrac{1}{16}\mathbf{V}_{2n+3}^{m-1} + \tfrac{1}{16}\mathbf{V}_{2n+4}^{m-1})] - (2v_{m-1} - 1)\cdot \\
&\quad [2(1 - u_{m-1})(\tfrac{3}{8}\mathbf{V}_4^{m-1} + \tfrac{3}{8}\mathbf{V}_5^{m-1} + \tfrac{1}{16}\mathbf{V}_1^{m-1} \\
&\quad + \tfrac{1}{16}\mathbf{V}_6^{m-1} + \tfrac{1}{16}\mathbf{V}_{2n+6}^{m-1} + \tfrac{1}{16}\mathbf{V}_{2n+7}^{m-1}) \\
&\quad + (2u_{m-1} - 1)(\mathbf{V}_5^{m-1} - \tfrac{1}{128}(7\mathbf{A}_5 + \mathbf{B}_5))] \| \\
&= \| \tfrac{1}{8}[(2u_{m-1} + v_{m-1} - 2u_{m-1}v_{m-1} - 1)\cdot \\
&\quad (2\mathbf{V}_6^{m-1} - \mathbf{V}_1^{m-1} - \mathbf{V}_{2n+4}^{m-1}) \\
&\quad + (u_{m-1} + 2v_{m-1} - 2u_{m-1}v_{m-1} - 1)\cdot \\
&\quad (2\mathbf{V}_5^{m-1} - \mathbf{V}_6^{m-1} - \mathbf{V}_{2n+6}^{m-1}) \\
&\quad + (2u_{m-1} + v_{m-1} - 2u_{m-1}v_{m-1} - 1)\cdot \\
&\quad (2\mathbf{V}_5^{m-1} - \mathbf{V}_4^{m-1} - \mathbf{V}_{2n+3}^{m-1}) \\
&\quad + (u_{m-1} + 2v_{m-1} - 2u_{m-1}v_{m-1} - 1)\cdot \\
&\quad (2\mathbf{V}_4^{m-1} - \mathbf{V}_1^{m-1} - \mathbf{V}_{2n+7}^{m-1})] \\
&\quad - \tfrac{1}{128}(2u_{m-1} - 1)(2v_{m-1} - 1)(7\mathbf{A}_5 + \mathbf{B}_5) \| \\
&\leq \tfrac{1}{8}[2(2u_{m-1} + v_{m-1} - 2u_{m-1}v_{m-1} - 1) \\
&\quad + 2(u_{m-1} + 2v_{m-1} - 2u_{m-1}v_{m-1} - 1)]M_{m-1} \\
&\quad + \tfrac{1}{128}(2u_{m-1} - 1)(2v_{m-1} - 1)32M_{m-1} \\
&= \tfrac{1}{4}(u_{m-1} + v_{m-1} - 1)M_{m-1} \\
&\leq \tfrac{1}{4}M_{m-1}.
\end{aligned}$$

We next consider the case $b = 3$, i.e., $0 < u \leq \frac{1}{2^m}$ and $\frac{1}{2^m} < v \leq \frac{1}{2^{m-1}}$. The proof of the case $b = 1$ is similar and, consequently, will be omitted.

In this case since $2^m u \leq 1$ and $1 < 2^m v$, but $2^{m-1}v \leq 1$, we have

$$u_{m-1} = 2^{m-1}u, \qquad v_{m-1} = 2^{m-1}v$$

and

$$u_m = 2^m u = 2u_{m-1}, \qquad v_m = 2^m v - 1 = 2v_{m-1} - 1.$$

The center face of the control mesh of $\mathbf{S}_3^m$ is $\{ \mathbf{V}_4^m, \mathbf{V}_5^m, \mathbf{V}_{2n+6}^m, \mathbf{V}_{2n+7}^m \}$. Hence, $\mathbf{L}_3^m(u_m, v_m)$ can be expressed as

$$\begin{aligned}
&\mathbf{L}_3^m(u_m, v_m) \\
&= (1 - v_m)[(1 - u_m)\mathbf{V}_4^m + u_m\mathbf{V}_5^m] \\
&\quad + v_m[(1 - u_m)\mathbf{V}_{2n+7}^m + u_m\mathbf{V}_{2n+6}^m] \\
&= 2(1 - v_{m-1})[(1 - 2u_{m-1})\mathbf{V}_4^m + 2u_{m-1}\mathbf{V}_5^m] \\
&\quad + (2v_{m-1} - 1)[(1 - 2u_{m-1})\mathbf{V}_{2n+7}^m + 2u_{m-1}\mathbf{V}_{2n+6}^m]
\end{aligned}$$

If we define $\mathbf{A}_4$ and $\mathbf{B}_4$ as follows

$$\begin{aligned}
\mathbf{A}_4 &= 2[(2\mathbf{V}_4^{m-1} - \mathbf{V}_1^{m-1} - \mathbf{V}_{2n+7}^{m-1}) \\
&\quad + (2\mathbf{V}_4^{m-1} - \mathbf{V}_3^{m-1} - \mathbf{V}_5^{m-1})] \\
\mathbf{B}_4 &= (2\mathbf{V}_1^{m-1} - \mathbf{V}_2^{m-1} - \mathbf{V}_6^{m-1}) \\
&\quad + (2\mathbf{V}_3^{m-1} - \mathbf{V}_2^{m-1} - \mathbf{V}_{2n+8}^{m-1}) \\
&\quad + (2\mathbf{V}_{2n+7}^{m-1} - \mathbf{V}_{2n+6}^{m-1} - \mathbf{V}_{2n+8}^{m-1}) \\
&\quad + (2\mathbf{V}_5^{m-1} - \mathbf{V}_6^{m-1} - \mathbf{V}_{2n+6}^{m-1})
\end{aligned}$$

then $\mathbf{V}_{2n+7}^m$ can be expressed as

$$\mathbf{V}_{2n+7}^m = \mathbf{V}_4^{m-1} - \frac{1}{128}(7\mathbf{A}_4 + \mathbf{B}_4)$$

with $\| \mathbf{A}_4 \| \leq 4M_{m-1}$ and $\| \mathbf{B}_4 \| \leq 4M_{m-1}$. Therefore, by using the expression of $\mathbf{L}_0^{m-1}(u_{m-1}, v_{m-1})$ given in the previous case and the above expression for $\mathbf{V}_{2n+7}^m$, we have

$$\begin{aligned}
&\| \mathbf{L}_0^{m-1}(u_{m-1}, v_{m-1}) - \mathbf{L}_3^m(u_m, v_m) \| \\
&= \| (1 - v_{m-1})[(1 - u_{m-1})\mathbf{V}_1^{m-1} + u_{m-1}\mathbf{V}_6^{m-1}] \\
&\quad + v_{m-1}[(1 - u_{m-1})\mathbf{V}_4^{m-1} + u_{m-1}\mathbf{V}_5^{m-1}] \\
&\quad - 2(1 - v_{m-1})[(1 - 2u_{m-1})(\tfrac{3}{8}\mathbf{V}_1^{m-1} + \tfrac{3}{8}\mathbf{V}_4^{m-1} \\
&\quad + \tfrac{1}{16}\mathbf{V}_2^{m-1} + \tfrac{1}{16}\mathbf{V}_3^{m-1} + \tfrac{1}{16}\mathbf{V}_5^{m-1} + \tfrac{1}{16}\mathbf{V}_6^{m-1}) \\
&\quad + 2u_{m-1}\tfrac{\mathbf{V}_1^{m-1}+\mathbf{V}_4^{m-1}+\mathbf{V}_5^{m-1}+\mathbf{V}_6^{m-1}}{4}] - (2v_{m-1} - 1)\cdot \\
&\quad [(1 - 2u_{m-1})(\mathbf{V}_4^{m-1} - \tfrac{1}{128}(7\mathbf{A}_4 + \mathbf{B}_4)) \\
&\quad + 2u_{m-1}(\tfrac{3}{8}\mathbf{V}_4^{m-1} + \tfrac{3}{8}\mathbf{V}_5^{m-1} + \tfrac{1}{16}\mathbf{V}_1^{m-1} + \tfrac{1}{16}\mathbf{V}_6^{m-1} \\
&\quad + \tfrac{1}{16}\mathbf{V}_{2n+6}^{m-1} + \tfrac{1}{16}\mathbf{V}_{2n+7}^{m-1})] \| \\
&= \| \tfrac{1}{8}[(2u_{m-1}v_{m-1} - u_{m-1})(2\mathbf{V}_4^{m-1} - \mathbf{V}_1^{m-1} - \mathbf{V}_{2n+7}^{m-1}) \\
&\quad + (2u_{m-1}v_{m-1} - u_{m-1})(2\mathbf{V}_5^{m-1} - \mathbf{V}_6^{m-1} - \mathbf{V}_{2n+6}^{m-1}) \\
&\quad + (2u_{m-1}v_{m-1} - 2u_{m-1} - v_{m-1} + 1)\cdot \\
&\quad (2\mathbf{V}_1^{m-1} - \mathbf{V}_2^{m-1} - \mathbf{V}_6^{m-1}) \\
&\quad + (2u_{m-1}v_{m-1} - 2u_{m-1} - v_{m-1} + 1)\cdot \\
&\quad (2\mathbf{V}_4^{m-1} - \mathbf{V}_3^{m-1} - \mathbf{V}_5^{m-1})] \\
&\quad - \tfrac{1}{128}(1 - 2u_{m-1})(2v_{m-1} - 1)(7\mathbf{A}_4 + \mathbf{B}_4) \| \\
&\leq \tfrac{1}{8}[2(2u_{m-1}v_{m-1} - u_{m-1}) \\
&\quad + 2(2u_{m-1}v_{m-1} - 2u_{m-1} - v_{m-1} + 1)]M_{m-1} \\
&\quad + \tfrac{1}{128}(1 - 2u_{m-1})(2v_{m-1} - 1)32M_{m-1} \\
&= \tfrac{1}{4}(8u_{m-1}v_{m-1} - 5u_{m-1} - 3v_{m-1} + 2)M_{m-1}
\end{aligned}$$

It is easy to see that if we define $f(u,v)$ as follows

$$f(u,v) = 8uv - 5u - 3v + 2$$

then the maximum value of $f(u,v)$ on $0 \leq u \leq \frac{1}{2}$ and $\frac{1}{2} < v \leq 1$ is $\frac{1}{2}$. Consequently, we have

$$\| \mathbf{L}_0^{m-1}(u_{m-1}, v_{m-1}) - \mathbf{L}_3^m(u_m, v_m \| \leq \frac{1}{8}M_{m-1}.$$

9 Appendix D: Proof of Lemma 7

By applying Lemmas 5 and 6 on the right side of (15) and then using (12) for the second term on the right side of (13), we get

$$\begin{aligned}
&\| \mathbf{L}(u,v) - \mathbf{S}(u,v) \| \\
&\leq \textstyle\sum_{k=0}^{m-2} \| \mathbf{L}_0^k(u_k, v_k) - \mathbf{L}_0^{k+1}(u_{k+1}, v_{k+1}) \| \\
&\quad + \| \mathbf{L}_0^{m-1}(u_{m-1}, v_{m-1}) - \mathbf{L}_b^m(u_m, v_m) \| \qquad (18) \\
&\quad + \| \mathbf{L}_b^m(u_m, v_m) - \mathbf{S}_b^m(u_m, v_m) \| \\
&\leq \textstyle\sum_{k=0}^{m-2} \frac{1}{\min\{n,8\}} M_k + \frac{1}{4}M_{m-1} + \frac{1}{3}M_m
\end{aligned}$$

When $n = 5$, using the fact that $\min\{n, 8\} = 5$ and $M_{k+1} \leq 0.72 M_k$ from Lemma 4, (18) becomes

$$\begin{aligned}
&\| \mathbf{L}(u,v) - \mathbf{S}(u,v) \| \\
&\leq \tfrac{1}{5}[\ \textstyle\sum_{k=0}^{m-2}\ (0.72)^k] M_0 + \tfrac{1}{4}(0.72)^{m-1} M_0 \\
&\quad + \tfrac{1}{3}(0.72)^m M_0 \\
&= \left[\tfrac{1-(0.72)^{m-1}}{1.4} + \tfrac{1}{4}(0.72)^{m-1} + \tfrac{1}{3}(0.72)^m\right] M_0 \\
&= \tfrac{6-(1.884)(0.72)^{m-1}}{8.4} M_0 \\
&\leq \tfrac{5}{7} M_0
\end{aligned}$$

The coefficient 5/7 follows from the fact that $\frac{6-(1.884)(0.72)^{m-1}}{8.4}$ is an increasing function of m and its maximum occurs at $m = +\infty$.

When $n = 3$, using the fact that $\min\{n, 8\} = 3$ and $M_{k+1} \leq \frac{2}{3} M_k$ from Lemma 4, (18) becomes

$$\begin{aligned}
&\| \mathbf{L}(u,v) - \mathbf{S}(u,v) \| \\
&\leq \tfrac{1}{3}[\ \textstyle\sum_{k=0}^{m-2}\ (\tfrac{2}{3})^k] M_0 + \tfrac{1}{4}(\tfrac{2}{3})^{m-1} M_0 + \tfrac{1}{3}(\tfrac{2}{3})^m M_0 \\
&= \left[1 - (\tfrac{2}{3})^{m-1} + \tfrac{1}{4}(\tfrac{2}{3})^{m-1} + \tfrac{1}{3}(\tfrac{2}{3})^m\right] M_0 \\
&\leq M_0
\end{aligned}$$

For $5 < n \leq 8$, we have

$$\begin{aligned}
&\| \mathbf{L}(u,v) - \mathbf{S}(u,v) \| \\
&\leq \tfrac{1}{n} \tfrac{1-(\frac{3}{4}+\frac{8n-46}{4n^2})^{m-1}}{1-(\frac{3}{4}+\frac{8n-46}{4n^2})} M_0 + \tfrac{1}{4}(\tfrac{3}{4} + \tfrac{8n-46}{4n^2})^{m-1} M_0 \\
&\quad + \tfrac{1}{3}(\tfrac{3}{4} + \tfrac{8n-46}{4n^2})^m M_0 \\
&= [\tfrac{4n}{n^2-8n+46} - (\tfrac{4n}{n^2-8n+46} - \tfrac{1}{2} - \tfrac{4n-23}{6n^2}) \\
&\quad (\tfrac{3}{4} + \tfrac{8n-46}{4n^2})^{m-1}] M_0 \\
&\leq \tfrac{4n}{n^2-8n+46} M_0
\end{aligned}$$

The last inequality follows from the observation that $\frac{4n}{n^2-8n+46} - \frac{1}{2} - \frac{4n-23}{6n^2} > 0$ and $\frac{3}{4} + \frac{8n-46}{4n^2} \leq \frac{7}{8}$ for $5 < n \leq 8$ and, therefore, the maximum of the coefficient of M_0 occurs at $m = +\infty$.

As for $n > 8$, we have

$$\begin{aligned}
&\| \mathbf{L}(u,v) - \mathbf{S}(u,v) \| \\
&\leq \tfrac{1}{8} \tfrac{1-(\frac{3}{4}+\frac{8n-46}{4n^2})^{m-1}}{1-(\frac{3}{4}+\frac{8n-46}{4n^2})} M_0 + \tfrac{1}{4}(\tfrac{3}{4} + \tfrac{8n-46}{4n^2})^{m-1} M_0 \\
&\quad + \tfrac{1}{3}(\tfrac{3}{4} + \tfrac{8n-46}{4n^2})^m M_0 \\
&= [\tfrac{n^2}{2(n^2-8n+46)} - (\tfrac{n^2}{2(n^2-8n+46)} - \tfrac{1}{2} - \tfrac{4n-23}{6n^2}) \cdot \\
&\quad (\tfrac{3}{4} + \tfrac{8n-46}{4n^2})^{m-1}] M_0 \\
&\leq \tfrac{n^2}{2(n^2-8n+46)} M_0
\end{aligned}$$

The last inequality follows from a similar observation as the above case. This completes the proof. □

References

[1] Biermann H, Kristjansson D, Zorin D, Approximate Boolean Operations on Free-Form Solids, *Proceedings of SIGGRAPH 2001*, 185-194.

[2] Catmull E, Clark J, Recursively Generated B-spline Surfaces on Arbitrary Topological Meshes, *Computer-Aided Design 10*, 6, 350-355, 1978.

[3] Cheng F, Yong J, Subdivision Depth Computation for Catmull-Clark Subdivision Surfaces, *Computer Aided Design & Applications 3*, 1-4, 2006.

[4] DeRose T, Kass M, Truong T, Subdivision Surfaces in Character Animation, *Proceedings of SIGGRAPH 1998*, 85-94.

[5] Doo D, Sabin M, Behavior of Recursive Division Surfaces near Extraordinary Points, *Computer-Aided Design 10*, 6, 356-360, 1978.

[6] Halstead M, Kass M, DeRose T, Efficient, Fair Interpolation Using Catmull-Clark Surfaces, *Proceedings of SIGGRAPH 1993*, 35-44.

[7] Lai S, Cheng F, Parametrization of General Catmull-Clark Subdivision Surfaces and its Applications, *Computer Aided Design & Applications 3*, 1-4, 2006.

[8] Litke N, Levin A, Schröder P, Trimming for Subdivision Surfaces, *Computer Aided Geometric Design 18*, 5, 463-481, 2001.

[9] Lutterkort D, Peters J, Tight linear envelopes for splines, *Numerische Mathematik 89*, 4, 735-748, 2001.

[10] Peters J, Patching Catmull-Clark Meshes, *Proceedings of SIGGRAPH 2000*, 255-258.

[11] Sederberg T W, Zheng J, Sewell D, Sabin M, Non-Uniform Recursive Subdivision Surfaces, *Proceedings of SIGGRAPH 1998*, 387-394.

[12] Stam J, Exact Evaluation of Catmull-Clark Subdivision Surfaces at Arbitrary Parameter Values, *Proceedings of SIGGRAPH 1998*, 395-404.

[13] Stam J, Evaluation of Loop Subdivision Surfaces, *SIGGRAPH'99 Course Notes*, 1999.

[14] Wu X, Peters J, An Accurate Error Measure for Adaptive Subdivision Surfaces, *Proc. Shape Modeling International 2005*, 1-6.

[15] Zorin, D., Schröder, P., and Sweldens, W. Interactive Multiresolution Mesh Editing. *Proceedings of SIGGRAPH 1997*, 259-268.

[16] Zorin D, Kristjansson D, Evaluation of Piecewise Smooth Subdivision Surfaces, *The Visual Computer*, 18(5/6):299-315, 2002.