Constructing G^1 quadratic Bézier curves with arbitrary endpoint tangent vectors

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Abstract

Quadratice Bézier curves are important geometric entities in many applications. However, it was often ignored by the literature the fact that a single segment of a quadratic Bézier curve may fail to fit arbitrary endpoint unit tangent vectors. The purpose of this paper is to provide a solution to this problem, i.e., constructing G^1 quadratic Bézier curves satisfying given endpoint (positions and arbitrary unit tangent vectors) conditions. Examples are given to illustrate the new solution and to perform comparison between the G^1 quadratic Bézier cures and other curve schemes such as the composite geometric Hermite curves and the biarcs."

Keywords: Quadratic Bézier curve, geometric continuity, endpoint condition, smoothness

1. Introduction

Quadratic Bézier curves are very important geometric entities in many applications [1, 2, 3, 6, 7, 11, et al]. For example, quadratic Bézier curves are often used as the generatrix curves of the surface of radars, and to approximate circular arcs [1, et al], which cannot be represented by polynomials in an exact way. The space and computation costs of quadratic Bézier curves are both smaller than any other free form curves of degree three or higher. When approximating a given curve, the number of segments of the resultant quadratic Bézier curve is usually much smaller than that required by a polyline.

In this paper, quadratic Bézier curves are constructed to satisfy the endpoint conditions, which include the two endpoint positions and two directions of the unit tangent vectors at the end points. How to construct curves to satisfy the endpoint conditions is a fundamental problem in computer aided geometric design [9, et al] and numerical computation and analysis [4, et al]. Hermite curves [4, 9, et al], biarcs [5, 8, 10, et al], and quadratic Bézier curves [1, 6, et al] are often the solutions. Among the above three kinds of curves, Hermite curves [4, et al] appear the most frequently in the literature of numerical computation and analysis. However, as pointed out by [9], they may have cusps (see Figure 1 from [9] as an example), which are not allowed in many applications. COH (composite optimized geometric Hermite) curves [9] have many advantages over traditional Hermite curves. However, they require at least three segments to cover all directions of the endpoint tangent vectors [9].

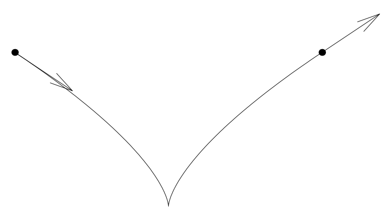

Figure 1. A Hermite curve with a cusp [9].

Biarcs [10] are usually made of only two segments. They are frequently used in CNC (Computer Numerical Control) to generate G^1 arc splines as the tool paths. Since biarcs are usually composite of two circular arcs, they cannot be exactly represented in a polynomial form. If some systems or applications prefer polynomials, they have to converted into polynomial representation, such as quadratic Bézier curves [1, et al].

Quadratic Bézier curves [1, 6, et al] can be used to satisfy the endpoint conditions as well. However, the fact that a single segment of a quadratic Bézier curve cannot cover all directions of the endpoint unit tangent vectors is ignored so far in many literatures [1, 6, et al]. Thus, some applications may fail due to the quadratic Bézier curves not being able to fit in with some requirements of the endpoint unit tangent vectors. Figure 2 gives such an example. In order to make the applications [1, 6, et al] available in a general way, it is definitely to find a solution for quadratic Bézier curves. Thus, in this paper, we give such a solution. We also prove that two segments of quadratic Bézier curves are enough to cover all directions of the endpoint unit tangent vectors.

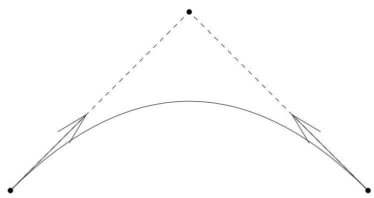

Figure 2. A single quadratic Bézier curve does not fit in with the directions of the given endpoint unit tangent vectors.

The remaining part of the paper is arranged as follows. The necessary and sufficient conditions for a single quadratic Bézier curve satisfying the given endpoint constrains (of both positions and directions of unit tangent vectors) are provided in Section 2. Section 3 addresses how to construct two segments of G^1 quadratic Bézier curves with the endpoint constrains of both positions and arbitrary endpoint unit tangent vectors. Some examples and discussion are given in Section 4. Concluding remarks are presented in the last section.

2. Necessary and sufficient conditions for a single quadratic Bézier curve

The definition of a quadratic Bézier curve is

$$\mathbf{C}(t) = \sum_{i=0}^{2} \mathbf{P}_i B_{i,2}(t), \quad t \in [0,1],$$

where $\mathbf{P}_0$, $\mathbf{P}_1$ and $\mathbf{P}_2$ are control points, and $B_{i,2}(t) = \frac{2}{i!(2-i)!} t^i (1-t)^{2-i}$. As shown in Figure 3, a G^1 quadratic Bézier curve $\mathbf{C}(t)$ is required to satisfy the following endpoint conditions:

$$\begin{cases} \mathbf{C}(0) = \mathbf{P}_0 = \mathbf{Q}_0, \\ \mathbf{C}(1) = \mathbf{P}_2 = \mathbf{Q}_1, \\ \mathbf{C}'(0) \text{ has the same direction with } \mathbf{V}_0, \text{ and} \\ \mathbf{C}'(1) \text{ has the same direction with } \mathbf{V}_1, \end{cases}$$

where $\mathbf{Q}_0$ and $\mathbf{Q}_1$ are given points, and $\mathbf{V}_0$ and $\mathbf{V}_1$ are given unit tangent vectors at $\mathbf{Q}_0$ and $\mathbf{Q}_1$, respectively.

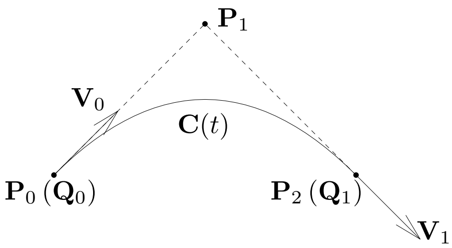

Figure 3. A G^1 quadratic Bézier curve.

To simplify the address, let $\mathbf{L}_i$ be the line passing through $\mathbf{Q}_i$ and having the direction $\mathbf{V}_i$, where $i = 0, 1$. According to the definition of a quadratic Bézier curve, we have that the endpoint tangent vectors of the Bézier curve $\mathbf{C}(t)$ are $\mathbf{C}'(0) = 2(\mathbf{P}_1 - \mathbf{P}_0)$ and $\mathbf{C}'(1) = 2(\mathbf{P}_2 - \mathbf{P}_1)$. Hence, we obtain that $\mathbf{P}_1$ should be the intersection point of $\mathbf{L}_0$ and $\mathbf{L}_1$. In the meanwhile, we have that

$$(\mathbf{P}_1 - \mathbf{P}_0) \cdot \mathbf{V}_0 > 0 \text{ and } (\mathbf{P}_2 - \mathbf{P}_1) \cdot \mathbf{V}_1 > 0.$$

Thus, we have the following theorem, which provides the necessary and sufficient conditions for a single G^1 quadratic Bézier curve.

Theorem 1 The necessary and sufficient conditions for a G^1 quadratic Bézier curve are

(1) if $\mathbf{L}_0$ and $\mathbf{L}_1$ has one unique intersection point $\mathbf{Q}$, then the following Boolean expression should be true

$$((\mathbf{Q} - \mathbf{Q}_0) \cdot \mathbf{V}_0 > 0) \text{ and } ((\mathbf{Q}_1 - \mathbf{Q}) \cdot \mathbf{V}_1 > 0);$$

(2) if $\mathbf{L}_0$ and $\mathbf{L}_1$ are coincident, then the following Boolean expression should be false

$$(\mathbf{V}_0 \cdot \mathbf{V}_1 > 0) \text{ and } ((\mathbf{Q}_1 - \mathbf{Q}_0) \cdot \mathbf{V}_1 < 0).$$

If the conditions in Theorem 1 are not satisfied, a single quadratic Bézier curve cannot satisfy the given endpoint conditions. An example is shown in Figure 2. Here addresses another example. If $\mathbf{V}_0$ has the same direction with $\mathbf{V}_1$, but not with the direction of $\mathbf{Q}_1 - \mathbf{Q}_0$, then the above endpoint conditions cannot be satisfied for any single quadratic Bézier curve.

3. Two segments of quadratic Bézier curves

If a single G[1] quadratic Bézier curve cannot satisfy the endpoint conditions, then we may turn to a Bézier curve with some degree higher than two such as a cubic Hermite curve [9, et al], or we may have to require some more segments. Our experience shows that two segments of quadratic Bézier curves are enough. The solution here is based on Section 2, and all the symbols in Section 2 are inherited here. $\mathbf{Q}_0$ and $\mathbf{Q}_1$ are two given points, which are required to be the starting point and the ending point, respectively. $\mathbf{V}_0$ and $\mathbf{V}_1$ are two given unit vectors. The tangent vector at the the starting point of the first quadratic Bézier curve $\mathbf{C}_1(s)$ should have the same direction with $\mathbf{V}_0$, and the tangent vector at the the ending point of the second quadratic Bézier curve $\mathbf{C}_2(t)$ should have the same direction with $\mathbf{V}_1$.

Our solution is as follows. It is illustrated in Figure 4 as well. Let $\mathbf{P}_{1,1} = \mathbf{Q}_0 + r\mathbf{V}_0$, $\mathbf{P}_{1,2} = \mathbf{Q}_1 - r\mathbf{V}_1$, and

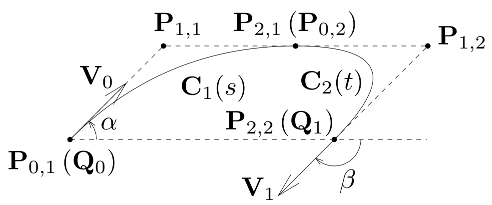

Figure 4. G[1] quadratic Bézier curves (two segments).

$\mathbf{P}_{2,1} = \frac{\mathbf{P}_{1,1}+\mathbf{P}_{1,2}}{2}$, where r is a positive real number (how to choose a value for r will be discussed later in the remaining part of this section). The control points of the first quadratic Bézier curve $\mathbf{C}_1(s)$ are $\mathbf{P}_{0,1} = \mathbf{Q}_0$, $\mathbf{P}_{1,1}$ and $\mathbf{P}_{2,1}$. And the control points of the second quadratic Bézier curve $\mathbf{C}_2(t)$ are $\mathbf{P}_{0,2} = \mathbf{P}_{2,1}$, $\mathbf{P}_{1,2}$ and $\mathbf{P}_{2,2} = \mathbf{Q}_1$. According to Theorem 1, we have the following theorem.

Theorem 2 For arbitrary $r \in (0, \frac{\|\mathbf{Q}_1-\mathbf{Q}_0\|}{3})$, the composite curve made by $\mathbf{C}_1(s)$ and $\mathbf{C}_2(t)$ satisfies the above endpoint conditions, and covers all possible directions of $\mathbf{V}_0$ and $\mathbf{V}_1$.

It seems that any positive real number r is enough for Theorem 2. Here $r < \frac{\|\mathbf{Q}_1-\mathbf{Q}_0\|}{3}$ is a safe requirement which avoids some degenerate cases. Our experience shows that the choice $r = 0.3\|\mathbf{Q}_1 - \mathbf{Q}_0\|$ is enough to produce a pleasing shape for the composite curve. In order to get a better shape, some more calculation may be necessary. In the remaining part of this section, we will discuss how to obtain r such that all edges of the control polygons of both $\mathbf{C}_1(s)$ and $\mathbf{C}_2(t)$ have the same length, i.e., $\|\mathbf{P}_{1,1} - \mathbf{P}_{0,1}\|$, $\|\mathbf{P}_{2,1} - \mathbf{P}_{1,1}\|$, $\|\mathbf{P}_{1,2} - \mathbf{P}_{0,2}\|$ and $\|\mathbf{P}_{2,2} - \mathbf{P}_{1,2}\|$ are equal to each other.

According to our solution, we already have

$$\|\mathbf{P}_{1,1} - \mathbf{P}_{0,1}\| = \|\mathbf{P}_{2,2} - \mathbf{P}_{1,2}\|$$

and

$$\|\mathbf{P}_{2,1} - \mathbf{P}_{1,1}\| = \|\mathbf{P}_{1,2} - \mathbf{P}_{0,2}\|.$$

Thus, all we need here is

$$\|\mathbf{P}_{2,1} - \mathbf{P}_{1,1}\| = \|\mathbf{P}_{1,1} - \mathbf{P}_{0,1}\|. \quad (1)$$

Equation (1) is equivalent to

$$\|\mathbf{P}_{1,2} - \mathbf{P}_{1,1}\| = 2\|\mathbf{P}_{1,1} - \mathbf{P}_{0,1}\|.$$

Thus, we have

$$(\mathbf{P}_{1,2} - \mathbf{P}_{1,1})^2 - 4r^2 = 0, \quad (2)$$

which is a quadratic equation with respect to r. Solve Equation (2), and we have the following conclusions.

As shown in Figure 4, let α be the angle from the direction of $(\mathbf{Q}_1 - \mathbf{Q}_0)$ to $\mathbf{V}_0$, and β be the angle from the direction of $(\mathbf{Q}_1 - \mathbf{Q}_0)$ to $\mathbf{V}_1$. Here, if the nonzero angle is measured in the counterclockwise, the value of the angle is positive; otherwise, it has a negative value. Let

$$h_1 = \sqrt{2 + \cos^2\alpha + \cos^2\beta - 2 \times \sin\alpha \times \sin\beta}$$

and

$$h_2 = 2 \times \cos(\beta - \alpha) - 2.$$

If $h_2 \neq 0$, we have

$$r = \frac{(\cos\alpha + \cos\beta + h_1)\|\mathbf{Q}_1 - \mathbf{Q}_0\|}{h_2}$$

or

$$r = \frac{(\cos\alpha + \cos\beta - h_1)\|\mathbf{Q}_1 - \mathbf{Q}_0\|}{h_2}.$$

If $h_2 = 0$, then Equation (2) degenerates into a linear equation. In this case, we have

$$r = \frac{\|\mathbf{Q}_1 - \mathbf{Q}_0\|}{2(\cos\alpha + \cos\beta)}.$$

In the above possible values of r, only the positive value can be used.

4. Examples and discussion

We have tested all directions of $\mathbf{V}_0$ and $\mathbf{V}_1$, of which the values of the angles (i.e., α and β) are integers in degrees. The shape of a composite curve made by two quadratic Bézier curves depends on the value of r. Our experience finds that the choice $r = 0.3\|\mathbf{Q}_1 - \mathbf{Q}_0\|$ is enough to produce a pleasing shape for our solution. The value of

r, which makes all edges of the control polygons of both $\mathbf{C}_1(s)$ and $\mathbf{C}_2(t)$ have the same length, may be another good choice. Figure 5 gives such an example. In this example, $\alpha = 0$, and $\beta = \frac{\pi}{3}$. From Figure 5(a) to Figure 5(d), the values of r are 0.1, 0.3, 0.75 and 0.3028, respectively. When $r = 0.3028$ as shown in Figure 5(d), all edges of the control polygons of both the quadratic Bézier curves have the same length. Here, 0.3 is very close to 0.3028, so the shape of the composite curve in Figure 5(b) is similar to the shape of the composite curve in Figure 5(d).

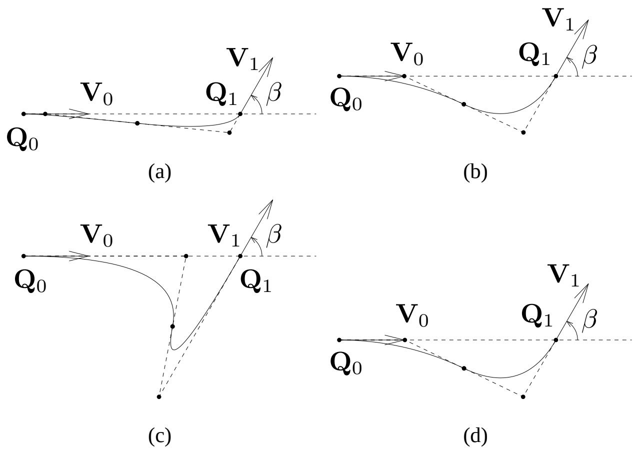

Figure 5. A comparison among the composite curves with different values of r: (a) $r = 0.1$, (b) $r = 0.3$, (c) $r = 0.75$, (d) $r = 0.3028$.

Another example is given in Figure 6. In this example, $\alpha = \beta = \frac{\pi}{3}$. Three kinds of curves are compared here. In Figures 6(a) and 6(b), G^1 quadratic Bézier curves are applied. All edges of the control polygons of both the quadratic Bézier curves have the same length when $r = 0.5$, as shown in Figure 6(b). Figures 6(c) and 6(d) illustrate the COH (composite optimized geometric Hermite) curve [9] and the biarc [10], respectively. As shown in the figures, the quadratic Bézier curves with $r = 0.3$ seems more similar to the COH curve and the biarc, and the quadratic Bézier curves with $r = 0.5$ seems to have the nicest shape among all curves in Figure 6. In this example, the COH curve has three segments, which requires larger space of computer storage than the two quadratic Bézier curves. The degree of a COH curve is three, which is higher than a quadratic Bézier curve. Thus, to obtain a point on curves, the COH curve requires more time cost than the quadratic Bézier curve. In the meanwhile, the biarc cannot be represented in a polynomial form.

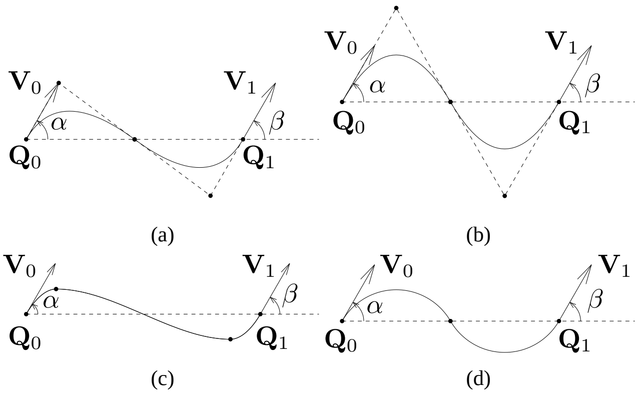

Figure 6. A comparison among the different kinds of curves: (a) two segments of quadratic Bézier curves with $r = 0.3$, (b) two segments of quadratic Bézier curves with $r = 0.5$, (c) a COH (composite optimized geometric Hermite) curve (3 segments) [9], (d) a biarc [10].

5. Conclusions

A solution for constructing G^1 quadratic Bézier curves is presented in this paper. We build a composite curve made by two quadratic Bézier curves to satisfy the endpoint constraints with both positions and directions of unit tangent vectors. The new solution can cover all directions of the endpoint tangent vectors. The composite curve has the flexibility to obtain different shape with different values of r. The default value of r and the value of r, which makes all edges of of the control polygons of both the quadratic Bézier curves have the same length, are provided as well. The new solution has some advantages over COH (composite optimized geometric Hermite) curves and biarcs. All those three kinds of curves are able to produce pleasing shape, and have their own favorite applications. The solution proposed in this paper makes quadratic Bézier curves have some more flexibility in some possible applications such as those in [1, 6].

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