First order absolute moment of Meyer-König and Zeller operators and their approximation for some absolutely continuous functions

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Abstract A sharp estimate is given for the first order absolute moment of Meyer-König and Zeller operators $M_n$. This estimate is then used to prove convergence of approximation of a class of absolutely continuous functions by the operators $M_n$. The condition considered here is weaker than the condition considered in a previous paper and the rate of convergence we obtain is asymptotically the best possible.

1 Introduction

For a function $f$ defined on $[0, 1]$, the Meyer-König and Zeller operators $M_n$ [5] are defined by

$$M_n(f, x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right) m_{n,k}(x), \quad 0 \leq x < 1,$$

$$M_n(f, 1) = f(1), \quad m_{n,k}(x) = \binom{n+k}{k} x^k (1-x)^{n+1}. \quad (1)$$

Let

$$K_n,x(t) = \begin{cases} 
\sum_{k\leq nt/(1-t)} m_{n,k}(x), & 0 < t < 1, \\
1, & t = 1, \\
0, & t = 0.
\end{cases}$$

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Then operators $M_n$ have the following Lebesgue-Stieltjes integral representation

$$M_n (f, x) = \int_0^1 f(t) d\nu_{n,x}(t).$$

(2)

Estimates of the first order absolute moment of the approximation operators play a key role in various investigations of convergence of the approximation operators (for example, cf. [3, 4, 6, 7, 8]). In this paper we give a sharp estimate for the first order absolute moment of the operators $M_n$. Furthermore, by means of this estimate and some analysis techniques we establish a convergence theorem on the approximation of a class of absolutely continuous functions by the operators $M_n$. The rate of convergence we obtain in this theorem is essentially the best possible.

2 Results and Proofs

For the first order absolute moment of Meyer-König and Zeller operators $M_n$, we have the following result.

Theorem 1. For $x \in (0,1]$, we have

$$M_n(t - x, t) = \frac{\sqrt{2x(1-x)}}{\sqrt{\pi n}} + O\left(\frac{1}{n^{n/2}}\right).$$

(3)

Proof. If $x = 1$, (3) is true. Let $0 < x < 1$ and write $r = x/(1 - x)$. By the fact that $M_n(t, t) = x$ we have

$$M_n(t - x, t) = \sum_{k=0}^{[nr]} \left(x - \frac{k}{n + k}\right) m_{n,k}(x) + \sum_{k=[nr]+1}^{\infty} \left(\frac{k}{n + k} - x\right) m_{n,k}(x)$$

$$= 2 \sum_{k=0}^{[nr]} \left(x - \frac{k}{n + k}\right) m_{n,k}(x) + M_n(t - x, t)$$

$$= 2 \sum_{k=0}^{[nr]} \left(\frac{n + k}{k}\right) x^{k+1}(1 - x)^{n+1} - 2 \sum_{k=0}^{[nr]} \frac{n + k}{n + k} \left(\frac{n + k}{k}\right) x^{k+1}(1 - x)^{n+1}$$

$$= 2 \sum_{k=0}^{[nr]} \left(\frac{n + k}{k}\right) x^{k+1}(1 - x)^{n+1} - 2 \sum_{k=0}^{[nr]-1} \frac{n + k}{k} x^{k+1}(1 - x)^{n+1}$$

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\[
2 \left( \frac{n}{n} + \frac{[nr]}{n} \right) x^{[nr]+1}(1-x)^{n+1}.
\]

Next we estimate
\[
2 \left( \frac{n}{n} + \frac{[nr]}{n} \right) x^{[nr]+1}(1-x)^{n+1}.
\]

Using Stirling's formula \[9\] \( n! = \sqrt{2\pi n}(n/e)^n e^{\theta} \), \( 0 < \theta < 1/12n \), we get
\[
2 \left( \frac{n}{n} + \frac{[nr]}{n} \right) = \frac{2(n + [nr])!}{n! [nr]!} = \frac{\sqrt{2}(n + [nr])^{n+[nr]+1/2}}{\pi n^{n+1/2} [nr]^{n+[nr]+1/2}} e^{\theta_1 - \theta_2 - \theta_3},
\]

where \( 0 < \theta_1 < \frac{1}{12(n + [nr])} \), \( 0 < \theta_2 < \frac{1}{12n} \), \( 0 < \theta_3 < \frac{1}{12[nr]} \).

Set \( c(\theta) = \theta_1 - \theta_2 - \theta_3 \), simple calculation derives
\[
-\frac{1}{12n} - \frac{1}{12[nr]} < c(\theta) \leq 0.
\]

Since \( r = x/(1-x) \), by straightforward calculation we have
\[
x^{[nr]+1/2}(1-x)^n = \frac{r^{[nr]+1/2}}{(1+r)^{n+[nr]+1/2}}.
\]

Furthermore we find that
\[
\frac{(n + [nr])^{n+[nr]+1/2}}{n^{n+1/2} [nr]^{n+[nr]+1/2}} \frac{r^{[nr]+1/2}}{(1+r)^{n+[nr]+1/2}} = \frac{1}{\sqrt{n}} \left( \frac{nr}{[nr]} \right)^{[nr]+1/2} \left( \frac{n + [nr]}{n + nr} \right)^{n+[nr]+1/2} \left( \frac{nr}{[nr]} \right)^{[nr]+1/2} \left( \frac{n + [nr]}{n + nr} \right)^{n+[nr]+1/2}.
\]

Thus it follows from (5-8) that
\[
2 \left( \frac{n}{n} + \frac{[nr]}{n} \right) x^{[nr]+1}(1-x)^{n+1} = \sqrt{2}(1-x)2 \left( \frac{n}{n} + \frac{[nr]}{n} \right) x^{[nr]+1/2}(1-x)^n
\]
\[
= \frac{\sqrt{2x(1-x)}}{\sqrt{\pi n}} \left( \frac{nr}{[nr]} \right)^{[nr]+1/2} \left( \frac{n + [nr]}{n + nr} \right)^{n+[nr]+1/2} e^{c(\theta)}.
\]

Write
\[
A(n,r) = \left( \frac{nr}{[nr]} \right)^{[nr]+1/2} \left( \frac{n + [nr]}{n + nr} \right)^{n+[nr]+1/2},
\]

and
\[
r = [nr] + \nu \quad (0 \leq \nu < 1).
\]
Then

\[ A(n, r) = \left(1 + \frac{\nu}{nr}\right)^{[nr]+1/2} \left(1 + \frac{\nu}{n + [nr]}\right)^{-(n+[nr]+1/2)}. \]

Thus

\[
\log A(n, r) = ([nr] + 1/2) \log \left(1 + \frac{\nu}{nr}\right) - (n + [nr] + 1/2) \log \left(1 + \frac{\nu}{n + [nr]}\right)
\]

\[ = ([nr] + 1/2) \left(\frac{\nu}{nr} + O \left(\frac{\nu}{nr}\right)^2\right) - (n + [nr] + 1/2) \left(\frac{\nu}{n + [nr]} + O \left(\frac{\nu}{n + [nr]}\right)^2\right)
\]

\[ = O \left([nr]^{-1}\right), \]

which means that

\[ A(n, r) = 1 + O \left([nr]^{-1}\right). \]  \hspace{2cm} (11)

Hence from (4), (9), (10), (11) and the fact that \(e^{c(\theta)} = 1 + O(n^{-1} + [nr]^{-1})\), we get

\[
M_n \left(|t - x|, x\right) = 2 \left(\frac{n + [nr]}{n}\right) x^{[nr]+1} (1-x)^{n+1}
\]

\[ = \frac{\sqrt{2x(1-x)}}{\sqrt{\pi n}} \left(1 + O(n^{-1} + [nr]^{-1})\right)
\]

\[ = \frac{\sqrt{2x(1-x)}}{\sqrt{\pi n}} + O \left(\frac{1}{n\sqrt{nx}}\right)\]

and Theorem 1 is proved.

Next we consider approximation of the operators \(M_n\) for a class of absolutely continuous functions \(\Phi_{DB}\) defined by

\[ \Phi_{DB} = \{f \left| f(t) - f(0) = \int_0^t h(u)du, \ t \in [0,1], \ h \text{ is bounded on } [0,1], \right. \]

\[ \left. \text{and } h(x+), \ h(x-) \text{ exist at } x \in (0,1)\} . \]

The following three quantities are needed in this paper. The readers are referred to Reference [8, p. 244] for their basic properties.

\[ \Omega_{x-}(h, \delta_1) = \sup_{t \in [x-\delta_1, x]} |h(t) - h(x)|, \quad \Omega_{x+}(h, \delta_2) = \sup_{t \in [x, x+\delta_2]} |h(t) - h(x)| , \]

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\[ \Omega(x, h, \lambda) = \sup_{t \in [x-x/\lambda, x+(1-x)/\lambda]} |h(t) - h(x)|, \]

where \(h\) is bounded on \([0, 1]\), \(x \in [0,1]\) is fixed, \(0 \leq \delta_1 \leq x\), \(0 \leq \delta_2 \leq 1 - x\), and \(\lambda \geq 1\).

We now state the approximation theorem as follows.

**Theorem 2.** Let \(f \in \Phi_{DB}\) and write \(\mu = h(x+) - h(x-)\). Then for \(n\) sufficiently large we have

\[
\left| M_n(f, x) - f(x) - \mu \frac{\sqrt{x}(1-x)}{\sqrt{2\pi n}} \right| \leq \frac{4 - 2x}{n} \sum_{k=1}^{\lfloor \sqrt{n} \rfloor} \Omega(x, h_x, k) + \frac{C|\mu|}{n^{\sqrt{n}/x}}. \tag{12} \]

where \(C\) is a constant independent of \(n\) and \(x\), \(\lfloor \sqrt{n} \rfloor\) is the greatest integer not exceeding \(\sqrt{n}\) and \(h_x(t)\) is defined by

\[
h_x(t) = \begin{cases} 
  h(t) - h(x+), & x < t \leq 1 \\
  0, & u = x \\
  h(t) - h(x-), & 0 \leq t < x,
\end{cases} \tag{13} \]

In view of the fact that \(\frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor \sqrt{n} \rfloor} \Omega(x, h_x, k) \to 0 \ (n \to \infty)\), from Theorem 2 we get the asymptotic formula

\[
M_n(f, x) = f(x) + \frac{\sqrt{x}(1-x)}{\sqrt{2\pi n}} \mu + o(n^{-1/2}),
\]

if \(f\) satisfies the assumptions of Theorem 2. In particular, (12) is true for \(f \in DBV[0,1]\) (that is, \(f\) is differentiable function whose derivative is of bounded variation, cf. [3]), since the class of functions \(DBV[0,1]\) is a subclass of the class \(\Phi_{DB}\). We also point out that Abel [1] presented the complete asymptotic expansion for the operators \(M_n\) under much stronger conditions.

Moreover, it is of interest to consider some further results. Let \(f\) satisfy the assumptions of Theorem 2 and \(\Omega(x, h_x, \lambda) = O(1/\lambda^\alpha)\) for some \(\alpha > 0\). Then from Theorem 2 we get

\[
M_n(f, x) = f(x) + \frac{\sqrt{x}(1-x)}{\sqrt{2\pi n}} \mu + \begin{cases} 
  O(n^{-\alpha}/\sqrt{n/2}), & \text{if } 0 < \alpha < 1 \text{ or } 1 < \alpha < 2 \\
  O(\log \sqrt{n}/n), & \text{if } \alpha = 1 \\
  O(n^{-3/2}), & \text{if } \alpha \geq 2
\end{cases}.
\]

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Proof of Theorem 2.

By Bojanic decomposition we have

\[ h(u) = \frac{h(x^+)-h(x^-)}{2} + \frac{h(x^+)-h(x^-)}{2} \text{sgn}(u-x) + h_x(u) + \delta_x(u) \left( \frac{h(x^-)+h(x^+)}{2} - \frac{h(x)}{2} \right), \]  

(14)

where \( \text{sgn}(u) \) is symbolic function, \( h_x \) is as defined in (13), and

\[ \delta_x(t) = \begin{cases} 
1, & t = x \\
0, & t \neq x.
\end{cases} \]

Note that \( M_n(t, x) = x, \int_x^t \text{sgn}(u-x)du = |t-x|, \) and \( \int_x^t \delta_x(u)du = 0. \) From (14) it follows by simple computation that

\[ f(t) - f(x) = \int_x^t h(u)du = \frac{h(x^+)-h(x^-)}{2} |t-x| + \int_x^t h_x(u)du. \]

Thus

\[ M_n(f, x) - f(x) = \frac{h(x^+)-h(x^-)}{2} M_n(|t-x|, x) + M_n \left( \int_x^t h_x(u)du, x \right). \]  

(15)

By Lebesgue-Stieltjes integral representation (2) we have

\[ M_n \left( \int_x^t h_x(u)du, x \right) = \int_0^1 \left( \int_x^t h_x(u)du \right) d_t K_{n,x}(t) \]

\[ = L(h, n, x) + Q(h, n, x), \]  

(16)

where

\[ L(h, n, x) = \int_0^x \left( \int_x^t h_x(u)du \right) d_t K_{n,x}(t), \]

\[ Q(h, n, x) = \int_0^1 \left( \int_x^t h_x(u)du \right) d_t K_{n,x}(t). \]

Integration by parts and note that \( K_{n,x}(0) = 0, \quad h_x(x) = 0 \) we have

\[ |L(h, n, x)| = \left| \int_0^x K_{n,x}(t)h_x(t)dt \right| \leq \int_0^x K_{n,x}(t)\Omega_{x-}(h_x, x-t)dt \]  

(17)
\[
\int_0^{x-x/\sqrt{n}} K_{n,x}(t) \Omega_{x-}(h_x, x-t)dt + \int_{x-x/\sqrt{n}}^x K_{n,x}(t) \Omega_{x-}(h_x, x-t)dt. \quad (18)
\]

By Lemma 2.1 of [2] there holds inequality

\[
M_n((t - x)^2, x) \leq \left(1 + \frac{2x}{n - 1}\right) \frac{x(1-x)^2}{n + 1}.
\]

Using this inequality, for \(0 \leq t < x\) we deduce that

\[
K_{n,x}(t) \leq \sum_{n+k \leq t} m_{n,k}(x) \leq \sum_{n+k \leq t} \left(\frac{k/(n+k) - x}{x-t}\right)^2 m_{n,k}(x)
\]

\[
\leq \frac{M_n((u-x)^2, x)}{(x-t)^2} \leq \frac{1}{(x-t)^2} \left(1 + \frac{2x}{n - 1}\right) \frac{x(1-x)^2}{n + 1} \leq \frac{2x(1-x)^2}{n(x-t)^2}.
\]

Thus by replacement of variable \(t = x - x/u\) we have

\[
\int_0^{x-x/\sqrt{n}} K_{n,x}(t) \Omega_{x-}(h_x, x-t)dt \leq \frac{2x(1-x)^2}{n} \int_0^{x-x/\sqrt{n}} \frac{\Omega_{x-}(h_x, x-t)}{(x-t)^2} dt
\]

\[
= \frac{2(1-x)^2}{n} \int_1^{\sqrt{n}} \Omega_{x-}(h_x, x/u)du
\]

\[
\leq \frac{2(1-x)^2}{n} \sum_{k=1}^{\lceil\sqrt{n}\rceil} \Omega_{x-}(h_x, x/k). \quad (19)
\]

On the other hand, by inequality \(K_{n,x}(t) \leq 1\) and the monotonicity of \(\Omega_{x-}(h_x, \lambda)\), it follows that

\[
\int_{x-x/\sqrt{n}}^x K_{n,x}(t) \Omega_{x-}(h_x, x-t)dt \leq \frac{x}{\sqrt{n}} \Omega_{x-}(h_x, x/\sqrt{n}) \leq \frac{2x}{n} \sum_{k=1}^{\lceil\sqrt{n}\rceil} \Omega_{x-}(h_x, x/k). \quad (20)
\]

From (19) and (20) and using the basic property \(\Omega_{x-}(h_x, \lambda) \leq \Omega(x, h_x, x/\lambda)\) (cf. [8, p. 244]) we get

\[
|L(h, n, x)| \leq \frac{2 - 2x + 2x^2}{n} \sum_{k=1}^{\lfloor\sqrt{n}\rfloor} \Omega(x, h_x, k). \quad (21)
\]

A similar estimate gives

\[
|Q(h, n, x)| \leq \frac{2 - 2x}{n} \sum_{k=1}^{\lfloor\sqrt{n}\rfloor} \Omega(x, h_x, k). \quad (22)
\]

Theorem 2 now follows from Eq. (15), (3), (16), (21), and (22).
3 Asymptotic Optimality of the Estimate in Theorem 2

In this section we show that the estimate in Theorem 2 is essentially the best possible.

Take function \( f(t) = |t - 1/2| \in \Phi_D B \) at point \( x = 1/2 \in (0, 1) \). Then \( f(1/2) = 0, \ r = x/(1-x) = 1, \ h(u) = \text{sgn}(u - 1/2), \ h_{1/2}(u) \equiv 0, \ h(x^+) - h(x^-) = 2, \) and (12) becomes

\[
M_n(|t - 1/2|, 1/2) - \frac{1}{2\sqrt{n\pi}} \leq \frac{2\sqrt{2C}}{n^{3/2}} \tag{23}
\]

On the other hand, by Straightforward computation and Stirling’s formula [9]

\[
n! = (2\pi n)^{1/2} \left(\frac{n}{e}\right)^n e^{\theta}, \quad \left(\frac{1}{12n+1} < \theta < \frac{1}{12n}\right),
\]

we get

\[
M_n(|t - 1/2|, 1/2) = 2 \left(\frac{n + n}{n}\right) \left(\frac{1}{2}\right)^{2n+2} = \left(\frac{2n!}{n!}\right) \left(\frac{1}{2}\right)^{2n+1}
\]

\[
= \frac{\sqrt{2\pi n} (2n/e)^n}{(\sqrt{2\pi n} (n/e)^n)^2} \left(\frac{1}{2}\right)^{2n+1} e^{\theta_2 - \theta_1} = \frac{1}{2\sqrt{\pi n}} e^{\theta_2 - \theta_1}, \tag{24}
\]

where

\[
\frac{1}{24n + 1} < \theta_1 < \frac{1}{24n}, \quad \frac{1}{12n + 1} < \theta_2 < \frac{1}{12n}.
\]

Simple computation gives

\[
\frac{1}{9n} < \frac{2}{12n + 1} - \frac{1}{24n} < 2\theta_2 - \theta_1 < \frac{1}{6n} - \frac{1}{24n + 1} < \frac{1}{6n}. \tag{25}
\]

Thus, from (24) and (25) we have

\[
M_n(|t - 1/2|, 1/2) - \frac{1}{2\sqrt{n\pi}} = \frac{1}{2\sqrt{n\pi}} \left(1 - e^{\theta_2 - \theta_1}\right) = \frac{1}{2\sqrt{n\pi}} e^{2\theta_2 - \theta_1} - \frac{1}{2\sqrt{n\pi}} e^{2\theta_2 - \theta_1} - \frac{1}{2\sqrt{n\pi}} e^{2\theta_2 - \theta_1} < \frac{2\theta_2 - \theta_1}{e^{2\theta_2 - \theta_1}} < \frac{1}{18\sqrt{\pi e}n^{3/2}}.
\]

Eqs. (23) and (26) mean that for \( f(t) = |t - 1/2| \), the following inequality holds

\[
\frac{3}{n} \sum_{k=1}^{\lfloor \sqrt{n} \rfloor} \Omega \left(\frac{1}{2}, h_{1/2}, k\right) + \frac{1/18\sqrt{\pi e}}{n\sqrt{n}} \leq \left|M_n \left(f, \frac{1}{2}\right) - f \left(\frac{1}{2}\right) - \frac{1}{2\sqrt{n\pi}} \right| \leq \frac{3}{n} \sum_{k=1}^{\lfloor \sqrt{n} \rfloor} \Omega \left(\frac{1}{2}, h_{1/2}, k\right) + \frac{2\sqrt{2C}}{n\sqrt{n}} \tag{27}
\]
Inequality (27) shows that the estimate (12) in Theorem 2 is asymptotically optimal.

References


