Constrained Interpolation Using Rational Cubic Spline Curves with Linear Denominators

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June 28, 2007

Abstract

Constrained interpolation with constraint on shape and second derivative is studied for rational cubic spline curves with linear denominators. Necessary and sufficient conditions for a C^1 interpolant to satisfy the constraint have been developed for both cases. The testing conditions are computationally efficient and easy to apply.

Keywords and phrases rational spline, constrained design, constrained interpolation, approximation, shape control

1 Introduction

Design of high quality, manufacturable surfaces is an important yet challenging task in today's manufacturing industries. Although significant progress has been made in the last decade in developing and commercializing production quality CAD tools, demand for more effective tools is still high due to the ever increase in model complexity and the needs to address and incorporate manufacturing requirements in the early stage of surface design. Within this content, *constrained design* has been identified as one of the surface design problems that need to be solved (Y. Chen [1]). This problem deals with control of the bound of curve/surface shape and curvature in the design process. A surface with regions of large curvature may be difficult to produce because the sheet metal may not be able to sustain the stamping tension. Hence, the capability to directly control the bound of curve/surface shape and curvature in the process of design is of significant importance. Having this ability would help reduce both the cost and cycle time of the product development process.

Research results in constrained design are scarce. The only work that seems to be available is a recent manuscript by S. Butt, M. S. Hussain and M. Sarfraz on constrained interpolation (S. Butt [4]). In this work, given a set of data points on the same side (above or below) of a linear function, necessary and sufficient conditions are determined for a rational C^1 cubic spline interpolant to be on the same side of the linear function. The main idea is to consider a rational cubic spline whose denominator for each segment is of degree two so that when the denominator is positive, one can multiply the equation of the straight line by the denominator of the rational cubic spline to convert the interpolant-above-straight-line problem to a positivity problem of cubic polynomials. In this case, the results of Schmidt and Hess (J.W. Schmidt [5]) can be applied directly. As far as constraint on curvature is concerned, nothing seems to be available in the literature yet.

In this paper we consider the constrained curve interpolation problem in a more general sense. We will consider constraint both on the shape and the second derivative of the interpolant. In the first case, we will study necessary and sufficient conditions for a C^1 continuous interpolant to be above (or below) a straight line and/or a quadratic curve in an individual knot interval. In the second case, we will study necessary and sufficient conditions for the second derivative of a C^1 continuous interpolant to be bounded above (or below) in an individual knot interval. The curve representation considered here is similar to the one considered by Sarfraz et al (M. Sarfraz [2,3]), i.e., rational cubic spline curves, with the exception that the denominator is of degree one instead of degree two. Efficient testing conditions will be developed for each case.

The rest of the paper is arranged as follows. In Section 2, we give the general form of the rational cubic spline curves considered in this paper and show some of its properties, including the tri-diagonal system of equations for the construction of such a C^1 rational cubic curve. In Section 3, we discuss the shape constraint problem. In Section 4, we consider the problem of controlling second derivative. In section 5, we give the existence conditions of the interpolation. Concluding remarks are given in Section 6.

2 Rational Cubic Interpolation

Let $f_i \in R, i = 0, 1, \dots n$, be a given set of data points, where $t_0 < t_1 < \cdots < t_n$ is the knot spacing. Also, let $d_i \in R, i = 0, 1, \dots n$, denote the first derivatives defined at the knots. We consider the C^1-continuous, piecewise rational cubic function defined by

$$p(t)|_{[t_i,t_{i+1}]} = \frac{p_i(t)}{q_i(t)}, \tag{1}$$

where

$$\begin{aligned}
p_i(t) &= (1-\theta)^3\alpha_i f_i + \theta(1-\theta)^2 V_i + \theta^2(1-\theta)W_i + \theta^3\beta_i f_{i+1}, \\
q_i(t) &= (1-\theta)\alpha_i + \theta\beta_i, \\
\theta &= (t-t_i)/h_i, \\
h_i &= t_{i+1} - t_i,
\end{aligned}$$

and

$$\begin{aligned}
V_i &= (2\alpha_i + \beta_i) f_i + \alpha_i h_i d_i, \\
W_i &= (\alpha_i + 2\beta_i) f_{i+1} - \beta_i h_i d_{i+1},
\end{aligned}$$

with $\alpha_i, \beta_i > 0$.

$p(t)$ is the standard cubic Hermite interpolant if $\alpha_i = \beta_i$. If $d_i, i = 0, 1, \cdots, n$, are not fixed, we can make $p(t)$ a C^2 rational cubic spline by requiring

$$p''(t_i+) = p''(t_i-)$$

for $i = 1, 2, \cdots, n-1$. The conditions lead to the following tri-diagonal system of linear equations:

$$\begin{aligned} & h_i \frac{\alpha_{i-1}}{\beta_{i-1}} d_{i-1} + (h_i(1 + \frac{\alpha_{i-1}}{\beta_{i-1}}) + h_{i-1}(1 + \frac{\beta_i}{\alpha_i}))d_i + h_{i-1}\frac{\beta_i}{\alpha_i} d_{i+1} \\ = \ & h_{i-1}(1 + 2\frac{\beta_i}{\alpha_i})\Delta_i + h_i(1 + 2\frac{\alpha_{i-1}}{\beta_{i-1}})\Delta_{i-1}; \qquad i = 1, 2, \cdots, n-1 \end{aligned} \tag{2}$$

where

$$\Delta_i = (f_{i+1} - f_i)/h_i.$$

If the knots are equally spaced, equation (2) becomes

$$\begin{aligned} & \frac{\alpha_{i-1}}{\beta_{i-1}} d_{i-1} + (2 + \frac{\alpha_{i-1}}{\beta_{i-1}} + \frac{\beta_i}{\alpha_i})d_i + \frac{\beta_i}{\alpha_i} d_{i+1} \\ = \ & (1 + 2\frac{\beta_i}{\alpha_i})\Delta_i + (1 + 2\frac{\alpha_{i-1}}{\beta_{i-1}})\Delta_{i-1}; \qquad i = 1, 2, \cdots, n-1. \end{aligned} \tag{3}$$

Furthermore, if $\alpha_i = \beta_i$, then (3) becomes the well known tri-diagonal system for cubic spline

$$d_{i-1} + 4d_i + d_{i+1} = 3(\Delta_i + \Delta_{i-1}); \qquad i = 1, 2, \cdots, n-1. \tag{4}$$

From (2) or (3), we can solve for d_i if α_i, β_i and the auxiliary end-condition are given.

In the following, we will study interpolation with constraint on shape and second derivative, respectively, for this rational cubic spline curve representation, and find the sufficient and necessary conditions for the parameters α_i, β_i to satisfy the interpolation requirement.

3 Constrained Interpolation

Given a function $g(x)$ and a data set $\{(t_i, f_i, d_i) : i = 0, 1, \cdots, n\}$ with

$$f_i \geq g(t_i), \qquad i = 0, 1, \cdots, n,$$

let $p(t)$ be a rational cubic Hermite function defined by (1) satisfying the following conditions:

$$p(t_i) = f_i, \quad p'(t_i) = d_i, \qquad i = 0, 1, \cdots, n.$$

If $p(t) \geq g(t)$ for all $t \in [t_0, t_n]$, then $p(t)$ is called a *constrained interpolant* above $g(t)$.

Within this content, we consider the following two cases.

Case 1. Let $g(t)$ be a piecewise linear function defined on $[t_0, t_n]$ with joints at the partition $\Delta : t_0 < t_1 < \cdots < t_n$ and

$$f_i \geq g(t_i), \qquad i = 0, 1, \ldots, n.$$

From (1) we know that $q_i(t) \geq 0$ for $t \in [t_i, t_{i+1}]$, so

$$p(t) = \frac{p_i(t)}{q_i(t)} \geq g(t)$$

is equivalent to

$$p_i(t) - q_i(t)g(t) \geq 0$$

Let

$$U_i(t) = p_i(t) - q_i(t)g(t) \tag{5}$$

We get

$$\begin{aligned} U_i(t) = {} & (1-\theta)^3\alpha_i f_i + \theta(1-\theta)^2 V_i + \theta^2(1-\theta)W_i + \theta^3\beta_i f_{i+1} \\ & - ((1-\theta)\alpha_i + \theta\beta_i)((1-\theta)g_i + \theta g_{i+1}) \geq 0 \end{aligned} \tag{6}$$

where g_i, g_{i+1} represent $g(t_i), g(t_{i+1})$ respectively. Since

$$\begin{aligned} & ((1-\theta)\alpha_i + \theta\beta_i)((1-\theta)g_i + g_{i+1}\theta) \\ = {} & (1-\theta)^2\alpha_i g_i + \theta(1-\theta)(\alpha_i g_{i+1} + \beta_i g_i) + \theta^2\beta_i g_{i+1} \\ = {} & (1-\theta)^3\alpha_i g_i + \theta(1-\theta)^2(\alpha_i g_{i+1} + \beta_i g_i + \alpha_i g_i) + \theta^2(1-\theta)(\alpha_i g_{i+1} + \beta_i g_i + \beta_i g_{i+1}) + \theta^3\beta_i g_{i+1} \end{aligned}$$

(6) becomes

$$U_i(t) = (1-\theta)^3\alpha_i(f_i - g_i) + \theta(1-\theta)^2 A_i + \theta^2(1-\theta)B_i + \theta^3\beta_i(f_{i+1} - g_{i+1}) \geq 0 \tag{7}$$

where

$$\begin{aligned} A_i &= V_i - (\alpha_i g_{i+1} + \beta_i g_i + \alpha_i g_i) \\ &= \alpha_i(2f_i - g_{i+1} - g_i + h_i d_i) + \beta_i(f_i - g_i), \\ B_i &= W_i - (\alpha_i g_{i+1} + \beta_i g_i + \beta_i g_{i+1}) \\ &= \beta_i(2f_{i+1} - g_{i+1} - g_i - h_i d_{i+1}) + \alpha_i(f_{i+1} - g_{i+1}). \end{aligned}$$

If $A_i \geq 0$, and $B_i \geq 0$, since

$$\begin{aligned} U_i(t_i) &= \alpha_i(f_i - g_i) \geq 0 \\ U_i(t_{i+1}) &= \beta_i(f_{i+1} - g_{i+1}) \geq 0 \end{aligned}$$

we have $U_i(t) \geq 0$ for all $t \in [t_i, t_{i+1}]$. Hence, we have the following

THEOREM 1. Given $\{(t_i, f_i, d_i), i = 0, 1, \cdots, n\}$ with $f_i \geq g_i$, the sufficient condition for the rational cubic Hermite spline $p(t)$ to be lying above the piecewise linear function $g(t)$ is that the parameters α_i and β_i satisfy the following linear inequalities :

$$\begin{aligned} A_i &= \alpha_i(2f_i - g_{i+1} - g_i + h_i d_i) + \beta_i(f_i - g_i) \geq 0, \quad (8) \\ B_i &= \alpha_i(f_{i+1} - g_{i+1}) + \beta_i(2f_{i+1} - g_{i+1} - g_i - h_i d_{i+1}) \geq 0. \quad (9) \end{aligned}$$

For a given data set $\{(t_i, f_i, d_i), i = 0, 1, \cdots, n\}$ the corresponding A_i, B_i in the above theorem are called the *criterion numbers* for the rational cubic interpolant to be above the straight line in the subinterval $[t_i, t_{i+1}]$.

Using the method of [5], we can get the sufficient and necessary condition for this interpolation process. Let $\theta = s/(s+1)$, it is easy to see that (7) is equivalent to

$$U_*(s) = \alpha s^3 + \beta s^2 + \gamma s + \delta; \qquad s \geq 0$$

where

$$\begin{aligned} \alpha &= \beta_i(f_{i+1} - g_{i+1}) \\ \beta &= B_i \\ \gamma &= A_i \\ \delta &= \alpha_i(f_i - g_i) \end{aligned}$$

Obviously, $\alpha \geq 0, \delta \geq 0$. So, using the result of [5], we have

THEOREM 2. Given $\{(t_i, f_i, d_i), i = 0, 1, \cdots, n\}$ and $f_i \geq g_i$, the rational cubic spline (1) lies above the straight line in $[t_i, t_{i+1}]$ if and only if the positive parameters α_i, β_i satisfy either

(a) $A_i \geq 0, B_i \geq 0$, or

(b)$4\beta_i(f_{i+1} - g_{i+1})A_i^3 + 4\alpha_i(f_i - g_i)B_i^3 + 27\alpha_i^2\beta_i^2(f_{i+1} - g_{i+1})^2(f_i - g_i)^2$
$- 18\alpha_i\beta_i(f_{i+1} - g_{i+1})(f_i - g_i)A_iB_i - A_i^2B_i^2 \geq 0.$

Case 2 Let $g(t)$ be a quadratic function, and $f_i \geq g(t_i)$. In the same way as in case 1, when $t \in [t_i, t_{i+1}]$, since

$$g_i(t) = (1-\theta)^2 g_i + \theta(1-\theta)(2g_i + g_i' h_i) + \theta^2 g_{i+1}$$

where

$$g_i = g(t_i),\ g_{i+1} = g(t_{i+1}),\ g_i' = g'(t_i),$$

it follows that

$$p(t) = \frac{p_i(t)}{q_i(t)} \geq g(t)$$

is equivalent to

$$U_i(t) = (1-\theta)^3\alpha_i(f_i - g_i) + \theta(1-\theta)^2 C_i + \theta^2(1-\theta)D_i + \theta^3\beta_i(f_{i+1} - g_{i+1}) \geq 0 \qquad (10)$$

where

$$
\begin{aligned}
C_i &= (2\alpha_i + \beta_i)(f_i - g_i) + \alpha_i h_i (d_i - g_i') \\
&= \alpha_i (2f_i - 2g_i + h_i d_i - h_i g_i') + \beta_i (f_i - g_i) \qquad (11) \\
D_i &= (2\beta_i + \alpha_i) f_{i+1} - \alpha_i g_{i+1}) - 2\beta_i g_i - \beta_i h_i (d_{i+1} + g_i') \\
&= \alpha_i (f_{i+1} - g_{i+1}) + \beta_i (2f_{i+1} - 2g_i - h_i d_{i+1} - h_i g_i') \qquad (12)
\end{aligned}
$$

For a given data set $\{(t_i, f_i, d_i), i = 0, 1, \cdots, n\}$, and a given quadratic (maybe piecewise) function g(t), we call the corresponding $C_i and D_i$ as defined in (11) and (12) the *criterion numbers* for the rational cubic interpolant to be above the quadratic curve in the subinterval $[t_i, t_{i+1}]$.

In the same way as in case 1, we can get

THEOREM 3. Let $\{(t_i, f_i, d_i), i = 0, 1, \cdots, n\}$ be a given data set, and $g(t)$ be a given quadratic function satisfying $f_i \geq g_i$. The sufficient condition for the rational cubic Hermite spline $p(t)$ to be lying above the quadratic curve $g(t)$ is that the parameters α_i and β_i satisfy the conditions $C_i \geq 0$ and $D_i \geq 0$.

Furthermore, by setting $\theta = s/(1+s)$, (10) may be write as

$$U^*(s) = \alpha s^3 + \beta s^2 + \gamma s + \delta$$

where

$$
\begin{aligned}
\alpha &= \beta_i (f_{i+1} - g_{i+1}) \\
\beta &= D_i \\
\gamma &= C_i \\
\delta &= \alpha_i (f_i - g_i)
\end{aligned}
$$

THEOREM 4. Let $\{(t_i, f_i, d_i), i = 0, 1, \cdots, n\}$ be a given data set, and g(t) be a given quadratic function satisfying $f_i \geq g_i$. The necessary and sufficient condition for the rational cubic Hermite spline $p(t)$ to be lying above the quadratic curve $g(t)$ in $[t_i, t_{i+1}]$ are that the parameters α_i and β_i satisfy either

(a) $C_i \geq 0, D_i \geq 0$, or

(b) $4\beta_i (f_{i+1} - g_{i+1}) C_i^3 + 4\alpha_i (f_i - g_i) D_i^3 + 27\alpha_i^2 \beta_i^2 (f_{i+1} - g_{i+1})^2 (f_i - g_i)^2$
$- 18\alpha_i \beta_i (f_{i+1} - g_{i+1})(f_i - g_i) C_i D_i - C_i^2 D_i^2 \geq 0.$

4 Constraint on the Second Derivative of the Interpolant

The second derivative of an interpolant has been used in estimating the strain energy and, consequently, smoothness of the interpolant. Smaller energy generally implies smoother shape. However, it is possible that the overall energy of an interpolant is small while great enough to generate abnormal shape at some points or even some small intervals. A better way would be to control the second derivative directly. An effective method can be developed for rational cubic interpolant with linear denominator to restrict its second derivative in a desired interval [N,M].

When $t \in [t_i, t_{i+1}]$, from (1) it is easy to get

$$\begin{aligned}
p''(t) &= \{h_i^2[(1-\theta)\alpha_i + \theta\beta_i]^3\}^{-1} \cdot \\
&\quad \{[(1-\theta)\alpha_i + \theta\beta_i]^2[6(1-\theta)\alpha_i f_i + (6\theta-4)V_i + (2-6\theta)W_i + 6\theta\beta_i f_{i+1}] \\
&\quad - 2(\beta_i - \alpha_i)[(1-\theta)\alpha_i + \theta\beta_i][-3(1-\theta)^2\alpha_i f_i + (1-4\theta+3\theta^2)V_i + 2(\theta-3\theta^2)W_i + 3\theta^2\beta_i f_{i+1}] \\
&\quad + 2(\beta_i-\alpha_i)^2[(1-\theta)^3\alpha_i f_i + \theta(1-\theta)^2 V_i + \theta^2(1-\theta)W_i + \theta^3\beta_i f_{i+1}]\}
\end{aligned}$$

Let $p''(t) \leq M$. We have

$$\begin{aligned}
Q(\theta) &= Mh_i^2[(1-\theta)\alpha_i + \theta\beta_i]^3 + \\
&\quad \{-[(1-\theta)\alpha_i + \theta\beta_i]^2[6(1-\theta)\alpha_i f_i + (6\theta-4)V_i + (2-6\theta)W_i + 6\theta\beta_i f_{i+1}] \\
&\quad + 2(\beta_i-\alpha_i)[(1-\theta)\alpha_i + \theta\beta_i] \cdot \\
&\quad [-3(1-\theta)^2\alpha_i f_i + (1-4\theta+3\theta^2)V_i + (2\theta-3\theta^2)W_i + 3\theta^2\beta_i f_{i+1}] \\
&\quad - 2(\beta_i-\alpha_i)^2[(1-\theta)^3\alpha_i f_i + \theta(1-\theta)^2V_i + \theta^2(1-\theta)W_i + \theta^3\beta_i f_{i+1}]\} \geq 0.
\end{aligned}$$

$Q(\theta)$ is a cubic polynomial of θ

$$Q(\theta) = a\theta^3 + b\theta^2 + c\theta + d \tag{13}$$

with

$$\begin{aligned}
a &= Mh_i^2(\beta_i-\alpha_i)^3 - 2(\beta_i-\alpha_i)^2[(\alpha_i+\beta_i)f_i + (2\alpha_i+5\beta_i)f_{i+1} + h_i\alpha_i d_i - 2\beta_i h_i d_{i+1}], \\
b &= 3Mh_i^2\alpha_i(\beta_i-\alpha_i)^2 - 6\alpha_i(\beta_i-\alpha_i)(\alpha_i f_i + \beta_i f_i + \beta_i f_{i+1} + d_i h_i \alpha_i), \\
c &= 3Mh_i^2\alpha_i^2(\beta_i-\alpha_i) - 6\alpha_i^2[(\alpha_i+\beta_i)f_i - (\alpha_i+\beta_i)f_{i+1} + d_i\alpha_i h_i + \beta_i h_i d_{i+1}], \\
d &= Mh_i^2\alpha_i^3 + 2\alpha_i^2[(\alpha_i+2\beta_i)f_i - (\alpha_i+2\beta_i)f_{i+1} + (\alpha_i+\beta_i)h_i d_i + \beta_i h_i d_{i+1}].
\end{aligned}$$

Let $\theta = s/(1+s)$. It is easy to see that (13) is equivalent to

$$Q^*(s) = \alpha s^3 + \beta s^2 + \gamma s + \delta \geq 0; \qquad s \geq 0$$

where

$$\alpha = \beta_i^3(Mh_i^2 - 2f_i - 10f_{i+1} + 4h_i d_{i+1}) + 2\alpha_i\beta_i^2(5f_{i+1} - 2f_i - 4h_i d_{i+1} - h_i d_i) + 6\alpha_i^2\beta_i f_{i+1}, \tag{14}$$

$$\beta = 3\alpha_i\beta_i^2(Mh_i^2 - 2f_i - 2f_{i+1}) + 6\alpha_i^2\beta_i(f_{i+1} - h_i d_{i+1}) + 6\alpha_i^3 f_{i+1}, \tag{15}$$

$$\gamma = 3\alpha_i^2\beta_i(Mh_i^2 + 2f_i - 2f_{i+1} + 2h_i d_i), \tag{16}$$

$$\delta = 2\alpha_i^2\beta_i(2f_i - 2f_{i+1} + h_i d_i + h_i d_{i+1}) + \alpha_i^3(Mh_i^2 + 2f_i - 2f_{i+1} + 2h_i d_i). \tag{17}$$

One may then construct the corresponding sufficient and necessary condition for the second derivative of the rational cubic Hermite interpolation function $p_i(t)$ defined by (1) to be less than or equal to M as above. It is possible, however, to find the sufficient and necessary condition in a different and yet easier way. Note that

$$Q'(\theta) = 3[(1-\theta)\alpha_i + \theta\beta_i]^2[(\beta_i - \alpha_i)Mh_i^2 + 2(\alpha_i f_i - V_i + W_i - \beta_i f_{i+1})].$$

Hence, $Q(\theta)$ is monotone in $[0,1]$. On the other hand, we have $Q(0) = \delta$ and $Q(1) = \alpha$ where δ and α are defined in (17) and (14), respectively. Therefore, we have

THEOREM 5. For the rational cubic Hermite interpolation function $p_i(t)$ defined by (1), the second derivative $p''(t)$ is less than or equal to M in $[t_i, t_{i+1}]$ if and only if the positive parameters α_i and β_i satisfy the conditions $\delta \geq 0$ and $\alpha \geq 0$ where δ and α are defined in (17) and (14), respectively.

As is known that when $\alpha_i = \beta_i,\ p(t)$ is the standard cubic Hermite interpolation function $H(x)$, and in this case $\alpha, \beta, \gamma, \delta$ defined by $(14)-(17)$ become

$$\begin{aligned}
\alpha &= Mh_i^2 + 6f_{i+1} - 6f_i - 4h_i d_{i+1} - 2h_i d_i &(18)\\
\beta &= 3Mh_i^2 + 6f_{i+1} - 6f_i - 6h_i d_{i+1} &(19)\\
\gamma &= 3Mh_i^2 - 6f_{i+1} + 6f_i + 6h_i d_i &(20)\\
\delta &= Mh_i^2 - 6f_{i+1} + 6f_i + 4h_i d_i + 2h_i d_{i+1} &(21)
\end{aligned}$$

From the standard cubic Hemite interpolation we know that when $t \in [t_i, t_{i+1}]$

$$H''(t) = h_i^{-2}(f_i\phi_0''(\theta) + f_{i+1}\phi_1''(\theta) + h_i d_i \psi_0''(\theta) + h_i d_{i+1}\psi_1''(\theta)) \qquad (22)$$

where

$$\begin{aligned}
\phi_0(\theta) &= (\theta-1)^2(2\theta+1),\\
\phi_1(\theta) &= \theta^2(3\theta-2),\\
\psi_0(\theta) &= \theta(\theta-1)^2,\\
\psi_1(\theta) &= \theta^2(\theta-1).
\end{aligned}$$

Let

$$U(\theta) = h_i^2 M - (f_i\phi_0''(\theta) + f_{i+1}\phi_1''(\theta) + h_i d_i \psi_0''(\theta) + h_i d_{i+1}\psi_1''(\theta)).$$

We have

$$\begin{aligned}
U(0) &= \delta,\\
U(1) &= \alpha,\\
U(1/3) &= \gamma/3,\\
U(2/3) &= \beta/3
\end{aligned}$$

where $\alpha, \beta, \gamma, \delta$ are defined in $(18)-(21)$. Because $U(\theta)$ is a linear function of θ, from the analysis above and Theorem 5 we have

THEOREM 6. For a standard cubic Hermite interpolation function $H(t)$, the sufficient and necessary condition for the second derivative $H''(t)$ to be less than or equal to M in $[t_i, t_{i+1}]$ is that the given data $\{f_i, f_{i+1}, d_i, d_{i+1}\}$ satisfy the following conditions:

$$Mh_i^2 + 6f_{i+1} - 6f_i - 4h_i d_{i+1} - 2h_i d_i \geq 0;$$
$$Mh_i^2 - 6f_{i+1} + 6f_i + 4h_i d_i + 2h_i d_{i+1} \geq 0.$$

5 Existence conditions of the interpolation

In this section, we first discuss the existence conditions for the interpolants in Sections 3. The existence of a constrained rational cubic interpolant $p_i(t)$ satisfying the constraints in $[t_i, t_{i+1}]$ depends on the existence of the solution parameters α_i, β_i of the inequality system (8)&(9) or (11)&(12) for case 1 or case 2, respectively. For simplicity of notations, we shall write the system as follows:

$$a_1\alpha_i + b_1\beta_i \geq 0 \tag{23}$$
$$a_2\alpha_i + b_2\beta_i \geq 0 \tag{24}$$

with $a_1,\ b_1,\ a_2$, and b_2 defined as follows for (8)&(9)

$$\begin{aligned} a_1 &= 2f_i - g_{i+1} - g_i + h_i d_i && (25)\\ b_1 &= f_i - g_i &&\\ a_2 &= f_{i+1} - g_{i+1} &&\\ b_2 &= 2f_{i+1} - g_{i+1} - g_i - h_i d_{i+1} && (26) \end{aligned}$$

and defined as follows for (11)&(12).

$$\begin{aligned} a_1 &= 2f_i - 2g_i + h_i d_i - h_i g_i' && (27)\\ b_1 &= f_i - g_i &&\\ a_2 &= f_{i+1} - g_{i+1} &&\\ b_2 &= 2f_{i+1} - 2g_i - h_i d_{i+1} - h_i g_i' && (28) \end{aligned}$$

By elementary analytic geometry, it is easy to get the following existence conditions for the interpolation.

THEOREM 7. For the constrained rational cubic interpolation discussed in section 3

1) if $f_i > g_i$ for all $i = 0, 1, \dots n$ (i.e. $b_1 > 0, a_2 > 0$), then the set of positive solution parameters α_i, β_i of the inequality system (23)&(24) is nonempty except when $a_1 < 0, b_2 < 0$ and $a_1 b_2 > a_2 b_1$.

2) if $f_i = g_i$ and $f_{i+1} > g_{i+1}$ (i.e. $b_1 = 0$ and $a_2 > 0$) for interval $[t_i, t_{i+1}]$, then the set of positive solution parameters α_i, β_i of the inequality system (19)&(20) is nonempty except when $a_1 < 0$.

3) if $f_i > g_i$ and $f_{i+1} = g_{i+1}$ (i.e. $b_1 > 0$ and $a_2 = 0$) for interval $[t_i, t_{i+1}]$, then the set of positive solution parameters α_i, β_i of the inequality system (19)&(20) is nonempty except when $b_2 < 0$.

4) if $f_i = g_i$ for all $i = 0, 1, \dots n$ (i.e. $b_1 = a_2 = 0$) then the set of positive solution parameters

α_i, β_i of the inequality system (19)&(20) is nonempty if and only if $a_1 \geq 0$ and $b_2 \geq 0$.

As far as the existence condition of the interpolant discussed in section 4 is concerned, we want the positive parameters α_i, β_i for each interpolating interval $[t_i, t_{i+1}]$ to satisfy $p''(t) \leq M$. From Theorem 5, by setting $\lambda_i = \alpha_i/\beta_i$, the conditions $\alpha \geq 0$ and $\delta \geq 0$ become

$$6f_{i+1}\lambda_i^2 + \lambda_i(10f_{i+1} - 4f_i - 8h_i d_{i+1} - 2h_i d_i) + (Mh_i^2 - 2f_i - 10f_{i+1} + 4h_i d_{i+1}) \geq 0 \qquad (29)$$

$$\lambda_i(Mh_i^2 + 2f_i - 2f_{i+1} + 2h_i d_i) + (4f_i - 4f_{i+1} + 2h_i d_i + 2h_i d_{i+1}) \geq 0 \qquad (30)$$

Hence we have

THEOREM 8. For the constrained rational cubic interpolation function discussed in section 4, the sufficient condition for $p(t)'' \leq M$ in $[t_i, t_{i+1}]$ is the inequality system (29) and (30) has positive solution λ_i.

6 Concluding Remarks

The techniques used in Sections 3 for constrained interpolation above a straight line or a quadratic curve can be used for the "below" case as well. Therefore, one may actually consider constrained interpolation between two curves. Similarly, the techniques used in Section 4 can be used for the case that the second derivative is greater than or equal to a given number. Thus one may also consider constrained interpolation in which the second derivative of the interpolating function is bounded both above and below.

The techniques used in Section 3 for C^1 constrained interpolation may be extended to cover C^2 constrained interpolation if condition (2) is used as the smoothness requirement. By combining the techniques used in Sections 3 and 4 one may even consider the construction of constrained convex rational cubic splines. These topics will be discussed in a different paper.

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