Explicit Construction of $C^2$ Surfaces for Meshes of Arbitrary Topology

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Abstract

Presented in this paper is an approach to construct a $C^2$-continuous surface for a mesh of arbitrary topology. The construction process is subdivision surface based, with modification performed on extra-ordinary patches to ensure $C^2$-continuity of the resulting surface. Implementation is easy because modification is patch-based. The resulting surface has an explicit expression of the form $WMG$ for each extra-ordinary patch where $W$ is a parameter vector, $M$ is a constant matrix and $G$ is the patch-wise control point vector. Therefore, computing derivatives, normals and curvatures for points in the domain of the given mesh is very easy and, consequently, the resulting surface is suitable for operations such as shape analysis, shape optimization, surface energy minimization, etc. The construction process includes constraints so that the shape of the resulting $C^2$ surface is very similar to the surface generated by subdivision. More importantly, the resulting $C^2$ surface satisfies the convex hull property. The approach presented in this paper is based on Catmull-Clark subdivision, however we believe the technique can be adopted for other subdivision schemes as well.

1 Introduction

It has been a long desire and a long effort of the computer graphics and geometric design community to have a nice approach to construct smooth surfaces from meshes of arbitrary topology. A nice approach should satisfy the following requirements:

- **simple**: no linear or non-linear system needs to be solved,
- **local**: changes to a control mesh only affect the resulting surface locally,
- **smooth**: the resulting surface is $C^2$ everywhere, including at any extra-ordinary points,
- **convex**: the resulting surface satisfies the convex hull property,
- **explicit**: the resulting surface has an explicit expression of the form $WMG$ for each patch, where $W$ is a parameter vector, $M$ is a constant matrix and $G$ is the control point vector, so that surface evaluation, and computation of the first and second derivatives, normal and curvature at any point can be easily done from the simple representation.

When the degree (valence) of each vertex of the given mesh is 4, the algorithm for generating tensor product B-spline surfaces is such a nice approach. However, for meshes not in this category, as far as we know, there is no such an approach reported in the literature yet, although there are approaches that satisfy almost all of the above requirements [18, 20, 22, 23, 6, 7, 8]. In this paper we propose a new smooth surface construction technique that satisfies all the above requirements. The concept of the new approach is similar to the one presented in Levin’s paper [6], that is, each extra-ordinary patch in a subdivision surface is replaced with a $C^2$ surface patch generated by blending two $C^2$ surface patches together. Both the new approach and Levin’s approach generate a $C^2$ surface that is similar to the surface generated by Catmull-Clark subdivision. The main difference is that the new approach does not need to solve any equation in the construction process, while Levin’s approach needs to solve a linear least square equation for each extra-ordinary patch. Second, our $C^2$ surface is constructed patch by patch, it does not require a global parametrization around an extra-ordinary point. Therefore the new approach
is local and easy to implement. Third, the resulting surfaces produced by Levin’s approach may not satisfy the convex hull property, which is a must-have property in many graphics and geometric design applications. The new approach guarantees the resulting surface is bounded by its convex hull. Finally, the new approach can represent a resulting surface with a simple matrix form $WMG$, where $W$ is a parameter vector, $M$ is a constant matrix and $G$ is the control point vector. With such an explicit matrix representation, one can easily find the location, partial derivatives, normal vector, and curvature for any point in the domain of the given mesh, including an extra-ordinary point.

2 Previous Work

The topic of smooth surface construction has been studied extensively [11, 12, 13, 15, 24, 7, 8, 9]. Many smooth surface construction methods have been proposed for meshes of arbitrary topology. Basically these methods can be divided into two categories: piecewise polynomial schemes [1, 2, 19] and non-polynomial schemes [22, 6]. The most famous type among the piecewise polynomial schemes is the subdivision schemes [4, 10, 19]. In the last decade, subdivision surfaces have become popular in graphics, geometric modeling and computer animation [3] because of their relatively high visual quality, numerical stability, simplicity in coding and, most importantly, their capability in modeling any complex shape with only one surface [21]. They are widely used for representing models of irregular topology. However, most of the general subdivision schemes suffer from irregularities at the extra-ordinary points. For example, although Catmull-Clark surfaces are $C^2$-continuous almost everywhere, they are only $C^1$-continuous at the extra-ordinary points.

Some techniques have been reported to improve the smoothness of a subdivision surface at extra-ordinary points. In [18], an algorithm is designed to generate $C^2$ surface everywhere. But the curvature at an extra-ordinary point is forced to be zero, resulting in a flat-spot. TURBS presented in [20] constructs $C^k$ continuous surfaces and in [22], $C^\infty$ surfaces can be constructed by blending polynomial patches with exponentials. Box spline is adapted to form $C^2$ surfaces on an infinite mesh with a single extra-ordinary point [23]. To directly improve the limit surface, Levin [6] perturbed Catmull-Clark surfaces using polynomial blending functions between local polynomial patches; Zorin [7] similarly perturbed Loop subdivision surfaces to be $C^2$ using a blending function that is itself a subdivision surface.

There are also other algorithms reported to improve smoothness by directly converting meshes to splines. For example, free-form splines [18, 16] are used to build $C^k$ surfaces. In [13, 25] curvature continuous surfaces are built from quad meshes using bi-degree 7 patches, setting extra parameters by minimizing deviation from bi-degree 3 patches. In [14, 26] guided subdivision is introduced, which is capable of constructing $C^k$ surfaces.

In general, non-polynomial schemes can yield $C^2$ or even smoother surfaces. For example, the approach presented in [6] can generate everywhere $C^2$ smooth surface, but it does not satisfy the convex hull property. Recently, a polar subdivision technique [8, 9] has been proposed. This new subdivision technique can generate smooth surfaces that are curvature continuous with good curvature distribution near extra-ordinary points. But this technique may only be applied to meshes with polar configurations.

2.1 Previous Works on Extraordinary Points

Many researches have been performed to improve the smoothness of a CCSS at extraordinary points.

Prautzsch [17] modifies the subdivision scheme near extraordinary points to generate a $C^2$ everywhere surface with zero curvature at extraordinary points.

Zorin [7] and Levin [6] present schemes to yield a $C^2$ continuous surface by blending the limit surface with a low degree polynomial defined over the characteristic map in the vicinity of each extraordinary point.

Loop and Schaefer [25] present a second order smooth filling of an n-valence Catmull-Clark spline ring with n biseptic patches, with shape optimization for free parameters.

Peters and Karciauskas [29] introduce a guided subdivision scheme that uses a Bezier surface as a guide for each subdivision step, and a $C^2$ accelerated Bi-3 guided subdivision that uses $2^m$ surfaces in the m-th level for surface patches surrounding extraordinary points. In the second case, they show that although this scheme is not practical for Catmull-Clark surfaces, it can be applied to a polar configuration.
However, these solutions are not completely satisfactory yet. Blending the limit surface with a precomputed curvature continuous surface patch is not flexible in surface representation. Filling the holes with bidegree-6 patches will result in higher gaussian curvature near the extraordinary points and make the limit surface unattractive. The bi-cubic subdivision scheme that generates $2^m$ subpatches in the $m$-th subdivision is also undesired.

3 Basic Idea

The basic idea of our approach is that for every patch $P_i$ around an extra-ordinary vertex $V$ of degree $n$, $1 \leq i \leq n$, we construct two $C^2$-continuous patches $S_i$ and $T_i$ (See Figure 1) in a way such that

- $S_i$ is $C^2$-continuously connected with $S_{i-1}$ and $S_{i+1}$, except at $V_\infty$, where it is $C^0$,
- $S_i$ is connected to $P_i$ at $C_i$ with $C^2$-continuity, where $C_i$ is the intersection curve of $S_i$, $T_i$ and $P_i$,
- $T_i$ is $C^2$-continuously connected with $T_{i-1}$ and $T_{i+1}$,
- all $T_i$’s are $C^2$-continuously connected at the extra-ordinary point $V_\infty$,
- $T_i$ is connected to $P_i$ at $C_i$ with $C^0$-continuity.

Note that if $S_i$ and $T_i$ are constructed this way, then a surface obtained by linearly blending $S_i$ and $T_i$ together is $C^2$-continuous everywhere. The key is how to construct $S_i$ and $T_i$, for $1 \leq i \leq n$.

4 Construction of $S_i$

For a given mesh, we assume that all the faces are quadrilaterals and all the extra-ordinary vertices are separated by at least two faces. If it is not the case, simply perform (at most) two Catmull-Clark subdivisions to reach such a status. We consider all the patches $P_i$ around an extra-ordinary vertex $V$ of valance $n$, $1 \leq i \leq n$. It is well known that $P_i$ depends on its surrounding $2n + 8$ vertices only [5]. See figure 2(a) for notation of these vertices. One can split $P_i$ into four pieces (See Figure 2(b)) by performing one subdivision on $P_i$. Three of these four pieces can be represented explicitly as follows.

Let $G_i = [V, E_1, \cdots, E_n, F_1, \cdots, F_n, I_1, \cdots, I_7]^T$. Vertices for $G_i$ can be identified similarly from the notation given in Figure 2(a). Let

$$W(u, v) = [1, u, v, u^2, uv, v^2, u^3, u^2v, uv^2, v^3, u^3v, u^2v^2, uv^3, u^3v^2, u^2v^3, u^3v^3].$$

Then $P_i$ can be defined as follows.

$$P_i(u, v) = \begin{cases} 
\text{something we do not need}, & [0, \frac{1}{2}] \times [0, \frac{1}{2}] \\
W(2u - 1, 2v)M_1K_1AG_i, & [\frac{3}{4}, 1] \times [0, \frac{1}{2}] \\
W(2u - 1, 2v - 1)M_2K_2AG_i, & [\frac{7}{8}, 1] \times [\frac{1}{2}, 1] \\
W(2u, 2v - 1)M_3K_3AG_i, & [0, \frac{1}{2}] \times [\frac{1}{2}, 1] 
\end{cases}$$

Figure 1: Basic idea.
where \( M_4 \) is the B-spline tensor matrix of size \( 16 \times 16 \), \( K_1, K_2, K_3 \) are constant picking matrices of size \( 16 \times 24 \), each of which picks 16 proper vertices from the mesh if one subdivision is performed on patch \( P_i \) (See Figure 2(b)). Matrix \( A \) is the extended Catmull-Clark subdivision matrix [5] which is of size \( 24 \times (2n + 8) \).

Now define \( C_i(t) = P_i(\cos t, \sin t) \), \( t \in [0, \pi/2] \). Let \( L_i(r, t) = P_i(r \cos t, r \sin t) \). Then \( L_i' (1, t) = \frac{\partial L_i(r, t)}{\partial r} \big|_{r=1} \), \( L_i'' (1, t) = \frac{\partial^2 L_i(r, t)}{\partial r^2} \big|_{r=1} \) are the first and second derivatives of \( P_i \) at \( C_i(t) \) with respect to \( r \), respectively.

Denote the limit point of \( V \) by \( V_\infty \). It is well known [5] that \( V_\infty = \frac{1}{n (n + 3)} (n^2 V + 4 \sum E_i + \sum F_i) \). Let \( R = [1, r, r^2, r^3] \), then we can construct a Bézier curve as follows such that it has the same first and second derivatives at \( C_i(t) \) as those of \( P_i \) at \( C_i(t) \).

\[
S_i(r, t) = R M_b [ V_\infty, L_i(1, t) - \frac{3}{2} L_i'(1, t) + \frac{9}{7} L_i''(1, t), L_i(1, t) - \frac{1}{4} L_i'(1, t), L_i(1, t)]^T,
\]

where \( 0 \leq r \leq 1, \ 0 \leq t \leq \pi/2 \) and \( M_b \) is the Bézier matrix.

From Eq. (2) and the definition of \( L_i(r, t) \), we know that \( L_i'(1, t) = \cos t P_i^u(\cos t, \sin t) + \sin t P_i^v(\cos t, \sin t) \) and \( L_i''(1, t) = \cos^2 t P_i^{uu}(\cos t, \sin t) + \sin^2 t P_i^{uv}(\cos t, \sin t) + 2 \cos t \sin t P_i^{uv}(\cos t, \sin t) \), where \( P_i^u, P_i^v, P_i^{uu}, P_i^{uv}, P_i^{vv} \) are the first and second partial derivatives of \( P_i \) (See Eq. (2)). We can see that \( S_i \) is a linear combination of \( G_i \) with parameters \( t, \cos t \) and \( \sin t \). Hence \( S_i \) can be represented in matrix form. Based on Eq. (1), we define \( \mathcal{W}_t = W(\cos t, \sin t) \) and \( \mathcal{W}(r, t) = [\mathcal{W}_t, r \mathcal{W}_t, r^2 \mathcal{W}_t, r^3 \mathcal{W}_t] \). If we plug \( L_i, L_i' \) and \( L_i'' \) into Eq. (3) and fully expand the formula, we get a matrix form representation for \( S_i \) as follows.

\[
S_i(r, t) = \mathcal{W}(r, t) \tilde{M}_n G_i, \ 0 \leq r \leq 1, \ 0 \leq t \leq \pi/2,
\]

where \( \tilde{M}_n \) is a constant matrix of size \( 64 \times (2n + 8) \) and \( \tilde{M}_n \) can be pre-calculated for each \( n \).

## 5 Proof of \( C^2 \) between \( S_i \)’s and \( P_i \)’s

\( S_i(r, t) \), when \( t \) is fixed, is a Bézier curve of degree three with \( S_i(1, t) = L_i(1, t) \), \( \frac{\partial S_i(r, t)}{\partial r} \big|_{r=1} = L_i'(1, t) \) and, \( \frac{\partial^2 S_i(r, t)}{\partial r^2} \big|_{r=1} = L_i''(1, t) \). When \( t \) varies, \( S_i(r, t) \) is a surface and we can similarly find \( L_i'(1, t), L_i''(1, t), L_i'''(1, t) \). For example, \( L_i'(1, t) = -\sin t P_i^u(\cos t, \sin t) + \cos t P_i^v(\cos t, \sin t) \). These are the directional partial derivatives of \( S_i(r, t) \) at \( C_i(t) \) in \( r \) and \( t \) directions. And by design, they are also the directional partial derivatives of \( P_i \) at \( C_i(t) \) in \( r \) and \( t \) directions. Hence \( S_i \) and \( P_i \) have the same position, same first and second partial derivatives at the curve \( C_i(t) \) in \( r \), \( t \) and \( rt \) directions. According to the second fundamental form of differential geometry, we obtain that \( S_i \) and \( P_i \) have the same first and second partial derivatives at any point of \( C_i(t) \) in any direction. Hence \( S_i(r, t) \) is connected with \( P_i \) at curve \( C_i(t) \) with \( C^2 \) smoothness.

To prove that when \( r \neq 0, S_i \) and \( S_{i-1} \) are connected with \( C^2 \), from the definition of \( S_i(r, t) \), we just need to show that in \( t \) direction, \( L_i(1, t), L_i'(1, t) \) and \( L_i''(1, t) \) are \( C^2 \) continuous with \( L_{i-1}(1, t), L_{i-1}'(1, t) \) and \( L_{i-1}''(1, t) \),
respectively. From the definition of \( L_i(r, t) \), we know that when \( r \) and \( t \) vary, \( L_i(r, t) \) becomes \( P_i \). Because \( P_i \) is \( C^2 \) everywhere except \((0, 0)\), by finding the corresponding derivatives, one can verify that, \( L_i(1, 0) = L_{i-1}(1, \pi/2) \), \( L_i'(1, 0) = L_{i-1}'(1, \pi/2) \), \( L_i''(1, 0) = L_{i-1}''(1, \pi/2) \), \( L_i''(1, 0) = L_{i-1}''(1, \pi/2) \), \( L_i''(1, 0) = L_{i-1}''(1, \pi/2) \), and \( L_i''(1, 0) = L_{i-1}''(1, \pi/2) \). Hence The \( C^2 \)-continuity between \( L_i(1, t) \) and \( L_{i-1}(1, t) \), \( L_i'(1, t) \) and \( L_{i-1}'(1, t) \), and, \( L_i''(1, t) \) and \( L_{i-1}''(1, t) \), is well known that \( P \) is as continuous as \( c \). Due to the fact that \( \omega = \cos(2\pi \omega/n) \), \( \Gamma_2, \Gamma_3, \Gamma_8, \Gamma_9, \Gamma_6 \) can be obtained by switching column \( k \), \( 1 \leq k \leq n/2 \), with column \( n-k+1 \) in the matrix \( \Gamma_0, \Gamma_1, \Gamma_4, \Gamma_5, \Gamma_7 \), respectively (please see supplemental materials).

To simplify the notation, we define \( D_i' = (D_i'' + D_i'''_1)/2 \) and \( D_i'' = (D_i''' + D_i''''_1)/2 \). With these two definitions, hereafter, when there is no possibility to get into confusion, we just say \( D_i' \) (or \( D_i'' \)) is the first (or second) partial derivative along the edge \( V \rightarrow E_i \). Due to the fact that \( c_\omega \) is \( c_{n-\omega} \) and \( c_{n-\omega} \), using Eq. (5), one can easily verify that when \( n \) is even, \( D_i' = -D_i'(n/2) \) and \( D_i'' = -D_i''(n/2) \). This means when \( n \) is even, all the first/second partial derivatives are symmetric with respect to the point \( V_{\infty} \).

### 7 Construction of \( T_i \)

Recall that the requirements for the construction of \( T_i \) are that \( T_i \) itself has to be \( C^2 \) everywhere, \( C^2 \) with its neighboring patches \( T_i \) and \( T_i+1 \) including at \((0,0)\), and at least \( C^0 \) with \( C_i(t) \). There are many ways to construct \( T_i \). One simple way is to construct it as a Bezier patch, using an approach similar to the one given in the above section. For example, if we use two coplanar circles for all the \( B_i(t) \)’s and \( H_i(t) \)’s in Figure 3(a) and
let \( R = [1, r, r^2, r^3] \), then the Bézier curve
\[
T_i(r, t) = RM_b[V_\infty, B_i, H_i, C_i]^T, \quad 0 \leq r \leq 1,
\]
becomes a surface when \( t \) varies, and this surface satisfies all the above requirements if the radius of \( H_i \) is two times the radius of \( B_i \). Note that two Bézier curves constructed from \([V_\infty, B, H, C]\) and \([V_\infty, \hat{B}, \hat{H}, \hat{C}]\) are \( C^2 \) smoothly connected at \( V_\infty \) if and only if (1) \( B, V_\infty, \) and \( \hat{B} \) are collinear, (2) \( V_\infty \) is the midpoint of \( B \) and \( \hat{B} \) and, (3) \( \hat{H} = H + 4(V_\infty - B) \). The above defined \( T_i(r, t) \) satisfies all the conditions because the two coplanar circles are smooth and symmetric with respect to \( V_\infty \). However, the resulting surface from this \( T_i(r, t) \) may not be the one the designer wants. So we need more constraints on \( B_i(t) \) and \( H_i(t) \). In the following, we will construct a \( T_i \) that is similar to the original subdivision surface \( P_i \) at the extra-ordinary point by requiring that \( T_i \) and \( P_i \) have the same location, same first and second derivatives at \( V_\infty \).

The basic idea is again to construct Bézier curves that pass through \( V_\infty \) and have the same partial derivatives at \( V_\infty \) as \( P_i \). This is done through four steps (see Figure 3(a)). First, we construct a B-spline curve \( B_i(t) \) around the extra-ordinary point using the first partial derivative vectors along each edge of the extra-ordinary point. Second, we construct another B-spline curve \( H_i(t) \) around the extra-ordinary point using the second partial derivative vectors along each edge of the extra-ordinary point. Third, find four control points for a Bézier curve that is similar to the original subdivision surface \( P_i \) at the extra-ordinary point by requiring that \( T_i \) and \( P_i \) have the same first and second derivatives at \( V_\infty \).

![Figure 3: Using Bézier curve to construct \( T_i \).](image-url)

First \( B_i(t) \) and \( H_i(t) \) can be explicitly constructed as follows. When \( n \) is even, we use the partial derivatives to define \( B_i \) and \( H_i \) directly:
\[
B_i(t) = V_\infty + \bar{g}(t)M_s\alpha_n[D'_{i-1}, D'_i, D''_{i+1}, D''_{i+2}]^T,
\]
and
\[
H_i(t) = V_\infty + \bar{g}(t)M_s\beta_n[D''_{i-1}, D''_i, D'''_{i+1}, D'''_{i+2}]^T.
\]
where \( \bar{g}(t) = [1, t, t^2, t^3] \), \( 0 \leq t \leq 1 \), \( \alpha_n \) and \( \beta_n \) are two constant coefficients, \( M_s \) is the B-spline matrix, \( 1 \leq i \leq n \). When \( n \) is odd, we add one more control point between each pair of consecutive derivatives, say the \( i \)th and \((i + 1)\)th derivatives, by reversing the \((i + (n + 1)/2)\)th derivatives (See Figure 3(b)). Each of \( B_i \) and \( H_i \) is then defined as a set of two piecewise B-spline curves, as follows. When \( n \) is odd and \( 0 \leq t \leq \frac{1}{2} \),
\[
B_i(t) = V_\infty + \bar{g}(2t)M_s\alpha_n[-D'_{i-1 + \frac{n-1}{2}}, D'_i, -D'_{i+1 + \frac{n+1}{2}}, D'_{i+1}]^T,
\]
\[
H_i(t) = V_\infty + \bar{g}(2t)M_s\beta_n[-D''_{i-1 + \frac{n-1}{2}}, D''_i, -D''_{i+1 + \frac{n+1}{2}}, D''_{i+1}]^T.
\]
When \( n \) is odd and \( \frac{1}{2} \leq t \leq 1 \),
\[
B_i(t) = V_\infty + \bar{g}(2t - 1)M_x\alpha_n[D'_i, -D'_i, \frac{1}{2}D'_i + D'_i, -D'_i, \frac{1}{2}D'_i + D'_i]T,
\]
\[
H_i(t) = V_\infty + \bar{g}(2t - 1)M_x\beta_n[D''_i, -D''_i, \frac{1}{2}D''_i + D''_i, -D''_i, \frac{1}{2}D''_i + D''_i]T,
\]
where \( \bar{g}(t) \), \( \alpha_n \), \( \beta_n \) and \( M_x \) are defined the same as the even case. In both cases, \( \alpha_n \) (or \( \beta_n \)) is chosen in a way such that when \( V_\infty + \alpha_nD'_k \) (or \( V_\infty + \beta_nD''_k \)), \( 1 \leq k \leq n \), is represented by a linear combination of the vertices of \( G_i \), the coefficients of the representation are non-negative. To satisfy this requirement, from Eq. (5), we can find that the proper ranges are \( 0 \leq \alpha_n \leq \bar{\alpha}_n \) and \( 0 \leq \beta_n \leq \bar{\beta}_n \), where
\[
\bar{\beta}_n = f/\left[ \frac{1}{8}A_1(\Gamma_2|c_2, c_1, c_n, c_n-2|T + \right.
\]
\[
\Gamma_2[c_2, c_2, c_1, c_n, c_n-1]|T^2),
\]
when \( n \) is even,
\[
\bar{\alpha}_n = f/\left[ \frac{1}{4}A_1(\Gamma_2|c_2+\frac{1}{2}, c_1+\frac{1}{2}, c_1, c_n-2|T + \Gamma_3[c_2, c_2+\frac{1}{2}, c_2+\frac{1}{2}, c_2+\frac{1}{2}]|T^2)ight],
\]
and when \( n \) is odd,
\[
\bar{\alpha}_n = f/\left[ \frac{1}{2}A_1(\Gamma_2|c_2+\frac{1}{2}, c_1, c_1, c_n-2|T + \Gamma_3[c_2+\frac{1}{2}, c_1+\frac{1}{2}, c_1, c_n-2]|T^2))
\]
where \( f = \frac{\delta(c+1)}{(4g-1)(4n+5)} \). All the symbols in the above equations have the same values as those in Eq. (5). For each \( n \), \( \bar{\alpha}_n \) and \( \bar{\beta}_n \) are constants and can be pre-calculated. \( \alpha_n \) and \( \beta_n \) can be used to adjust the final surface appearance around an extra-ordinary point as well. In our testing, we choose \( \alpha_n = (1/2)\bar{\alpha}_n \) and \( \beta_n = \bar{\beta}_n \).

Now we can define \( T_i \) using basic Bézier curves as follows.
\[
T_i(r, t) = R M_{\beta} \bar{V}_\infty + \frac{1}{2} B_i(\frac{2t}{\pi}),
\]
\[
\frac{1}{2} V_\infty + \frac{1}{2} B_i(\frac{2t}{\pi}), C_i(\frac{2t}{\pi})]T,
\]
where \( 0 \leq r \leq 1, \ 0 \leq t \leq \pi/2 \).

From Eq. (5) we know that all \( D'_i \) and \( D''_i \) can be represented by a linear combination of \( G_i \). Hence \( B_i \) and \( H_i \) can be represented by a linear combination of \( G_i \) as well. We already know \( C_i \) and \( V_\infty \) can be represented by a linear combination of \( G_i \) in Section 4. Hence if fully expanded, \( T_i(r, t) \) can be represented with the following matrix form.
\[
T_i(r, t) = \tilde{W}(r, t)\tilde{M}_n G_i, \quad 0 \leq r \leq 1, \quad 0 \leq t \leq \pi/2,
\]
where \( \tilde{W} \) is defined in Section 4 and \( \tilde{M}_n \) is a constant matrix of size \( 64 \times (2n + 8) \). \( \tilde{M}_n \) can be pre-computed for each \( n \).

8 Proof of \( C^2 \) among all \( T_i \)’s

Define \( B(t) \) to be the curve consisting of all the \( B_i \)’s, and \( H(t) \) to be the curve consisting of all the \( H_i \)’s, \( 1 \leq i \leq n \). It is obvious that \( B(t) \) and \( H(t) \) are \( C^2 \) everywhere because they are piecewise B-spline curves. In addition, as proven in section 4, \( C(t) \) is also \( C^2 \) everywhere. Define \( T(r, t) \) to be the union of all \( T_i \)’s, \( 1 \leq i \leq n \). Because \( T_i \), as defined in Eq. (6), only depends on \( B_i(t), H_i(t) \) and \( C_i(t) \), \( T(r, t) \) is only depending on \( B(t), H(t) \) and \( C(t) \), which are all \( C^2 \) continuous curves. Therefore, \( T(r, t) \) is \( C^2 \) continuous everywhere, except \((0, 0)\). This means \( T_i \) is \( C^2 \) continuous with \( T_{i-1} \) and \( T_{i+1} \) everywhere, except \((0, 0)\).

To prove \( T \) is \( C^2 \) at \((0, 0)\), we just need to prove that, for any \( t \), there exists a 3D plane \( P_t \), such that \( P_t \) passes through \( V_\infty \), \( B(t) \) and \( H(t) \), and the intersection curve of \( T \) and \( P_t \) is \( C^2 \) at \( V_\infty \). Note that for any \( t \), \( T(0, 0) = T(0, t) = V_\infty \). From Eq. (6), for any \( t \), \( 0 \leq t \leq \pi/2 \) and \( i \), \( 1 \leq i \leq n \), we have
\[
T_i(0, t) = B_i(\bar{t}) - V_\infty, \quad \text{and} \quad T_{ii}^r(0, t) = H_i(\bar{t}) - V_\infty.
\]
where $t = 2t/\pi$.

When $n$ is even, because of the symmetry of $B(t)$ and $H(t)$, we have that $V_\infty$ is the midpoint of $B_i(t)$ and $B_{i+n/2}(t)$, and $V_\infty$ is the midpoint of $H_i(t)$ and $H_{i+n}(t)$. As a result, $V_\infty$, $B_i(t)$, $B_{i+n/2}(t)$, $H_i(t)$ and $H_{i+n/2}(t)$ are on the same plane $\mathcal{P}_t$. Because $T_{rr}^{r}(0, t) = B_i+n/2(t) - V_\infty = -T_i(t) - V_\infty = -T_i^{r}(0, t)$, we have that the intersection curve of the plane $\mathcal{P}_t$ and the surface $T$ is $C^2$ at $V_\infty$.

When $n$ is odd, it can be proven similarly except there are two cases. Again because of the symmetry of $B(t)$ and $H(t)$, when $0 \leq t \leq \pi/4$, we have that $V_\infty$ is the midpoint of $B_i(t)$ and $B_{i+(n-1)/2}(t + 1/2)$, and $V_\infty$ is the midpoint of $H_i(t)$ and $H_{i+(n-1)/2}(t + 1/2)$. As a result, $V_\infty$, $B_i(t)$, $B_{i+(n-1)/2}(t + 1/2)$, $H_i(t)$ and $H_{i+(n-1)/2}(t + 1/2)$ are on the same plane $\mathcal{P}_t$. Because $T_{rr}^{r}(0, t+\pi/4) = B_i+(n-1)/2(t+1/2) - V_\infty = -T_i^{r}(0, t)$ and $T_{rr}^{r}(0, t+1/2) - V_\infty = -T_i^{r}(0, t)$, we have that the intersection curve of the plane $\mathcal{P}_t$ and the surface $T$ is $C^2$ at $V_\infty$. When $\pi/4 \leq t \leq \pi/2$, we know that $V_\infty$ is the midpoint of $B_i(t)$ and $B_{i+(n+1)/2}(t - 1/2)$, and $V_\infty$ is the midpoint of $H_i(t)$ and $H_{i+(n+1)/2}(t - 1/2)$. As a result, $V_\infty$, $B_i(t)$, $B_{i+(n+1)/2}(t - 1/2)$, $H_i(t)$ and $H_{i+(n+1)/2}(t - 1/2)$ are on the same plane $\mathcal{P}_t$. Because $T_{rr}^{r}(0, t-\pi/4) = B_i+(n+1)/2(t - 1/2) - V_\infty = -T_i^{r}(0, t)$ and $T_{rr}^{r}(0, t-1/2) - V_\infty = -T_i^{r}(0, t)$, we have that the intersection curve of the plane $\mathcal{P}_t$ and the surface $T$ is $C^2$ at $V_\infty$. Therefore $C^2$ continuity of $T(s, t)$ at $(0, 0)$ is proven.

Also from Eq. (6), for any $i$ and any $t$, we have

$$T_i^t(0, t) = 0, \quad T_i^{tt}(0, t) = 0, \quad \text{and} \quad T_i^{rr}(0, t) = \frac{2}{\pi} B_i^t(\hat{t}).$$

One can easily verify that $B_i^t$, which is the first derivative of $B_i$ with respect to parameter $t$, is symmetric relative to $V_\infty$ as well. Note that, when $r = 0$, $T_i(r, t)$ becomes a point for all $t$. As a result, when $r = 0$, the $t$ direction collapses into a single point. Although for any $t$, $T_i^t(0, t) = T_i^{tt}(0, t) = 0$, the curvature at $T_i(0, 0)$ is not necessarily equal to 0 because the partial derivatives at $T_i(0, 0)$ in the $r$ direction (which are $T_r^r(0, 0) = B_i(0) - V_\infty$ and $T_r^r(0, 0) = H_i(0) - V_\infty$) are not necessarily 0. Hence it is not a flat spot at the extra-ordinary point. To calculate the normal vector at $T_i(0, 0)$, instead of using $T_i^t(0, 0)$, which is 0, we can use $T_i^r(0, 0)$ and $T_i^r(0, \pi/2)$.

### 9 Blending $T_i$ with $S_i$

To construct a $C^2$ patch $Q_i(r, t)$ in the $i$th face around an extra-ordinary vertex $V$ of valance $n$, we first construct $T_i$ and $S_i$ using the methods given in the previous sections and then blend them together smoothly with a $C^2$ continuous blending function as follows.

$$Q_i(r, t) = r^2S_i(r, t) + (1 - r^2)T_i(r, t) = r\mathcal{W}_iG_i + (1 - r)\mathcal{W}\mathcal{M}_nG_i = \mathcal{W}\mathcal{M}_nG_i,$$

where $0 \leq r \leq 1$, $0 \leq t \leq \pi/2$, $\mathcal{W} = [\mathcal{W}_i, r\mathcal{W}_i, r^2\mathcal{W}_i, r^3\mathcal{W}_i, r^4\mathcal{W}_i]$ and $\mathcal{M}_n$ is a constant coefficient matrix of size $80 \times 2(n + 8)$. $\mathcal{W}_i$ is defined in section 4. $\mathcal{M}_n$ can be pre-computed for each $n$ involved.

Although other weight functions can be used in the blending process, in our testing, we simply use linear weights and they give satisfactory results and also simplify the calculation of matrix $\mathcal{M}_n$. If $Q(r, t)$ is defined to be the union of all the $Q_i(r, t)$, then $Q(r, t)$ is $C^2$ everywhere including $(0, 0)$ because all the $S_i(r, t)$’s and all the $T_i(r, t)$’s are connected with $C^2$ smoothness. Eq. (8) is the most important result of this paper. It gives us a direct and explicit way to construct a $C^2$ smooth surface for any extra-ordinary patch. It also gives us a simple way to calculate the partial derivatives and curvature of an extra-ordinary patch at any parameter point, including $(0, 0)$, by simply calculating the partial derivatives of $\mathcal{W}$. Therefore Eq. (8) can be effectively used for surface evaluation, shape analysis, optimization, energy calculation ... etc.

Now we can define a new $C^2$ patch $\hat{P}_i(u, v)$ to replace the whole patch $P_i(u, v)$, as follows.

$$\hat{P}_i(u, v) = \begin{cases} P_i(u, v), & \text{when } u^2 + v^2 \geq 1, \\ Q_i(r, t), & \text{when } u^2 + v^2 \leq 1, \end{cases}$$

$$8$$
where $0 \leq u, v \leq 1$ and $u = r \cos t$, $v = r \sin t$.

It is clear that $\hat{P}_i(u, v)$ is $C^2$ itself and $C^2$ with its neighboring patches. Note that from Eq. (2) one can see that $P_i(u, v)$, when $u^2 + v^2 = 1$, can also be represented by a matrix form $W M_n G_i$, where $W$ is defined in section 4, $M_n$ is a constant matrix of size $16 \times (2n + 8)$ and can be pre-calculated as well. Hence at any parameter point $(u, v)$, $\hat{P}_i(u, v)$ and its derivatives can be calculated explicitly using just simple matrix operations.

## 10 Proof of Satisfying Convex Hull Property

From Eq. (9) and the definition of $Q_i(r, t)$ we can see that we only need to show that $S_i$ and $T_i$ satisfy the convex hull property. From Eq. (3), we can see that $S_i$ depends on $V_\omega$, $L_i(1, t)$, $L'_i(1, t)$ and $L''_i(t, 1)$. $V_\omega$ and $L_i(1, t)$ are on the surface $P_i$, hence they are within the convex hull of $G_i$, i.e., $V_\omega = \sum a_k G_{i,k}$ such that $\sum a_k = 1$ and $a_k \geq 0$ for $1 \leq k \leq 2n + 8$. $L_i(1, t) = \sum a_k G_{i,k}$ such that $\sum a_k = 1$ and $\bar{a}_k \geq 0$ for $1 \leq k \leq 2n + 8$. Also because $L'_i(1, t)$, and $L''_i(t, 1)$ are derivatives, they do not have absolute locations. If $G_i$ is translated to another location, $L'_i(1, t)$, and $L''_i(t, 1)$ will be the same. Hence we have $L'_i(1, t) = \sum \bar{a}_k G_{i,k}$ such that $\sum \bar{a}_k = 0$ for $1 \leq k \leq 2n + 8$. If we plug them into Eq. (3), we have $S_i = \sum_{k=1}^{2n+8} F_k(r, t) G_{i,k}$, where $F_k(r, t) = (1 - r)^3 a_k + r(r^2 - 3r + 3) \bar{a}_k - r(1 - r)(2 - r) \bar{a}_k + (1/2)r(1 - r)^2 \bar{a}_k$. It is easy to verify that for any $r$ and $t$, $\sum_{k=1}^{2n} F_k(r, t) = 1$ due to the fact that $\sum a_k = \sum \bar{a}_k = 1$ and $\sum \bar{a}_k = 0$. Hence to prove $S_i$ satisfies the convex hull property, we just need to prove $F_k(r, t)$ is always non-negative.

From Eq. (4), we have $F_k(r, t) = \tilde{W}(r, t) \tilde{M}_{n,k}$ where $\tilde{M}_{n,k}$ is the $k$th column of the constant matrix $\tilde{M}_n$.

Hence we know that $F_k(r, t)$ is a polynomial of $r$, $\cos t$ and $\sin t$ defined in a bounded (hence, compact) domain of $[0, 1] \times [0, \pi/2]$. As a result there exist extremes for the continuous function $F_k(r, t)$. The extremes are located either at points where the first partial derivatives are zero or on the domain boundary. Using a scientific visualization tool, such as Matlab, we can visualize all the values of $F_k(r, t)$ in the domain of $[0, 1] \times [0, \pi/2]$. We have done so using Matlab for $3 \leq n \leq 1000$, and found that for any $(r, t) \in [0, 1] \times [0, \pi/2]$, $0 \leq F_k(r, t) \leq 1$. Hence, $S_i$ satisfies the convex hull property.

From Eq. (6), we can see that $T_i$ depends on $V_\omega$, $B_i$, $H_i$, and $C_i$. $V_\omega$ and $C_i$ are on the surface $P_i$, hence they lie inside of the convex hull of $G_i$ and can be represented similarly by a linear combination of $G_i$ with non-negative coefficients whose sum is one. $B_i$ and $H_i$ are B-spline curves defined by partial derivatives of $P_i$. All the derivatives $D'_i$ and $D''_i$ can be represented similarly by a linear combination of $G_i$, but with sum of coefficients to be zero (see Eq. (5)). Recall that in the definition of $B_i$ (or $H_i$), $\alpha_n$ (or $\beta$) is chosen in a way such that when $V_\omega + \alpha_n D'_i$ (or $V_\omega + \beta_n D''_i$), $1 \leq k \leq n$, is represented by a linear combination of the vertices of $G_i$, the coefficients of the representation are non-negative. Also, we can see that the sum of all the coefficients of the representation of $V_\omega + \alpha_n D'_i$ (or $V_\omega + \beta_n D''_i$) is one. Hence for any $k \in [1, n]$, both $V_\omega + \alpha_n D'_i$ and $V_\omega + \beta_n D''_i$ are within the convex hull of $G_i$. Therefore, $B_i$ and $H_i$ satisfy the convex hull property because they are B-spline curves defined by control points that are within the convex hull of $G_i$. As a result, $B_i$ and $H_i$ can be represented similarly by a linear combination of $G_i$ with non-negative coefficients whose sum is one.

With $V_\omega$, $B_i$, $H_i$, and $C_i$ all being able to be represented by a linear combination of $G_i$ with non-negative coefficients whose sum is one, using an approach similar to the proof of $S_i$’s convex hull property, one can verify that $T_i$ is within the convex hull of $G_i$ as well.

## 11 Test Results

The proposed approach has been implemented in C++ using OpenGl as the supporting graphics system on the Windows platform. Quite a few examples have been tested with the method described here (see Figure 4). All the examples have extra-ordinary vertices. With $M_n$ pre-calculated for all different valences of $n$, the implementation is actually very easy. Although $M_n$ is a big matrix, the computation needed for each point is not big at all because $M_n G_i$ needs to be done only once.

Our method is designed to ensure the resulting $C^2$ surface is similar to the subdivision surface. Figures 4(a-d) show two cases of comparison between a $C^2$ surface and its corresponding Catmull-Clark subdivision surface.
(CCSS). In either case, it is not obvious to tell the difference between the $C^2$ surface and its corresponding CCSS at all, although some very minor differences indeed exist.

Figures 4(f-h) demonstrate surface evaluation around an extra-ordinary vertex of degree 13, using our approach and CCSS approach [5]. All the displayed corresponding points are evaluated using the same parameters. Figures 4(j-l) show the isophotes around extra-ordinary points using also our approach and CCSS approach. Ten isophotes are displayed around each extra-ordinary point and each isophote is corresponding to a circle in parameter space. The radii for the $C^2$ isophotes are the same as those for the CCSS isophotes. From these figures we can see that, when a point in the parameter space tends to $(0, 0)$, the points generated by our approach are closer to the extra-ordinary point than points generated by a subdivision approach. When there are more points closer to the extra-ordinary point, there is more room for the generated surface to overcome the oscillation problem around an extra-ordinary point. As a result, our method produces smoother surface in the neighborhood of an extra-ordinary vertex. Figures 4(e, i, m) demonstrate that our method satisfies the convex hull property. Figure 4(e) is a mesh that some of its edges overlap three times. Note that in such a case, when the surface is evaluated, the edges stay where they are.
12 Summary

An approach for the construction of a $C^2$-continuous surface from a mesh of arbitrary topology is presented. The construction is subdivision surface based, with each extraordinary patch modified so that the resulting surface is not only $C^2$ continuous everywhere, but has an explicit representation for each extraordinary patch as well. Implementation is easy because the construction process is patch-based. The explicit representation for an extraordinary patch is a simple matrix form $W M G$ where $W$ is a parameter vector, $M$ is a constant coefficient matrix and $G$ is the control point vector. Therefore, evaluation of the surface position and computation of partial derivatives, normal vector, and curvature for any parameter point, including an extra-ordinary point, is very easy and, consequently, the resulting surface is suitable for operations such as shape analysis, shape optimization, surface energy minimization ... etc. The construction process includes constraints to ensure the shape of the resulting $C^2$ surface is very similar to the limit surface generated by Catmull-Clark subdivision. More importantly, the resulting $C^2$ surface satisfies the convex hull property. With all these properties, we believe the new approach will have broad applications in computer graphics and geometric design. Our future work will focus on its applications.

References


