3D Extension of Aesthetic Plane Curve and Its B-Spline Approximation

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Abstract

Curves are basic design elements in determining the shape and silhouette of an industrial product. Being able to build aesthetic and attractive curves certainly would increase a designed's ability in designing good quality 3D shapes. Such a capability depends on if there are ways/standards for one to determine if a curve is an aesthetic curve and, as well as, to create an aesthetic curve.

We have found the general equations of aesthetic curves. But these equations are for planar curves only. In this paper, we improve this work by first showing the necessary and sufficient condition for a curve to have self-affinity and then extending the aesthetic curves into 3-dimensional space. The process of computing a B-spline approximation of an 3D aesthetic curve is also shown.

Keywords: aesthetic curve, spacial aesthetic curve, self-affinity

1 Introduction

"Aesthetic curves" were first introduced by Harada [1] as curves whose logarithmic distribution diagrams of curvature (LDDCs) are close to a straight line. Miura et al. [2, 3] derived analytical expressions for curves whose LDDCs are strictly given by a straight line and called those expressions *general equations of aesthetic curves.* Yoshida and Saito [4] further analyzed properties of the curves represented by the *general equations* and developed a new method to interactively generate such a curve by specifying two end points, tangent vectors at those points, three control points and an α: slope of the straight line that defines the LDDC. In this research, we call the curves represented by the general equations of

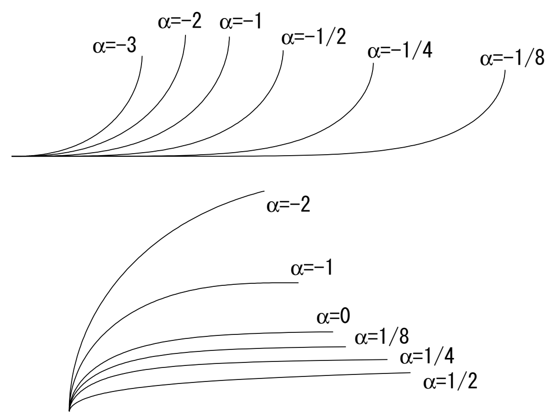

Figure 1: Aesthetic plane curves with various α values

aesthetic curves the *aesthetic curves.*

Aesthetic curves include logarithmic (equiangular) curves ($\alpha = 1$), clothoid curves ($\alpha = -1$) and involute curves ($\alpha = 2$) as special cases. It is possible to generate and deform aesthetic curves even if they are represented by integral forms using their unit tangent vectors as integrands ($\alpha \neq 1, 2$). These curves are expected to play important role in practical applications. However, the general equations at this moment can be used for plane curves only, they can not be used for 3D space curves. In this paper, we will first show the necessary and sufficient condition for a plane curve to have self-affinity and then extend the aesthetic curves into 3-dimensional space with guaranteed self-affinity. We call the derived curves *aesthetic space curves.* We will also show how to compute a B-spline approximation of an aesthetic space curve.

2 Aesthetic plane curves

We will show several important properties of aesthetic curves in this section. Recall that an aesthetic curve is a curve whose LDDC is defined by a straight line.

2.1 General equations of aesthetic curves

Given an aesthetic curve, we assume arc length of the curve is represented by s and radius of curvature is represented by ρ. The horizontal axis of LDDC measures $\log \rho$ and the vertivcal axis measures $\log(ds/d(\log \rho)) = \log(\rho\, ds/d\rho)$. Since LDDC is defined by a straight line, there exists a constant α such that the folloing equation is satisfied:

$$\log(\rho \frac{ds}{d\rho}) = \alpha \log \rho + C \qquad (1)$$

where C is a constant. We call this the fundamental equation of aesthetic curves. Eq.(1) can be written as

$$\frac{1}{\rho^{\alpha-1}} \frac{ds}{d\rho} = e^C = C_0 \qquad (2)$$

Hence there is some constant c_0 such that

$$\rho^{\alpha-1} \frac{d\rho}{ds} = c_0 \qquad (3)$$

Figure 1 shows several planar aesthetic curves with various α values.

2.2 Self-affinity of plane curves

We define **self-affinity** of a plane curve as follows [3]. Given a plane curve, if we can regenerate it by removing an arbitrary head portion from the curve and then scaling the remaining partg with some factors in the tangent and normal directions at some point of the curve, then the curve is said to have self-affinity.

A plane curve satisfying Eq.(3) has self-affinity [2].

2.3 Necessary and sufficient condition for self-affinity

For a given curve $\boldsymbol{C}(s)$ parametrerized by the arc length parameter $s \geq 0$, we assume derivative of its curvature and derivative of its radius of curvature are both continuous. In other words, we assume the curve has C^3 continuity. In addition, we assume the radius of curvature $\rho(s)$ is non-zero.

By scaling the curve with different factors in the tangent and normal directions (affine transformation of the plane curve [3]) at various points of the curve, we look for cases where the scaled curve contains a portion that is *congruent* to the original curve. We therefore reparameterize the given curve $\boldsymbol{C}(s)$ using a new parameter $t = as + b$ where a and b are positive constants as shown in Figure 2.3. To scale the curve uniformly in the tangent direction is equivalent to relate a point $\boldsymbol{C}(t_0 = as_0 + b)$ to another point $\boldsymbol{C}(s_0)$ as shown in Figure 2.3. In this relationship the scaling factor in the tangent direction f_t is given by $1/a$.

Although a and b are constants, they are related to the scaling factors in the tangent and normal directions f_t and f_n and they depend on the shape of the curve. Hence we can not specify them independently.

The start point of the curve $\boldsymbol{C}(t)$ is given by $\boldsymbol{C}(b)$, the point corresponding to $s = 0$. Hence $\boldsymbol{C}(t)$ is a curve without a head porition of the origianl curve $\boldsymbol{C}(s)$.

The condition for a curve to have self-affinity can be described as follows. For an aribitray constant $b > 0$, let $a > 0$ be a constant determined by b. Then the following equation is satisfied for

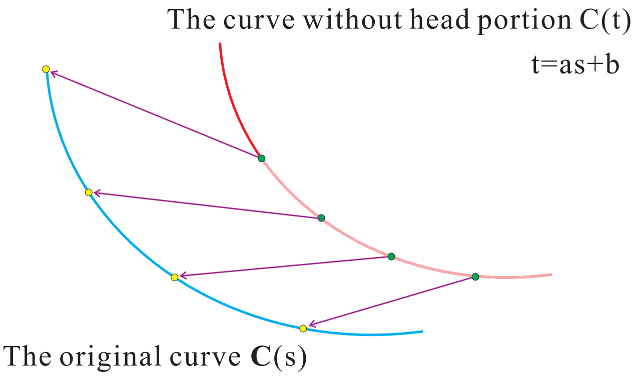

Figure 2: Correspondence between the original curve and a reparameterized version of the curve

any $s \geq 0$:

$$\frac{\rho(s)}{\rho(as+b)} = f_n \tag{4}$$

where f_n is a constant dependent on b and is a scaling factor in the normal direction. f_n is given by setting s to 0 in the above equation, as follows:

$$f_n = \frac{\rho(0)}{\rho(b)} \tag{5}$$

2.3.1 In case of $f_n = 1$

To make the subsequent derivation simpler, we first discuss the case when $f_n = 1$. From Eq.(4) we have

$$\rho(s) = \rho(as+b) \tag{6}$$

By the lemma proven in the appendix, $\rho(s)$ turns out to be a constant and the curve is given by an arc or a straight line ($\rho(s) = \infty$).

In the following, $f_n \neq 1$ is assumed. Rewrite Eq.(4) as

$$\rho(s) - f_n\, \rho(as+b) = 0 \tag{7}$$

Since the radius of curvature $\rho(s)$ is differentiable, we have

$$\begin{aligned}\frac{d\rho(s)}{ds} &- a\, f_n \left.\frac{d\rho(t)}{dt}\right|_{t=as+b} \\ &= \frac{d\rho(s)}{ds} - \frac{f_n}{f_t}\left.\frac{d\rho(t)}{dt}\right|_{t=as+b} = 0\end{aligned} \tag{8}$$

By substituting 0 for s and rewriting the above equation,

$$f_t = f_n \frac{\frac{d\rho(b)}{dt}}{\frac{d\rho(0)}{ds}} \tag{9}$$

Hence, as Eq.(5) is satisfied, both f_n and f_t are determined uniquely by the values of the radius of curvature and its derivative at the start point of the curve footnoteFrom $a = 1/f_t$, a is also uniquely determined by b.

2.3.2 In case of $f_n/f_t = 1$

First, for some $b > 0$, if $f_n/f_t = 1$ then from Eq. 8 we have

$$\frac{d\rho(s)}{ds} = \left.\frac{d\rho(t)}{dt}\right|_{t=as+b} \tag{10}$$

From this equation and the lemma in the appendix, it follows that

$$\frac{d\rho(s)}{ds} = c_0 \tag{11}$$

for some constant c_0. By integrating the above equation, one gets

$$\rho(s) = c_0\, s + c_1 \tag{12}$$

where c_1 is a constant of integration. Eq.(12) represents the relationship between the radius of curvature and the arc length of the logarithmic spiral and the curva has a special self-affinity, i.e., self-affinity when f_t is equal to f_n.

2.3.3 In case of $f_n/f_t \neq 1$

Next, consider the case $f_n/f_t \neq 1$. Since $f_n \neq 1$, there is some $\alpha \neq 1$ such that

$$\frac{f_n}{f_t} = f_n^{1-\alpha} \tag{13}$$

Then

$$\begin{aligned}\frac{d\rho(s)}{ds} &= f_n^{1-\alpha}\left.\frac{d\rho(t)}{dt}\right|_{t=as+b} \\ &= \left\{\frac{\rho(s)}{\rho(as+b)}\right\}^{1-\alpha}\left.\frac{d\rho(t)}{dt}\right|_{t=as+b}\end{aligned} \tag{14}$$

Hence

$$\rho(s)^{\alpha-1}\frac{d\rho(s)}{ds} = \rho(as+b)^{\alpha-1}\left.\frac{d\rho(t)}{dt}\right|_{t=as+b} \tag{15}$$

Threfore, if α is independent of b, then by the lemma, we obtain the following equation which is quivalent to Eq.(3)

$$\rho(s)^{\alpha-1}\frac{d\rho(s)}{ds} = c_0 \tag{16}$$

where c_0 is a constant. By integrating the above equation, the first and second general equations are derived [2].

2.3.4 Independence of α on b

In this subsection, we prove that α is independent of b. Here we consider the case where b is small enough and $\Delta b > 0$. Let a to be $1+\Delta a$ or $1-\Delta a$ ($\Delta a > 0$), depending on and uniquely determined by Δb. We relax the condition that b is positive and consider the case where $b = 0$ and let Δb be equal to 0. Then Eq.(4) relates itself. Hence $a = 1$, or $\Delta a = 0$. Then $f_n = 1$. For the curve without the portion corresponding to the domain $0 \le s < \Delta b$, Eq.(4) is satisfied and from Eq.(13), there exists α such that

$$\frac{\rho(s)}{\rho((1\pm\Delta a)s+\Delta b)} = f_n = \left\{\frac{f_n}{f_t}\right\}^{1-\alpha} \tag{17}$$

a is a continuous function of b and we can make the value of Δa smaller without limit if we make Δb smaller.

In Eq.(4), by repeatedly substituting $(1\pm\Delta a)s + \Delta b$ for s, we have

$$\begin{aligned} f_n &= \frac{\rho(s)}{\rho((1\pm\Delta a)s+\Delta b)} \\ f_n &= \frac{\rho((1\pm\Delta a)s+\Delta b)}{\rho((1\pm\Delta a)^2 s+\Delta b((1\pm\Delta a)+1))} \\ &\cdots \\ f_n &= \frac{\rho((1\pm\Delta a)^{m-1}s+\cdots+1))}{\rho((1\pm\Delta a)^m s+\cdots+1))} \end{aligned}$$

where $\pm$ is appropriately selected for the given curve to satisfy $\Delta a > 0$. From these equations,

$$\frac{\rho(s)}{\rho((1\pm\Delta a)^m s+\Delta b((1\pm\Delta a)^{m-1}+\cdots+1))} = f_n^m$$

Hence the scaling factor in the tangent direction for $b = \Delta b((1\pm\Delta a)^{m-1}+\cdots+1)$ is equal to $1/(1\pm\Delta a)^m = f_t^m$ and

$$f_n^m = \left\{\frac{f_n^m}{f_t^m}\right\}^{1-\alpha} \tag{18}$$

Therefore α is equal to that for Δb.

We will prove by contradiction that α is a constant. From Eq.(13), α can be expressed as a continuous function of b: $\alpha = \alpha(b)$. For some $b_0 > \Delta b > 0$, $\alpha_0 = \alpha(b_0)$ and we assume that α_0 is different from $\alpha = \alpha(\Delta b)$. For a small positive ϵ, we furthermore assume that

$$|\alpha_0 - \alpha| > 2\epsilon \tag{19}$$

Since $\alpha(b)$ is a continuous function, there exists some δ such that for any $b > 0$ satisfying $|b_0 - b| < \delta$ we have

$$|\alpha(b_0) - \alpha(b)| < \epsilon \tag{20}$$

As Δa is small, $1\pm\Delta a > 0$ and $\Delta b((1\pm\Delta a)^{m-1}+\cdots+1))$ increases monotonously from Δb and can become larger than any value by increasing m. Hence there exists m such that

$$\begin{aligned} b_l &= \Delta b((1\pm\Delta a)^{m-1}+\cdots+1)) < b_0 \\ &< b_u = \Delta b((1\pm\Delta a)^{m}+\cdots+1)) \end{aligned} \tag{21}$$

Since $b_u - b_l = \Delta b(1\pm\Delta a)^m$, if

$$\Delta b(1\pm\Delta a)^m < 2\delta \tag{22}$$

we get $|b_0 - b_l| < \delta$ or $|b_0 - b_u| < \delta$. Eq.(22) can be rewritten into $1\pm\Delta a < (2\delta/\Delta b)^{\frac{1}{m}}$ and Δa becomes smaller if we make Δb smaller and there exists Δb satisfying this equation. Hence Eq.(20) is satisfied which contradicts (19). Therefore α is constant for any b.

The results of the above discussion can be summed up as follows: a necessary and sufficient condition for a plane cuve to have self-affinity is that for some constant α, Eq.(16) is satisfied. When $\alpha = 1$, Eq.(16) becomes Eq.(11) and it contains the case of self-affinity.

2.4 Self-affinity ratio

α is the slope of the LDDC and, as discussed in the previous section, it is related to the scaling factors in the tangent and normal directions: f_t and f_n. Therefore, it characterizes the curve. Let γ be the reciprocal of α. Then from Eq.(13) we have,

$$\gamma = \frac{1}{\alpha} = \frac{\log f_n}{\log f_t} \tag{23}$$

This means $f_n = f_t^\gamma$.

For a fractal with self-affinity, a way to measure its degree of affinity is defined as follows [5]. When the whole figure is consisted of similar figures of number $1/b$ scaled by $1/a$ with $b = a^D$, the degree of affinity is given by

$$D = \frac{\log b}{\log a} \tag{24}$$

Eq.(23) is similar to the above definition and Eq.(23) can be interpreted as that it is necessary to have f_n curves to fill up the space in the normal direction if we scale the curve by $1/f_t$. γ can be interpreted as a dimension and we call it *self-affinity ratio*.

3 Extension into 3-dimensional space

The aesthetic curves considered so far are plane curves only. We will extend them into 3-dimensional space by using the Frenet-Serret formula (see, for example, [6]).

3.1 The Frenet-Serret formula

For a space curve $\boldsymbol{C}(s)$ parameterized by s, let its unit tangent vector be $\boldsymbol{t}$, unit principal normal vector be $\boldsymbol{n}$ and unit binormal vector be $\boldsymbol{b}$. These vectors are related by the Frenet-Serret formula as follows:

$$\begin{aligned} \frac{d\boldsymbol{C}(s)}{ds} &= \boldsymbol{t}, & \frac{d\boldsymbol{t}}{ds} &= \kappa\boldsymbol{n}, \\ \frac{d\boldsymbol{n}}{ds} &= -\kappa\boldsymbol{t}+\tau\boldsymbol{b}, & \frac{d\boldsymbol{b}}{ds} &= -\tau\boldsymbol{n} \end{aligned} \tag{25}$$

where κ and τ are the curvature and torsion, respectively. In the following we define self-affinity of a space curve. An *aesthetic space curve* is a space curve with self-affinity.

Given a space curve, similar to self-affinity of a plane curve, we say the curve has self-affinity if we can regenerate it by removing an arbitrary head portion from the curve and then scaling the remaining portion with some (different) factors in the tangent, principal normal and binormal directions at some point of the curve.

Since the curvature and torsion, or their reciprocals: the radius of curvature and radius of torsion can be independently specified with respect to the radius of torsion $\mu = 1/\tau$, we assume an equation similar to Eq.(1), as follows, is satisfied:

$$\log(\mu\frac{ds}{d\mu}) = \beta\log\mu + C' \tag{26}$$

where β is a constant. Like Eq.(3), we would have

$$\mu^{\beta-1}\frac{d\mu}{ds} = c_1 \tag{27}$$

for some constant c_0. Using arguments similar to those given in subsection 2.3 to show that the necessary and sufficient condition for a plane curve to have self-affinity is the equation given in Eq.(3), we can prove that the necessary and sufficient condition for a space curve to have self-affinity is the set of equations given in Eqs.(3) and (27)

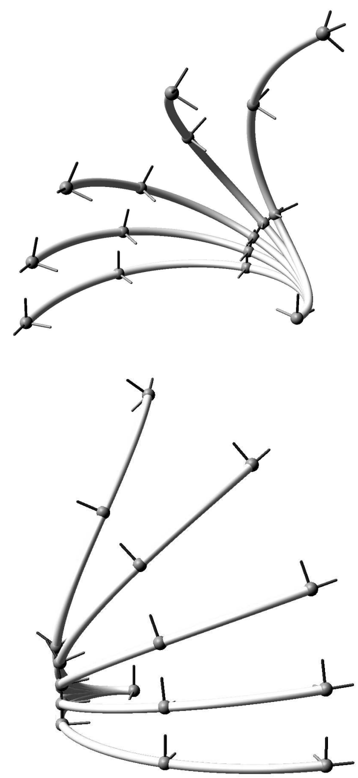

Figure 3: Examples of the aesthetic space curves

The Frenet-Serret formula can be considered as a set of differential equations and an example calculated by their numerial integration is shown in Figure 3. The left and right figures show the same five curves from different viewpoints and the curves drawn at the bottom are identical to a logarithmic spiral whose torsion is always 0 and radius of curvature is given by a linear function of the arc length. The other curves have the same start point and radius of curvature as the logarithmic spiral and their torsion is given by a linear function of the arc length with $\beta = 1$. The upper curves have smaller coefficient in the linear function of the arc length (larger torsion). For each curve, at the start point and end point, and two other points on the curve, we draw the tangent, principal normal and binormal vectors of the moving frame (Frenet frame) as short slim cyliders.

4 B-spline approximation

It is generally useful to use the *evolute* of a curve as well as the curve itself to evaluate the quality of a curve for aesthetic design [4]. The radius of curvature of an aesthetic curve changes smoothly and its evolute is given by another aesthetic curve wih smoothly changing curvatre. We will use as objective functions 1) *position errors* for the *least squares method* and 2) *position* and *curvature errors* for the *conjugate gradient method.*

4.1 Positional errors

Let $\boldsymbol{C}(s)$ be an aesthetic curve and let $\boldsymbol{C}_b(t)$ be a cubic B-spline approximation of $\boldsymbol{C}(s)$. $\boldsymbol{C}_b(t)$ is constructed as follows. We sample $\boldsymbol{C}(s)$ at m uniformly distributed parameter space points $\boldsymbol{Q}_i = \boldsymbol{C}(s_i)$ and minimize the following objective function:

$$R_p = \sum_{i=0}^{m-1} |\boldsymbol{C}_b(t_i) - \boldsymbol{Q}_i|^2 \tag{28}$$

Let the unit interval $0 \leq t \leq 1$ be the domain of $\boldsymbol{C}_b$ and let $\boldsymbol{P}_i$, $i = 0, ..., n$, be its control points (hence the number of segments of the B-spline curve is $n-2$). We use mutiple knots for the start and end points to make the start and end points identical to the first and last control points $\boldsymbol{P}_0$ and $\boldsymbol{P}_n$, respectively. The parameter value t_i of the ith sampled point is given by $t_i = s_i/l$ where l is the total length of the curve $\boldsymbol{C}(s)$. The tangent vectors at the start and end points of $\boldsymbol{C}(s)$ are $\boldsymbol{t}_s$ and $\boldsymbol{t}_e$, respectively. In order to make the positions and tangent vectors of the start and end points identical to the original curve, the following conditions are imposed. $\boldsymbol{P}_0 = \boldsymbol{C}(0)$, $\boldsymbol{P}_n = \boldsymbol{C}(l)$, $\boldsymbol{P}_1 = \boldsymbol{P}_0 + \alpha \boldsymbol{t}_s$, $\boldsymbol{P}_{n-1} = \boldsymbol{P}_n - \beta \boldsymbol{t}_e$.

The variables of Eq.(28) are the scalars α, β and the x and y coordinates of the control points $\boldsymbol{P}_i$, $i = 2, ..., n-2$. It is possible to solve the problem by the least squares method since the objective function given by Eq.(28) is a quadratic function of these parameters.

4.2 Curvature errors

As in the previous section, we use the same type of cubic B-spline curve for approximation, but minimize the following objective function to consider the errors of curvature as well:

$$\begin{aligned} R_{pc} &= \sum_{i=0}^{m-1} \{|\boldsymbol{C}_b(t_i) - \boldsymbol{Q}_i|^2 \\ &\qquad + w|\Psi(t_i) - \kappa(s_i)|^2\} \end{aligned} \tag{29}$$

where w is a weight to control the significance of the curvature error, $\Psi(t_i)$ is the curvature of the B-spline curve and $\kappa(s_i)$ is the curvature of the original curve at the correspoinding point. Since $\Psi(t)$ is given by $|d\boldsymbol{C}_b(t)/dt \times d^2\boldsymbol{C}_b(t)/d^2t|/|d\boldsymbol{C}_b(t)/dt|^3$, Eq.(29) can not be solved by the least squares method. We use one of the numerical search methods: the *conjugate gradient method* to minimize the objective function. We use α, β, and the control points that minimize Eq.(28) as the initial values.

4.3 Approximation results

In the case of $\alpha = 1$ (logarithmic spiral), Figure 4 shows the approximation results by the least squares method and Table 1 shows the approximation errors. Table 2 shows the errors by the conjugate gradient method ($w = 1$). The number of sampled points for approximation was 100 and that for error calculation was 1000. In the tables, *rms* means the root-mean square average and the errors are normalizd by setting the total length to 1. In the tables p means position, ρ stands for radius of curvature and e means position of the evolutes. Results of these tables show that errors of these two methods are comparable and it is not necessary to include errors of curvature in the objective function. We can obtain good approximation of the curve as well as high accuracy of its evolute if we use large enough sampled points and the curve segments. This is because 1) it is possible to sample any number of points on the curve and 2) it is possible to obtain accurate length of the aesthetic curves, hence it is not necessary to optimize t_i.

5 Conclusions

In this research, we have derived necessary and sufficient conditions for a plane curve and a space curve to have self-affinity and extended the planar aesthetic curves into 3-dimensional space with self-affinity based on the Frenet-Serret formula

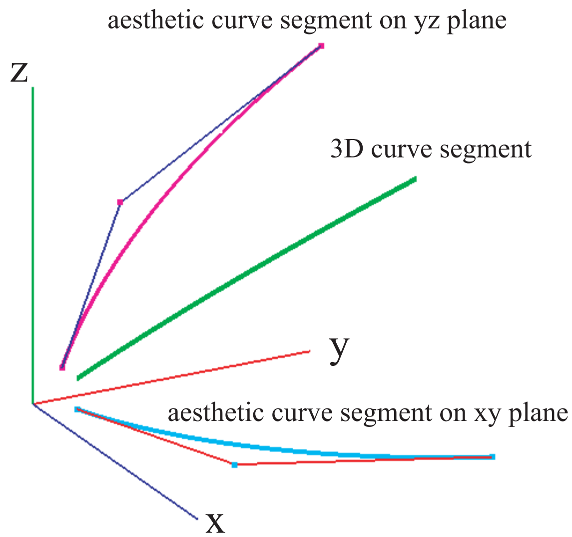

Figure 4: Approximation and its evolute

Table 1: Least square

seg@	1	3	7
rms_p	3.966×10^{-3}	1.909×10^{-4}	5.924×10^{-6}
e_p^{max}	5.878×10^{-3}	4.405×10^{-4}	2.006×10^{-5}
rms_ρ	6.750×10^{-2}	1.204×10^{-2}	1.743×10^{-3}
e_ρ^{max}@	1.660×10^{-1}	3.653×10^{-2}	6.559×10^{-3}
rms_e	6.911×10^{-2}	1.219×10^{-2}	1.747×10^{-3}
e_e^{max}	1.660×10^{-1}	3.668×10^{-2}	6.559×10^{-3}

and derived the aesthetic space curve. For a space aesthetic curve, the radius of torsion, i.e., the reciprocal of torsion to the power of some constant is given by a linear function of the arc length similar to the radius of curvature. Self-affinity of a space aesthetic curve is guarnateed.

For future work, we are planning on an automatic classification of curves with the following functions: 1) to determine if the rhythm is simple (monotonic) or complex (consisting of plural rhythms), 2) to calculate the slope of the line that approximates the LDDC. It seems to us that there is a lot of possible applications of the general equations of aesthetic curves in the field of computer aided geometric design. For example, it would be possible for one to use the equations to deform curves to change their appearance, say, from sharply bending to loosely bending. Another example is smoothing for reverse engineering. Even

Table 2: Conjugate gradient ($w = 1$)

seg@	1	3	7
rms_p	3.984×10^{-3}	1.909×10^{-4}	5.942×10^{-6}
e_p^{max}	6.012×10^{-3}	4.393×10^{-4}	2.046×10^{-5}
rms_ρ	6.800×10^{-2}	1.205×10^{-2}	1.737×10^{-3}
e_ρ^{max}	1.814×10^{-1}	3.653×10^{-2}	6.539×10^{-3}
rms_e	6.952×10^{-2}	1.220×10^{-2}	1.741×10^{-3}
e_e^{max}	1.814×10^{-1}	3.668×10^{-2}	6.539×10^{-3}

if only noisy data of curves are available, we can still use the equations as some sort of rulers to smooth out the noise and yield aesthetically high quality curves. We will also develop a CAD system using planar and space aesthetic curves.

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Appendix

A Lemma

Given a function $f(s)$ patameterized by arc length s. For an arbitrary constant $b > 0$, let $a > 0$ be a constant determined by b. With these a and b, if the following equation is satisfied for any $s \geq 0$

$$f(a\,s + b) = f(s) \tag{30}$$

Then $f(s)$ is a constant function.

Proof: Assume $f(s)$ is not a constant function. Then there exists some $s_0 > 0$ such that

$$f(s_0) \neq f(0) \tag{31}$$

If $b = s_0$. Then for some $a_0 > 0$ we have

$$f(a_0\,s + s_0) = f(s) \tag{32}$$

Substituting 0 for s in the above equation we get $f(s_0) = f(0)$ which contradicts Eq.(31). Therefore, $f(s)$ is a constant function [1].

[1] The lemma means that for an arbitrary $b > 0$, $a = a(b) > 0$, when the given function is scaled by a about the origin and is translated by b, if the function is congruent with the original function, then the funtion is constant.