The Modal Logic S4F, the Default Logic, and the Logic Here-and-There

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Abstract

The modal logic S4F provides an account for the default logic of Reiter, and several modal nonmonotonic logics of knowledge and belief. In this paper we focus on a fragment of the logic S4F concerned with modal formulas called *modal defaults*, and on sets of modal defaults — *modal default theories*. We present characterizations of S4F-expansions of modal default theories, and show that strong and uniform equivalence of modal default theories can be expressed in terms of the logical equivalence in the logic S4F. We argue that the logic S4F can be viewed as the general default logic of nested defaults. We also study special modal default theories called *modal programs*, and show that this fragment of the logic S4F generalizes the logic *here-and-there*.

Introduction

The emergence of the logic *here-and-there* as the foundation of *answer-set programming* (ASP) has been one of the most exciting developments in nonmonotonic reasoning in recent years. ASP is concerned with the use of general logic programs in knowledge representation, databases and constraint satisfaction. It is based on the concept of an *answer-set* of a logic program (Gelfond & Lifschitz 1988; 1991) and the idea that to model a problem, one constructs a program so that its answer sets represent solutions (Marek & Truszczyński 1999; Niemelä 1999; Gelfond & Leone 2002).

While researchers proposed several accounts of the semantics of answer sets of logic programs by relating this formalism to other nonmonotonic logics (Bidoit & Froidevaux 1991; Lifschitz & Schwarz 1993), several key issues remained open. First, the syntax of logic programs was restricted to program rules only. Second, there were no results addressing the fundamental problem of equivalence of logic programs. It all changed when Pearce (1997) discovered connections between ASP and the logic *here-and-there*, a non-standard propositional logic introduced by Heyting (1930). Subsequent research by Ferraris and Lifschitz (2005) expanded the syntax of logic programs and resulted in the class of *general logic programs*. It also established a relation between general logic programs and theories in the logic here-and-there. Under this connection, answer-sets of programs correspond precisely to *equilibrium* models of theories in the logic here-and-there (Pearce 1997; Ferraris & Lifschitz 2005).

Even more interestingly, the logic here-and-there allows one to characterize the *strong equivalence* of programs. One of the key questions for any logic is: given a theory $P \cup R$, when can one replace P with Q so that the result, $Q \cup R$, is logically equivalent to $P \cup R$, no matter what R is. For classical logic, the equivalence of theories P and Q is the necessary and sufficient condition guaranteeing this property. It is not so for logic programs with the answer-set semantics. There are logic programs P, Q and R such that P and Q have the same answer sets but $P \cup R$ and $Q \cup R$ do not! We say that programs P and Q are *strongly equivalent* if for every logic program R, $P \cup R$ and $Q \cup R$ have the same answer sets (Lifschitz, Pearce, & Valverde 2001). Several characterizations of strong equivalence have been discovered (Lifschitz, Pearce, & Valverde 2001; Lin 2002; Turner 2003; Eiter, Fink, & Woltran 2006). In particular, it is known that two general programs are strongly equivalent if they are equivalent in the logic here-and-there (Ferraris & Lifschitz 2005).

Our goal is to show that the modal logic S4F can be regarded as a more general alternative to the logic here-and-there. It has been known since early 1990s that the modal logic S4F provides an account for the default logic of Reiter, and several modal nonmonotonic logics of knowledge and belief (Truszczyński 1991; Schwarz & Truszczyński 1994; Lifschitz & Schwarz 1993; Lifschitz 1994). In this paper we focus on a fragment of the logic S4F concerned with modal formulas called *modal defaults* and on sets of modal defaults — *modal default theories*.

We present characterizations of S4F-expansions of modal default theories We show that the nonmonotonic logic S4F generalizes the (disjunctive) default logic in the same way as general logic programs generalize (disjunctive) logic programs. We extend the characterization of strong equivalence of programs in terms of the equivalence in the logic here-and-there to a characterization of strong equivalence of modal default theories in terms of the equivalence in the logic S4F. Similarly, we extend the characterizations of the uniform equivalence of programs given by Eiter, Fink and Woltran (2006) to the case of *modal programs* — a modal counterpart to general logic programs and a special class of modal defaults

Modal Logics and Modal Nonmonotonic Logics

We refer to (Marek & Truszczyński 1993) for a detailed discussion of topics covered in this section. We consider the propositional modal language determined by a set At (possibly infinite) of propositional atoms, a constant $\bot$, the usual boolean connectives $\neg$, $\vee$, $\wedge$, $\rightarrow$, and a single modal operator K. The constant $\bot$ represents a "generic" *contradiction* and K is read as "known". An inductive definition of a formula, given in the BNF notation, is as follows:

$$\varphi ::= \bot \,|\, p \,|\, K\varphi \,|\, \neg\varphi \,|\, \varphi \vee \varphi \,|\, \varphi \wedge \varphi \,|\, \varphi \rightarrow \varphi,$$

where $p \in At$. We denote the language consisting of such formulas by $\mathcal{L}_K$ (in the paper we fix the set At and so, we drop references to At from the notation). We write $\mathcal{L}$ for the set of K-free (modal-free) formulas in $\mathcal{L}_K$.

Modal logics in the language $\mathcal{L}_K$ differ from each other in the properties of the modality K. Typically, a modal logic $\mathcal{S}$ is defined by its *entailment* relation $\models_{\mathcal{S}}$, specified by *Kripke interpretations*, or in terms of proofs based on a set of modal axioms of $\mathcal{S}$.

To model knowledge and belief sets formed by rational agents with perfect introspection capabilities based on incomplete information, researchers introduced modal *nonmonotonic* logics (McDermott 1982). Let $\mathcal{S}$ be a modal logic with the entailment relation $\models_{\mathcal{S}}$. For every theory $I \subseteq \mathcal{L}_K$, a theory $T \subseteq \mathcal{L}_K$ is an $\mathcal{S}$-*expansion* of I if

$$T = \{\varphi \in \mathcal{L}_K | I \cup \neg K\overline{T} \models_{\mathcal{S}} \varphi\},$$

where $\neg K\overline{T} = \{\neg K\varphi | \varphi \in \mathcal{L}_K \setminus T\}$. The *nonmonotonic logic* $\mathcal{S}$ is a formalism, in which the semantics of a theory $I \subseteq \mathcal{L}_K$ is given by its $\mathcal{S}$-expansions.

If $A \subseteq \mathcal{L}$ is a propositional theory then A has a *unique* S5-expansion, where S5 is a well-known modal logic whose entailment relation is given by Kripke interpretations with the universal accessibility relation. We denote this unique expansion by $St(A)$[1]. We have the following result.

Theorem 1 *If $\mathcal{S}$ is a modal logic contained in S5 and $A \subseteq \mathcal{L}$, then $St(A)$ is the unique $\mathcal{S}$-expansion of A.*

Logic S4F

In this work we are concerned with the modal logic S4F (Segerberg 1971; Voorbraak 1991; Marek & Truszczyński 1993). It is a modal logic in the language $\mathcal{L}_K$ with the semantics specified by *Kripke* S4F-*interpretations* (or simply, S4F-*interpretations*), that is, tuples $\langle V, W, \pi\rangle$, where

1. V and W are *nonempty* and disjoint sets of *worlds*, and
2. π is a function assigning to each world $w \in V \cup W$ a set of atoms $\pi(w)$, representing a *propositional* truth valuation for w.

Given an S4F-interpretation $\mathcal{M} = \langle V, W, \pi\rangle$, we define the *satisfaction relation* $\mathcal{M}, w \models \varphi$, where $w \in V \cup W$ and $\varphi \in \mathcal{L}_K$, as follows:

1. $\mathcal{M}, w \not\models \bot$
2. $\mathcal{M}, w \models p$ if $p \in \pi(w)$ (for $p \in At$)
3. If $w \in V$, then $\mathcal{M}, w \models K\varphi$ if $\mathcal{M}, v \models \varphi$ for every $v \in V \cup W$
4. If $w \in W$, then $\mathcal{M}, w \models K\varphi$ if $\mathcal{M}, v \models \varphi$ for every $v \in W$
5. The induction over boolean connectives is standard. For instance, $\mathcal{M}, w \models \varphi \wedge \psi$ if $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$.

An S4F-interpretation $\mathcal{M} = \langle V, W, \pi\rangle$ is an S4F-model of $\varphi \in \mathcal{L}_K$, written $\mathcal{M} \models \varphi$, if for every $w \in V \cup W$, $\mathcal{M}, w \models \varphi$. We write $\varphi \models_{\mathrm{S4F}} \psi$ if every S4F-model of φ is an S4F-model of ψ. The notation extends in a standard way to *modal theories*, that is, subsets of $\mathcal{L}_K$. The logic S4F is the logic of the entailment relation $\models_{\mathrm{S4F}}$. We note that $\models_{\mathrm{S4F}}$ has a proof-theoretic characterization based on the necessitation inference rule and axiom schemata K, T, 4 and F (Segerberg 1971; Marek & Truszczyński 1993).

The importance of the logic S4F for knowledge representation stems from properties of its nonmonotonic version. Truszczyński (1991) and Schwarz and Truszczyński (1994) showed that the nonmonotonic logic S4F captures, under some direct and intuitive encodings, the (disjunctive) logic programming with the answer set semantics (Gelfond & Lifschitz 1991), the (disjunctive) default logic (Reiter 1980; Gelfond *et al.* 1991), the logic of grounded knowledge (Lin & Shoham 1990), the logic of minimal belief and negation as failure (Lifschitz 1994) and the logic of minimal knowledge and belief (Schwarz & Truszczyński 1994).

For an S4F-interpretation $\mathcal{M} = \langle V, W, \pi\rangle$, we write $L_{\mathcal{M}}$ ($U_{\mathcal{M}}$) for the set of all formulas from $\mathcal{L}$ (propositional formulas) that hold in every truth assignment $\pi(v)$, where $v \in V$ ($\pi(w)$, where $w \in W$, respectively). The following result is a restatement of a characterization of S4F-expansions developed by Schwarz in (1992).

Theorem 2 *Let $I \subseteq \mathcal{L}_K$. A theory $T \subseteq \mathcal{L}_K$ is an* S4F-*expansion of I if and only if there is an* S4F-*model $\mathcal{M}$ of I such that $T = St(U_{\mathcal{M}})$; $L_{\mathcal{M}} = U_{\mathcal{M}}$; and for every* S4F-*model $\mathcal{N}$ of I with $U_{\mathcal{N}} = U_{\mathcal{M}}$, $U_{\mathcal{N}} \subseteq L_{\mathcal{N}}$.*

Modal Defaults

In this paper, we are interested in formulas from $\mathcal{L}_K$, which we call *modal defaults*. An inductive definition of a *modal default*, given in the BNF notation, is as follows:

$$\varphi ::= K\psi \,|\, K\varphi \,|\, \neg\varphi \,|\, \varphi \vee \varphi \,|\, \varphi \wedge \varphi \,|\, \varphi \rightarrow \varphi,$$

where $\psi \in \mathcal{L}$. In other words, modal defaults are formulas built according to the same rules as arbitrary modal formulas are. The difference is in the base case. It consists of formulas $K\psi$, where $\psi \in \mathcal{L}$. We define a *modal default theory* to be a set of modal defaults. The terminology reflects the fact that, as we argue later, standard defaults and default theories introduced by Reiter (1980) are special cases of modal defaults and modal default theories, respectively.

For modal defaults, the concepts of an S4F-interpretation and S4F-entailment can be replaced by simpler ones. An S4F-*pair* is a pair $\langle L, U\rangle$, where $L, U \subseteq \mathcal{L}$ are propositional theories closed under propositional entailment.

[1] The notation reflects the fact that expansions are *stable* theories (Stalnaker 1980; McDermott 1982).

For an S4F-*pair* $\langle L, U\rangle$ and a modal default φ, we define two satisfiability relations $\langle L, U\rangle \models_l \varphi$ and $\langle L, U\rangle \models_u \varphi$ inductively as follows:

1. For $\varphi = K\psi$, where $\psi \in \mathcal{L}$, we define $\langle L, U\rangle \models_u \varphi$ if $\psi \in U$; and we define $\langle L, U\rangle \models_l \varphi$ if $\psi \in L \cap U$
2. We handle boolean connectives in the standard way. For instance, for $\varphi = \neg\psi$, where ψ is a modal default, we define $\langle L, U\rangle \models_u \varphi$ if $\langle L, U\rangle \not\models_u \psi$; and $\langle L, U\rangle \models_l \varphi$ if $\langle L, U\rangle \not\models_l \psi$
3. For $\varphi = K\psi$, where ψ is a modal default, we define $\langle L, U\rangle \models_u \varphi$ if $\langle L, U\rangle \models_u \psi$; and we define $\langle L, U\rangle \models_l \varphi$ if $\langle L, U\rangle \models_l \psi$ and $\langle L, U\rangle \models_u \psi$

We write $\langle L, U\rangle \models \varphi$[2] if $\langle L, U\rangle \models_l \varphi$ and $\langle L, U\rangle \models_u \varphi$.

If $\mathcal{M}$ is an S4F-interpretation then $\langle L_{\mathcal{M}}, U_{\mathcal{M}}\rangle$ is an S4F-pair. One can also show that for every S4F-pair $\langle L, U\rangle$ there is an S4F-interpretation $\mathcal{M} = \langle V, W.\pi\rangle$ such that $L_{\mathcal{M}} = L$ and $U_{\mathcal{M}} = U$.

The following result describes the connection between the satisfiability of modal defaults in S4F-interpretations, and the satisfiability relations involving S4F-pairs.

Proposition 3 *Let $\mathcal{M} = \langle V, W, \pi\rangle$ be an* S4F-*interpretation and φ a modal default.*

1. *For every $v \in V$:*
   $\mathcal{M}, v \models \varphi$ if and only if $\langle L_{\mathcal{M}}, U_{\mathcal{M}}\rangle \models_l \varphi$
2. *For every $w \in W$:*
   $\mathcal{M}, w \models \varphi$ if and only if $\langle L_{\mathcal{M}}, U_{\mathcal{M}}\rangle \models_u \varphi$
3. *$\mathcal{M} \models \varphi$ if and only if $\langle L_{\mathcal{M}}, U_{\mathcal{M}}\rangle \models \varphi$.*

This result implies the following characterization of S4F-expansions of modal default theories.

Theorem 4 *Let $I \subseteq \mathcal{L}_K$ be a modal default theory. A theory $T \subseteq \mathcal{L}_K$ is an* S4F-*expansion of I if and only if there is $U \subseteq \mathcal{L}$ such that U is closed under propositional entailment, $T = St(U)$, $\langle U, U\rangle \models I$, and for every* S4F-*pair $\langle L, U\rangle$, $\langle L, U\rangle \models I$ implies $U \subseteq L$.*

Proof: ($\Rightarrow$) Let T be an S4F-expansion of I. By Theorem 2, there is an S4F-model $\mathcal{M}$ of I such that $T = St(U_{\mathcal{M}})$, $L_{\mathcal{M}} = U_{\mathcal{M}}$, and for every S4F-model $\mathcal{N}$ of I with $U_{\mathcal{N}} = U_{\mathcal{M}}$, $U_{\mathcal{N}} \subseteq L_{\mathcal{N}}$. Let $U = U_{\mathcal{M}}$. Clearly, U is closed under propositional entailment and $T = St(U)$. Moreover, by Proposition 3, $\langle U, U\rangle \models I$. Let us consider an S4F-pair $\langle L, U\rangle$ such that $\langle L, U\rangle \models I$. Let $\mathcal{N}$ be an S4F-interpretation such that $L_{\mathcal{N}} = L$ and $U_{\mathcal{N}} = U$ (we observed earlier that such S4F-interpretations exist). Then $\mathcal{N} \models I$ and $U_{\mathcal{N}} = U_{\mathcal{M}}$. By Theorem 2, $U_{\mathcal{N}} \subseteq L_{\mathcal{N}}$. Since $U = U_{\mathcal{N}}$ and $L = L_{\mathcal{N}}$, $U \subseteq L$. Thus, U has all the properties required.
($\Leftarrow$) Let $\langle U, U\rangle$ be an S4F-pair such that $T = St(U)$, $\langle U, U\rangle \models I$, and for every S4F-pair $\langle L, U\rangle$ such that $\langle L, U\rangle \models \varphi$, $U \subseteq L$. Let $\mathcal{M}$ be any S4F-interpretation such that $L_{\mathcal{M}} = U_{\mathcal{M}} = U$. Since $U_{\mathcal{M}} = U$, $T = St(U_{\mathcal{M}})$. By Proposition 3, $\mathcal{M} \models I$. Let us consider an S4F-model $\mathcal{N}$ of I such that $U_{\mathcal{N}} = U_{\mathcal{M}}$ (that is, $U_{\mathcal{N}} = U$). By Proposition 3, $\langle L_{\mathcal{N}}, U_{\mathcal{N}}\rangle \models I$. Hence, $U_{\mathcal{N}} \subseteq L_{\mathcal{N}}$ and so, by Theorem 2, T is an S4F-expansion of I, □

We can further simplify this result by restricting attention to special S4F-pairs. First, we note the following result.

Proposition 5 *For every modal default φ and for every* S4F-*pair $\langle L, U\rangle$:*

1. *$\langle L, U\rangle \models_u \varphi$ if and only if $\langle L \cap U, U\rangle \models_u \varphi$*
2. *$\langle L, U\rangle \models_l \varphi$ if and only if $\langle L \cap U, U\rangle \models_l \varphi$.*

We now define an *se-interpretation* as an S4F-pair $\langle L, U\rangle$ such that $L \subseteq U$. If I is a modal default theory then an se-interpretation $\langle L, U\rangle$ such that $\langle L, U\rangle \models I$ is an *se-model* of I[3]. Since S4F-pairs $\langle L \cap U, U\rangle$ are se-interpretations, Proposition 5 implies that se-interpretations suffice to characterize the S4F-entailment. For a modal default theory I and a modal default φ we write $I \models_{se} \varphi$ if every se-model of I is an se-model of φ.

Corollary 6 *Let $I \subseteq \mathcal{L}_K$ be a modal default theory and let $\varphi \in \mathcal{L}_K$ be a modal default. Then $I \models_{\mathrm{S4F}} \varphi$ if and only if $I \models_{se} \varphi$.*

Corollary 7 *Let $I \subseteq \mathcal{L}_K$ be a modal default theory. A theory $T \subseteq \mathcal{L}_K$ is an* S4F-*expansion of I if and only if there is $U \subseteq \mathcal{L}$ such that U is closed under propositional entailment, $T = St(U)$, $\langle U, U\rangle$ is an se-model for I, and for every se-model $\langle L, U\rangle$ for I, $U = L$.*

Let $I \subseteq \mathcal{L}_K$. We call an se-interpretation $\langle U, U\rangle$ an *se-expansion* of I if $\langle U, U\rangle \models I$ and for every se-model $\langle L, U\rangle$ of I, $L = U$. Our results show that there is a one-to-one correspondence between S4F-expansions and se-expansions.

Corollary 8 *Let $I \subseteq \mathcal{L}_K$ be a modal default theory. A theory $T \subseteq \mathcal{L}_K$ is an* S4F-*expansion of I if and only if there is an se-expansion $\langle U, U\rangle$ of I such that $T = St(U)$.*

Strong Equivalence for the Nonmonotonic Logic S4F

Strong equivalence is a fundamental property in the studies of nonmonotonic logics. It captures the concept of modularity, or "equivalence for replacement". Theories $I, I' \subseteq \mathcal{L}_K$ are *strongly equivalent* (in the nonmonotonic logic S4F) if for every theory $J \subseteq \mathcal{L}_K$, $I \cup J$ and $I' \cup J$ have the same S4F-expansions (or equivalently, the same se-expansions). Thus, if $I, I' \subseteq \mathcal{L}_K$ are strongly equivalent, one can be replaced by the other within a larger theory without affecting the nonmonotonic semantics of the theory. This notion was studied for logic programs (Lifschitz, Pearce, & Valverde 2001; Lin 2002) and default logic (Turner 2003), and in abstract algebraic settings (Truszczyński 2006).

We will now characterize strong equivalence of modal default theories I and I'. We will use the following notation: for $U \subseteq \mathcal{L}_K$, we write $KU = \{K\varphi | \varphi \in U\}$. We will also use the following simple lemma.

Lemma 9 *Let $\langle L, U\rangle$ be an se-interpretation. For every modal default φ, if $\langle L, U\rangle \models \varphi$ then $\langle U, U\rangle \models \varphi$.*

[2] We overload the symbol $\models$. Its exact meaning is determined by the context, that is, the structure appearing to its left (a Kripke interpretation $\mathcal{M}$, a pair $\mathcal{M}, w$ or an S4F-pair).

[3] The term "se" stands for *strong equivalence* reflecting connections to models used by Turner (2003) to characterize *strong equivalence* of logic programs.

We now present the main result of this section.

Theorem 10 *Let $I, I' \subseteq \mathcal{L}_K$ be modal default theories. The following conditions are equivalent:*

1. *I and I' are strongly equivalent*
2. *I and I' are equivalent in the logic* S4F
3. *I and I' have the same se-models.*

Proof: By Corollary 6, the conditions (2) and (3) are equivalent. If I and I' are equivalent in the logic S4F then, for every theory $J \subseteq \mathcal{L}_K$, $I \cup J$ and $I \cup J'$ are equivalent in the logic S4F and so, have the same S4F-expansions. Thus, I and I' are strongly equivalent, and so (2) implies (1).

It follows that to complete the proof it suffices to show that (1) implies (3). By Corollary 8, (1) implies
($1'$) for every modal default theory J, $I \cup J$ and $I' \cup J$ have the same se-expansions.

Let us assume that (3) fails. By symmetry, we may assume that some se-model $\langle L, U\rangle$ of I is not an se-model of I'.

Case 1. $\langle U, U\rangle \not\models I'$. Then, $\langle U, U\rangle \not\models I' \cup KU$. Thus, $\langle U, U\rangle$ is not an se-expansion of $I' \cup KU$.

Next, we note that $\langle U, U\rangle \models I$ (it follows by Lemma 9 from the fact that $\langle L, U\rangle \models I$). Moreover, from the definition we have $\langle U, U\rangle \models KU$. Thus, $\langle U, U\rangle \models I \cup KU$.

Let us consider an se-model $\langle L', U\rangle$ of $I \cup KU$. Since $\langle L', U\rangle \models KU$, $U \subseteq L'$, and since $\langle L', U\rangle$ is an se-interpretation, $L' = U$. Thus, $\langle U, U\rangle$ is an se-expansion of $I \cup KU$. This is a contradiction with ($1'$).

Case 2. $\langle U, U\rangle \models I'$. Since $\langle L, U\rangle \not\models I'$, it follows that $L \neq U$ and so, $U \setminus L \neq \emptyset$.

Since $L \subseteq U$, it follows that $\langle U, U\rangle$ is an se-model of

$$P' = I' \cup KL \cup \{K\alpha \rightarrow K\beta | \alpha, \beta \in U \setminus L\}.$$

Let us consider an se-model $\langle L', U\rangle$ of P'. Then, it follows that $L \subseteq L' \subseteq U$. Since $L' \neq L$ (we recall that $\langle L, U\rangle$ is not an se-model of I'), $L' \setminus L \neq \emptyset$.

Let $\alpha \in L' \setminus L$. Since $\alpha \in U$, $\alpha \in U \setminus L$. Thus, $\langle L', U\rangle \models K\alpha \rightarrow K\beta$, for every $\beta \in U \setminus L$. Since $\alpha \in L'$,

$$\langle L', U\rangle \models_l K\alpha \text{ and } \langle L', U\rangle \models_u K\alpha.$$

Consequently, $\langle L', U\rangle \models K\beta$, for every $\beta \in U \setminus L$. Thus, since $\langle L', U\rangle \models KL$, $\langle L', U\rangle \models KU$. It follows that $U \subseteq L'$ and so $L' = U$. Thus, $\langle U, U, \rangle$ is an se-expansion of P'. By ($1'$), $\langle U, U\rangle$ is an se-expansion of

$$P = I \cup KL \cup \{K\alpha \rightarrow K\beta | \alpha, \beta \in U \setminus L\}.$$

On the other hand, we have $\langle L, U\rangle \models P$. Thus, $\langle U, U\rangle$ is not an se-expansion of P, a contradiction with ($1'$). □

Uniform Equivalence for the Nonmonotonic Logic S4F

The notion of *uniformly equivalent* logic programs was introduced by Eiter, Fink and Woltran (2006). Here we generalize it to modal default theories in the logic S4F.

Let $P, Q \subseteq \mathcal{L}_K$ be modal theories. We say that P and Q are *uniformly equivalent* if for every set $X \subseteq \mathcal{L}$, $P \cup KX$ and $Q \cup KX$ have the same S4F-expansions. If P and Q are modal default theories then for every $X \subseteq \mathcal{L}$, $P \cup KX$ and $Q \cup KX$ are modal default theories, too. It turns out that se-interpretations can also be used to characterize uniform equivalence. Namely, we have the following theorem.

Theorem 11 *Default modal theories $P, Q \subseteq \mathcal{L}_K$ are uniformly equivalent if and only if the following three conditions hold:*

1. *for every se-interpretation $\langle U, U\rangle$, $\langle U, U\rangle \models P$ if and only if $\langle U, U\rangle \models Q$*
2. *for every se-interpretation $\langle L, U\rangle$, if $L \subset U$ and $\langle L, U\rangle \models P$ then there is an se-interpretation $\langle L', U\rangle$ such that $L \subseteq L' \subset U$ and $\langle L', U\rangle \models Q$*
3. *for every se-interpretation $\langle L, U\rangle$, if $L \subset U$ and $\langle L, U\rangle \models Q$ then there is an se-interpretation $\langle L', U\rangle$ such that $L \subseteq L' \subset U$ and $\langle L', U\rangle \models P$*

Modal Programs

We will now consider *modal programs*, a special class of modal default theories consisting of *modal rules*. A formal inductive definition of a *modal rule*, given in the BNF notation, is as follows:

$$\varphi ::= Kp \,|\, K\varphi \,|\, \neg\varphi \,|\, \varphi \vee \varphi \,|\, \varphi \wedge \varphi \,|\, \varphi \rightarrow \varphi,$$

where $p \in At \cup \{\bot\}$. In other words, modal rules are built in the same way as modal defaults, but from expressions of the form Kp, where $p \in At \cup \{\bot\}$ and not $K\psi$, $\psi \in \mathcal{L}$. A *modal program* is a set of modal rules.

If $X \subseteq \mathcal{L}$, we write $Th(X)$ for the set of all propositional consequences of X. A *simple se-interpretation* is any se-interpretation of the form $\langle Th(L), Th(U)\rangle$, where $L, U \subseteq At$. To characterize S4F-expansions of modal programs it suffices to restrict to simple se-interpretations. Indeed, the following results state the key property of modal rules.

Theorem 12 *If φ is a modal rule and $\langle L, U\rangle$ is an se-interpretation then $\langle L, U\rangle \models \varphi$ if and only if $\langle Th(L \cap At), Th(U \cap At)\rangle \models \varphi$.*

Proof: The assertion can be proved by a standard inductive argument on the complexity of a formula. □

Corollary 13 *If I and I' are modal programs, then I and I' have the same se-models if and only if they have the same simple se-models.*

These results allow us to strengthen the characterization of strong equivalence in the case of modal programs.

Corollary 14 *Let $I, I' \subseteq \mathcal{L}_K$ be modal programs. The following conditions are equivalent:*

1. *I and I' are strongly equivalent*
2. *I and I' have the same simple se-models.*

Proof: By Corollary 13, I and I' have the same simple se-models if and only if they have the same se-models. Thus, the assertion follows directly from Theorem 10. □

Using a similar argument we can also strengthen the characterization of the uniform equivalence of modal programs (Theorem 11) by consistently replacing se-interpretations with simple se-interpretations.

We conclude this section by noting that all results concerning modal programs can be restated in terms of the modal logic SW5. The semantics of the logic SW5 is determined by S4F-interpretations $\langle V, W, \pi\rangle$, where $|V| = 1$. Alternatively, this logic can be characterized in a proof-theoretic way in terms of axiom schemata K, T, 4 and W5 (Segerberg 1971; Marek & Truszczyński 1993). For modal programs we have the following results.

Theorem 15 *If $I, I' \subseteq \mathcal{L}_K$ are modal programs and φ is a modal rule then:*

1. *$I \models_{\mathrm{S4F}} \varphi$ if and only if $I \models_{\mathrm{SW5}} \varphi$*
2. *A theory T is an* S4F*-expansion of I if and only if T is an* SW5*-expansion of I*
3. *I and I' are strongly equivalent if and only if I and I' are equivalent in the logic* SW5.

The Logic Here-and-There

In recent years, the the logic *here-and-there* (Heyting 1930) has emerged as the logic behind the answer-set programming (Pearce 1997; Ferraris & Lifschitz 2005). We will now show that the logic here-and-there can be embedded in the logic S4F. Even more, as the embedding represents theories of the logic here-and-there as modal programs, the logic S4F can be replaced by the logic SW5. Our results in this section are related to results of Lin and Zhou (2007), where an embedding of the logic here-and-there in the logic GK (Lin & Shoham 1990) was presented.

The language of the logic here-and-there has three primitive binary connectives $\wedge$, $\vee$ and $\rightarrow$, and a constant $\bot$ to represent a generic contradiction. A formal BNF definition of formulas in the propositional language of the logic here-and-there is:

$$\varphi ::= \bot \,|\, p \,|\, \varphi \vee \varphi \,|\, \varphi \wedge \varphi \,|\, \varphi \rightarrow \varphi,$$

where $p \in At$. The negation of a formula φ, $\neg\varphi$, is a shorthand for the formula $\varphi \rightarrow \bot$. We denote the set of all formulas defined in this way by $\mathcal{L}_{ht}$ (formally, it is only a subset of $\mathcal{L}$, as $\neg$ is not a primary connective in the logic here-and-there, as introduced here).

The semantics of the logic here-and-there is given by HT-interpretations. An *HT-interpretation* is a pair $\langle L, U\rangle$, where $L \subseteq U \subseteq At$ are sets of atoms. We define the satisfiability relation $\langle L, U\rangle \models_{ht} \varphi$, where $\varphi \in \mathcal{L}_{ht}$, by induction as follows:

1. $\langle L, U\rangle \not\models \bot$
2. For $\varphi = p$, where $p \in At$, we define $\langle L, U\rangle \models_{ht} p$ if $p \in L$
3. $\langle L, U\rangle \models_{ht} \varphi \wedge \psi$ if $\langle L, U\rangle \models_{ht} \varphi$ and $\langle L, U\rangle \models_{ht} \psi$
4. $\langle L, U\rangle \models_{ht} \varphi \vee \psi$ if $\langle L, U\rangle \models_{ht} \varphi$ or $\langle L, U\rangle \models_{ht} \psi$
5. $\langle L, U\rangle \models_{ht} \varphi \rightarrow \psi$ if (i) $\langle L, U\rangle \not\models_{ht} \varphi$ or $\langle L, U\rangle \models_{ht} \psi$; and (ii) $U \models \varphi \rightarrow \psi$ (in standard propositional logic).

An HT-interpretation $\langle U, U\rangle$ is an *equilibrium model* of $A \subseteq \mathcal{L}_{ht}$ if $\langle U, U\rangle \models_{ht} A$ and for every $L \subseteq U$, if $\langle L, U\rangle \models_{ht} A$ then $L = U$ (Pearce 1997). Equilibrium models correspond to *stable* models of general logic programs (Ferraris & Lifschitz 2005).

We will now show that the logic here-and-there can be embedded in the logic S4F. To this end, for every propositional formula $\varphi \in \mathcal{L}_{ht}$ we define a formula $\varphi_{\neg K \neg K}$ to be a modal rule obtained from φ by replacing each $a \in At \cup \{\bot\}$ in φ with $\neg K \neg K a$ (intuitively, $\neg K \neg K$ represents a modality exhibiting properties of the *belief* modality). We note that all formulas $\varphi_{\neg K \neg K}$ are modal rules.

Next, for every propositional formula $\varphi \in \mathcal{L}_{ht}$ we define the corresponding modal rule φ_{mp} inductively as follows:

1. $a_{mp} = Ka$ for $a \in At \cup \{\bot\}$
2. $(\varphi \wedge \psi)_{mp} = \varphi_{mp} \wedge \psi_{mp}$ and $(\varphi \vee \psi)_{mp} = \varphi_{mp} \vee \psi_{mp}$
3. $(\varphi \rightarrow \psi)_{mp} = (\varphi_{mp} \rightarrow \psi_{mp}) \wedge (\psi \rightarrow \varphi)_{\neg K \neg K}$.

We extend this notation to sets of formulas: for a set $A \subseteq \mathcal{L}_{ht}$, we define $A_{mp} = \{\varphi_{mp} | \varphi \in A\}$.

We have the following result establishing the connection between the logic here-and-there and the logic SW5 (or S4F as the two logics coincide on modal programs). To state it, we use the notation $A \models_{sse} \varphi$ to denote that every simple se-model of A is a simple se-model of φ.

Theorem 16 *Let $A \subseteq \mathcal{L}_{ht}$ and $\varphi \in \mathcal{L}_{ht}$. The following conditions are equivalent:*

1. *$A \models_{ht} \varphi$*
2. *$A_{mp} \models_{se} \varphi_{mp}$*
3. *$A_{mp} \models_{sse} \varphi_{mp}$*
4. *$A_{mp} \models_{\mathrm{SW5}} \varphi_{mp}$*
5. *$A_{mp} \models_{\mathrm{S4F}} \varphi_{mp}$.*

Corollary 17 *Let $A \subseteq \mathcal{L}_{ht}$ and $U \subseteq At$. The following conditions are equivalent:*

1. *U is a stable model of A*
2. *$\langle U, U\rangle$ is an equilibrium model of A*
3. *$\langle Th(U), Th(U)\rangle$ is an se-expansion of A*
4. *$St(U)$ is an* SW5*-expansion of A_{mp}*
5. *$St(U)$ is an* S4F*-expansion of A_{mp}.*

Logic of Nested Defaults

Let $\frac{\alpha:\beta_1,\ldots,\beta_m}{\gamma_1|\ldots|\gamma_n}$ be a disjunctive default (Gelfond *et al.* 1991). By encoding it with a modal default

$$K\alpha \wedge K\neg K\beta_1 \wedge \ldots \wedge K\neg K\beta_k \rightarrow K\gamma_1 \vee \ldots \vee K\gamma_n$$

we obtain an embedding of (disjunctive) default theories in $\mathcal{L}_K$ which establishes a one-to-one correspondence between extensions and S4F-expansions (Truszczyński 1991). Thus, the class of modal default theories with the semantics of S4F-expansions (or se-expansions) can be regarded as a generalization of the disjunctive default logic. In fact, we can regard it as a general default logic of nested defaults as it covers, for instance, the case of formulas of the form

$$K\alpha \wedge K\neg K\beta_1 \wedge \ldots \wedge K\neg K\beta_k \rightarrow K\gamma_1 \vee \ldots \vee K\gamma_n$$

where α, β_i and γ_i are arbitrary modal defaults rather than formulas from $\mathcal{L}$.

We also note that by exploiting the embedding given above, our results on strong equivalence of modal default theories can be specialized to results on strong equivalence of disjunctive default theories, first obtained by Turner (2003). Our results on uniform equivalence of modal default theories generalize those obtained by Truszczyński (2006).

Conclusions

In this paper we investigated the modal (nonmonotonic) logic S4F under the restriction to modal default theories. We presented new characterizations of S4F-expansions of modal default theories and showed that two modal default theories are strongly equivalent if and only if they are equivalent in the logic S4F. We also generalized to the setting of the logic S4F and modal default theories the concept of uniform equivalence, and we derived characterizations of modal default theories that are uniformly equivalent.

We also studied modal programs — a subclass of the class of modal default theories. We showed that the logic here-and-there can be viewed as a fragment of the logic S4F. In particular, general logic programs with the semantics of stable models can be encoded as modal programs under the semantics of the nonmonotonic modal logic S4F. We also showed that for the class of modal programs, the logic S4F can be replaced by a stronger modal logic, the logic SW5.

One might argue that the modal logic S4F has an advantage over the logic here-and-there as the logic of general logic programs. Its language contains all boolean connectives and they are interpreted in the standard way. In the logic here-and-there, negation is a derived operator and implication has a non-standard interpretation in HT-models.

Our paper is related to (Lin & Zhou 2007). However, our logic explicitly handles nested modalities and, more importantly, requires only one modal operator and not two.

The results of the paper provide further evidence of the importance of the logic S4F for knowledge representation.

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