Nonmonotonic reasoning is sometimes simpler![1]

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Abstract

We establish the complexity of decision problems associated with the nonmonotonic modal logic S4. We prove that the problem of existence of an S4-expansion for a given set A of premises is Σ_2^P-complete. Similarly, we show that for a given formula φ and a set A of premises, it is Σ_2^P-complete to decide whether φ belongs to at least one S4-expansion for A, and it is Π_2^P-complete to decide whether φ belongs to all S4-expansions for A. This refutes a conjecture of Gottlob that these problems are PSPACE-complete. An interesting aspect of these results is that reasoning (testing satisfiability and provability) in the monotonic modal logic S4 is PSPACE-complete. To the best of our knowledge, the nonmonotonic logic S4 is the first example of a nonmonotonic formalism which is computationally easier than the monotonic logic that underlies it (assuming PSPACE does not collapse to Σ_2^P).

1 Introduction

First nonmonotonic logics were proposed in late 70s and early 80s in an attempt to construct knowledge representation tools for situations where only partial, incomplete information is available. Among these early nonmonotonic logics are *circumscription* [11], *default logic* [20] and a whole family of *modal nonmonotonic logics* [13, 12], with *autoepistemic logic* [15] as its most prominent representative. (In fact, at the time of its introduction autoepistemic logic was not known to belong to the family of logics introduced in [13, 12]. It was proved to be the case by Schwarz [22].)

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As the discipline of nonmonotonic logics matured, the issues of their practical applicability came to the focus of attention. In particular, the complexity of nonmonotonic reasoning has been recently a subject of an extensive research activity. Usually, a nonmonotonic logic is obtained from some monotonic logic by modifying its semantics, for example, by restricting to minimal models only (circumscription), or proof theoretically by means of some fixed-point construction (default logic, autoepistemic logic, nonmonotonic modal logics of [13, 12]). Checking whether a model is minimal or if a theory is a solution to a fixed-point equation is costly. In fact, when a nonmonotonic logic is defined by a fixed-point construction, it is not at all obvious that it is decidable. Hence, perhaps not unexpectedly but still rather disappointingly, in all cases studied so far nonmonotonic reasoning turns out to be more complex (unless the polynomial hierarchy collapses at some low level) than the underlying monotonic reasoning. For example, Eiter and Gottlob proved that propositional circumscription is Π_2^P-complete [1] and Gottlob proved that several problems associated with reasoning in default logic are Σ_2^P- or Π_2^P-complete [4]. Hence, under the assumption that the polynomial hierarchy does not collapse, propositional circumscription and default logic are substantially more complex than the reasoning in propositional logic.

Let us look now at the case of nonmonotonic modal logics introduced by McDermott and Doyle [13, 12]. These logics are defined by means of a fixed-point construction. The basic notion is that of an expansion. Let $\mathcal{S}$ be a propositional modal logic and let A be a set of sentences. A theory T is called an $\mathcal{S}$-*expansion*, if

$$T = Cn_{\mathcal{S}}(A \cup \{\neg K\varphi \colon \varphi \notin T\}), \tag{1}$$

where $Cn_{\mathcal{S}}$ stands for the consequence operator in the logic $\mathcal{S}$. Depending on the approach, an arbitrarily selected $\mathcal{S}$-expansion for A or the intersection of all $\mathcal{S}$-expansion for A is regarded as a set of nonmonotonic consequences of A. Hence, reasoning with the nonmonotonic logic $\mathcal{S}$ consists of checking whether a given formula is in some (or all) $\mathcal{S}$-expansions for A. Since the definition of an $\mathcal{S}$-expansion involves the consequence operator in the (monotonic) modal logic $\mathcal{S}$, it is natural to expect that a decision procedure for the nonmonotonic logic $\mathcal{S}$, if exists, is at least as complex as any decision procedure for the monotonic modal logic $\mathcal{S}$.

McDermott [12] considered nonmonotonic modal logics S5, S4 and T. He proved that the nonmonotonic logic S5 coincides with the monotonic logic S5. Hence, the computational complexity of nonmonotonic S5 coincides with that of monotonic S5, which is co-NP complete for the provability problem and NP-complete for the satisfiability problem [8]. Gottlob [4] conjectured that, although nonmonotonic logics K, T and S4 do not collapse into monotonic logics, their computational complexity coincide with the complexity of the corresponding monotonic modal logics (which all are known to be PSPACE-complete, see [8]).

Some modal logics have received special attention since their nonmonotonic versions are particularly suitable for knowledge representation purposes. The logic KD45 is one of them. It is commonly accepted as a logic of belief [9, 7]. Halpern and Moses [7] proved that the satisfiability problem for KD45 is NP-complete (and, correspondingly, the derivability problem is co-NP complete). The nonmonotonic logic KD45 is (essentially) equivalent to one of the most celebrated nonmonotonic formalisms, autoepistemic logic of Moore [15]. It was proved in [22] that for each set A of premises, a consistent theory T is a stable expansion of A if and only if T is a KD45-expansion of A. Gottlob [4] and Niemelä [18] proved that the problems of existence of a stable expansion, and of membership in some (all) stable expansions are located on the second level of the polynomial hierarchy. Hence, they are more complex (unless the polynomial hierarchy collapses to NP) than reasoning in the underlying modal logic KD45, for which the satisfiability problem is NP-complete. More results of that type, on nonmonotonic systems that are variations of autoepistemic logic, are given in [2].

Another logic with important applications in knowledge representation is logic S4. This logic is often considered as a logic of knowledge (see [14]). It was shown in [22] that if we use the nonmonotonic logic S4 instead of autoepistemic logic, then we can avoid some of the problems caused by counterintuitive behavior of stable expansions. In addition, the nonmonotonic logic S4 can be regarded as a generalization of both default logic [25] and autoepistemic logic [23]. Thus, it is important, to investigate its computational aspects. This is the main objective of this paper.

McDermott presented in [12] a decision procedure for the nonmonotonic logic S4, based on the tableau method, but no complexity analysis was given.

General results characterizing $\mathcal{S}$-expansions for a wide class of modal logics $\mathcal{S}$ were obtained in [22, 16]. These characterizations are algorithmic under the assumptions that a set A of premises is finite, and that the derivability in $\mathcal{S}$ is decidable. It was used in [16] to design algorithms for problem such as: does an $\mathcal{S}$-expansion of A exist? does a given formula belong to all (some) $\mathcal{S}$-expansions of A? The algorithms presented in [16] depend on procedures for checking derivability of formulas in the underlying (monotonic) modal logic $\mathcal{S}$. Therefore they are at least as complex. In particular, in the case of S4 they are at least as complex as any decision procedure for the monotonic logic S4.

Can we do better? In fact, we can! The main result of this paper states that reasoning in the nonmonotonic logic S4 resides on the second level of the polynomial hierarchy while S4-satisfiability is PSPACE-complete [8]. Hence, the nonmonotonic logic S4 is much less computationally complex (assuming PSPACE does not collapse to Σ_2^P) than monotonic S4. To the best of our knowledge it is the first example of a nonmonotonic logic with this property.

The key idea is to exploit the concept of a *range* [16]. It was observed in [16] that, sometimes, different monotonic modal logics generate the same nonmonotonic version. For example, logics K45 and KD45 are different, but consistent K45- and KD45-expansions coincide. If we have two logics $\mathcal{S}_1$ and $\mathcal{S}_2$ with the property that for each set A of premises, consistent $\mathcal{S}_1$-expansions of A coincide with consistent $\mathcal{S}_2$-expansions of A, then we say that logics $\mathcal{S}_1$ and $\mathcal{S}_2$ *are in the same range*. The concept of a range suggests the following approach to the problem of reasoning with nonmonotonic S4: find a computationally simple modal logic $\mathcal{S}$ in the same range as S4. Then, the decision procedure from [16] for the nonmonotonic logic $\mathcal{S}$ would be, in the same time, a decision procedure for the nonmonotonic logic S4. Since it would be based on a computationally simple logic $\mathcal{S}$, rather than on a highly complex logic S4, it might actually be simpler than any decision procedure for monotonic S4.

With this approach we immediately run into a problem. So far we have been unable to find a logic in the same range as S4 and computationally simpler than S4. However, for our complexity considerations a weaker notion of a range is sufficient. Namely such which only requires that logics in the same range generate the same expansions for all *finite* sets of premises. In [16] it was shown that there is a logic, namely the logic S4F (also known as S4.3.2), which is equivalent, in this weaker sense, to S4. That is, for every finite theory A, consistent S4-expansions of A coincide with consistent S4F-expansions of A. (In general, the nonmonotonic logics S4 and S4F are different. An example of an infinite set A of premises such that not every S4F-expansion of A is an S4-expansion of A is also given in [16].)

The logic S4F, although not as well-known as S4, has been investigated in the literature. Segerberg [21] studied its mathematical properties, and Lenzen [10] discussed it from the epistemological point of view. The nonmonotonic logic S4F has several interesting properties. In particular, similarly to the nonmonotonic logic S4, it can be regarded as a generalization of default and autoepistemic logics. A detailed study of the properties of the nonmonotonic logic S4F is given in [23]. The logic S4F has a more complex axiomatization than S4. Despite this, we will show that it is computationally much simpler than S4 (assuming PSPACE does not collapse to NP). In the paper, we will use the logic S4F, according to the plan outlined above, to establish the complexity of reasoning with the nonmonotonic logic S4.

The paper is organized as follows. Basic definitions and results are gathered in Section 2. Results on the complexity of reasoning in the nonmonotonic modal logic S4F are presented in Section 3. The last section contains main results of the paper establishing the complexity of reasoning in the nonmonotonic modal logic S4.

2 Preliminaries

We assume familiarity with basic notions of the complexity theory such as the polynomial hierarchy, the classes P, NP, Δ_k^P, Σ_k^P, Π_k^P, PSPACE, the notion of completeness of a problem in a class. For a good presentation of the topic of complexity the reader is referred to Garey and Johnson [3]. Let us recall that class NP coincides with Σ_1^P, class co-NP coincides with Π_1^P. Class Δ_2^P (Σ_2^P) consists of problems which can be decided in polynomial time by a deterministic algorithm (a non-deterministic algorithm) with an oracle for a problem belonging to NP. Class Π_2^P consists of problems whose complements are in Σ_2^P. For a detailed presentation of the topic of the polynomial hierarchy the reader is referred to [3].

We will present now basic notions and results on modal logics. More on the subject can be found in Hughes and Cresswell [6]. We will consider the language of modal logic obtained by extending a language of propositional calculus, say $\mathcal{L}$, by a *necessity* modal operator K. We will denote it by $\mathcal{L}_K$. Let $\mathcal{S}$ be a modal logic.

By $A \vdash_{\mathcal{S}} \psi$ we denote the fact that a formula ψ is derivable from a set A of formulas by means of axioms of $\mathcal{S}$ and the inference rules modus ponens and necessitation. Note that our notion of $\vdash_{\mathcal{S}}$ differs from that in [6] and some other books and papers on modal logic. Namely, in [6], $A \vdash_{\mathcal{S}} \psi$ denotes that ψ is derivable from A and *theorems* of $\mathcal{S}$ by means of modus ponens only. Thus, in our sense, $p \vdash_{\mathcal{S}} Kp$ while, under the stronger interpretation of $\vdash_{\mathcal{S}}$, this statement is false. However, this is mostly just a terminological difference and all our results may be reformulated under the alternative notation. By $Cn_{\mathcal{S}}$ we denote the provability operator in the modal logic $\mathcal{S}$. That is, $Cn_{\mathcal{S}}(A) = \{\psi : A \vdash_{\mathcal{S}} \psi\}$.

In this paper we focus our attention on two modal logics: S4 and S4F. Logic S4 is specified by the following three axiom schemata:

K: $Kp \supset (K(p \supset q) \supset Kq)$;

T: $Kp \supset p$;

4: $Kp \supset KKp$.

Logic S4F (also known as S4.3.2) is specified by the schemata K, T and 4 and, in addition by the schema

F: $(p \wedge MKq) \supset K(Mp \vee q)$,

where $M\varphi$ abbreviates, as usual, the formula $\neg K \neg \varphi$.

We assume familiarity with Kripke models (see [6, 17] for a detailed presentation of modal logics). Here we briefly recall the terminology we use. A *Kripke model* is a triple $\mathcal{M} = \langle M, R, V \rangle$, where M is a nonempty set (called the set of *worlds* of $\mathcal{M}$), R is a binary relation on M, and for each $\alpha \in M$, V_α is a valuation of propositional variables of the language. By $(\mathcal{M}, \alpha) \models \psi$ we denote that a formula ψ is true in a world α of $\mathcal{M}$. Let us recall that, by definition, $(\mathcal{M}, \alpha) \models K\psi$ if and only if for each $\beta \in M$ such that $\alpha R \beta$, $(\mathcal{M}, \beta) \models \psi$.

We write $\mathcal{M} \models \psi$ and say that ψ is *valid* in $\mathcal{M}$ if for each $\alpha \in M$, $(\mathcal{M}, \alpha) \models \psi$. Let A be a modal theory. We write $\mathcal{M} \models A$ and say that A is *valid* in $\mathcal{M}$, if $\mathcal{M} \models \psi$ for every $\psi \in A$.

Let $\mathcal{K}$ be a class of Kripke models. We say that $\mathcal{S}$ is *characterized* by $\mathcal{K}$, if the following holds: for each set A of formulas, and for each ψ, $A \vdash_{\mathcal{S}} \psi$ if and only if for each Kripke model $\mathcal{M} \in \mathcal{K}$, $\mathcal{M} \models A$ implies $\mathcal{M} \models \psi$. It is well known that logic S4 is characterized by the class of models with reflexive and transitive accessibility relation. We will refer to such models as S4-*models*. Logic S4F is characterized by the class $\mathcal{C}(\mathrm{S4F})$ of models of the form

$$\langle M, (M_1 \times M) \cup (M_2 \times M_2), V \rangle,$$

where $M = M_1 \cup M_2$, $M_1 \cap M_2 = \emptyset$ and $M_2 \neq \emptyset$. Observe that we do not require that M_1 be nonempty. Hence, speaking informally, $\mathcal{C}(\mathrm{S4F})$ consists of clusters and of pairs of clusters one concatenated on top of the other. Models in $\mathcal{C}(S4F)$ are referred to as S4F-*models*. Finally, models of the form $\langle M, M \times M, V \rangle$ characterize the logic S5 (defined by the modal axioms K, T, 4 and 5: $\neg Kp \supset K \neg Kp$). They are called *universal* S5-*models*.

Remark 1 Usually, the characterization results are proved in a weaker form, namely, that $\vdash_{\mathcal{S}} \psi$ if and only if for each $\mathcal{M} \in K$, $\mathcal{M} \models \psi$. Characterization results used in this paper can be obtained from the weaker ones, see [12, 17].

It is well known, that if $\not\vdash_{\mathrm{S4}} \psi$, then there is a *finite* S4-model $\mathcal{M}$ and a world α in $\mathcal{M}$ such that $(\mathcal{M}, \alpha) \not\models \psi$. This result is known as the *finite model property* for S4. We will now prove a generalization of this result.

Proposition 2 Let A be a finite set of formulas. If $A \not\vdash_{\mathrm{S4}} \psi$, then there is a Kripke model $\mathcal{M} = \langle M, R, V \rangle$ such that M is finite, R is transitive and reflexive, $\mathcal{M} \models A$, and for some $\alpha \in M, (\mathcal{M}, \alpha) \not\models \psi$.

Proof: Let Φ be a conjunction of all the formulas of A. Clearly, $\Phi \not\vdash_{\mathrm{S4}} \psi$. Since $\Phi \vdash_{\mathrm{S4}} K\Phi$, it follows that $\not\vdash_{\mathrm{S4}} K\Phi \supset \psi$. Applying the finite model property for S4, we get a finite model $\mathcal{N} = \langle N, Q, W\rangle$, where Q is reflexive and transitive, and a world $\alpha \in N$ such that $(\mathcal{N}, \alpha) \not\models K\Phi \supset \psi$. Then $(\mathcal{N}, \alpha) \models K\Phi$ and $(\mathcal{N}, \alpha) \not\models \psi$. Let $\mathcal{M} = \langle M, R, V\rangle$ be the submodel of $\mathcal{N}$ generated by α. That is, $M = \{\beta : \alpha R \beta\}$, $R = Q \cap (M \times M)$, V is the restriction of W on M.

It is easy to prove by induction on the length of a formula that for every formula φ and for every world $\alpha' \in M$, $(\mathcal{M}, \alpha') \models \varphi$ if and only if $(\mathcal{N}, \alpha') \models \varphi$. By the definition of $\mathcal{M}$, we also have $\alpha \in M$. Consequently, $(\mathcal{M}, \alpha) \models K\Phi$ and $(\mathcal{M}, \alpha) \not\models \psi$. Also by the definition of $\mathcal{M}$, for every $\alpha' \in M$, $\alpha Q \alpha'$ and, hence, $\alpha R \alpha'$. Thus, $\mathcal{M} \models \Phi$ or, equivalently, $\mathcal{M} \models A$. Hence, the assertion follows. □

We say that a theory A is *consistent* with a modal logic $\mathcal{S}$ if $A \not\vdash_{\mathcal{S}} \bot$. The following result on consistency of theories with logics S4, S4F and S5 is well-known (see [24, 8, 17] for more details).

Proposition 3 Let A be a finite modal theory, let k be the number of occurrences of the modal operator K in A. Then the following conditions are equivalent:

1. A is consistent with S4
2. A is consistent with S4F
3. A is consistent with S5
4. A is valid in a universal S5-model with $k+1$ worlds.

Proof: The equivalence of (1) and (3) is proved, for example, in [24]. Since S4F contains S4 and is contained in S5, the equivalence of (1), (2) and (3) follows. For the proof that (3) is equivalent to (4), assume that the conjunction of all formulas in A is Φ. Now, note that Φ is valid in a universal S5-model if and only if $K\Phi$ is satisfiable in some world of the model, and that Φ is consistent with S5 if and only if $\not\vdash_{\mathrm{S5}} \neg K\Phi$ (deduction theorem for S5, see e.g. [12]). The result then follows from a theorem by Ladner [8]. □

Next, we present basic notions and results on modal nonmonotonic logics (a detailed treatment of the subject can be found in [17] and [16]). Let $\mathcal{S}$ be a modal logic. Let $A \subseteq \mathcal{L}_K$. A theory $T \subseteq \mathcal{L}_K$ is an $\mathcal{S}$-expansion for A if

$$T = Cn_{\mathcal{S}}(A \cup \{\neg K\varphi : \varphi \notin T\}). \tag{2}$$

For a wide range of modal logics $\mathcal{S}$, $\mathcal{S}$-expansions have been effectively characterized in [22, 16]. We will recall this characterization in the case of logics S4 and S4F. Given a theory $A \subseteq \mathcal{L}_K$, we define

$$A^K = \{\varphi : K\varphi \text{ is a subformula of a formula from } A\}.$$

Any formula built of propositional variables and modal atoms (formulas of the form $K\varphi$) from the set $\{K\varphi : \varphi \in A^K\}$ is called an *A-formula*.

A pair (Φ, Ψ) of subsets of A^K is called *introspection-consistent* with A if:

(C1) $\Phi \cap \Psi = \emptyset$ and $\Phi \cup \Psi = A^K$;

(C2) $A \cup \{\neg K\varphi : \varphi \in \Phi\} \cup \{K\psi : \psi \in \Psi\} \cup \Psi$ is propositionally consistent;

(C3) For each $\varphi \in \Phi$, $A \cup \{\neg K\varphi : \varphi \in \Phi\} \cup \{K\psi : \psi \in \Psi\} \cup \Psi \not\vdash \varphi$ ($\vdash$ denotes propositional provability relation).

A *propositional valuation* assigns a truth value to each propositional letter, and to each formula of the form $K\psi$. Note that no relationship is assumed between truth values of $K\psi$ and ψ. That is, formulas beginning with K are treated here as atoms. Propositional valuations are extended, in the standard way, to the set of all modal formulas.

Let a pair (Φ, Ψ) of subsets of A^K be introspection-consistent with A. Let $V_A(\Phi, \Psi)$ consist of all propositional valuations v of $\mathcal{L}_K$ such that

$$v(A \cup \{\neg K\varphi : \varphi \in \Phi\} \cup \{K\psi : \psi \in \Psi\} \cup \Psi) = 1.$$

Denote by $\mathcal{M}_A(\Phi, \Psi)$ the Kripke model

$$\langle V_A(\Phi, \Psi), V_A(\Phi, \Psi) \times V_A(\Phi, \Psi), U_A(\Phi, \Psi) \rangle,$$

where for every $v \in V_A(\Phi, \Psi)$, $U_A(\Phi, \Psi)(v) = v|\mathcal{L}$ ($v|\mathcal{L}$ denotes the restriction of a valuation v to the set of formulas in $\mathcal{L}$). Clearly, $\mathcal{M}_A(\Phi, \Psi)$ is a universal S5-Kripke model. Finally, define

$$T_A(\Phi, \Psi) = \{\varphi \in \mathcal{L}_K : \mathcal{M}_A(\Phi, \Psi) \models \varphi\}.$$

The following theorem [22, 16] gathers useful properties of $T_A(\Phi, \Psi)$. The next one provides a characterization of S4- and S4F-expansions.

Theorem 4 Let $A \subseteq \mathcal{L}_K$. If a pair (Φ, Ψ) of subsets of A^K is introspection-consistent with A, then

1. The theory $T_A(\Phi, \Psi)$ is consistent, closed under propositional provability and necessitation, and for every $\varphi \notin T_A(\Phi, \Psi)$, $\neg K\varphi \in T_A(\Phi, \Psi)$.

2. For every A-formula ξ, $\xi \in T_A(\Phi, \Psi)$ if and only if

$$A \cup \{\neg K\varphi : \varphi \in \Phi\} \cup \{K\psi : \psi \in \Psi\} \cup \Psi \vdash \xi.$$

Theorem 5 Let $\mathcal{S}$ stand for S4 or S4F. A consistent with $\mathcal{S}$ theory T is an $\mathcal{S}$-expansion for A if and only if $T = T_A(\Phi, \Psi)$ for some pair (Φ, Ψ) introspection-consistent with A and such that for every $\psi \in \Psi$,

$$A \cup \{\neg K\varphi : \varphi \in \Phi\} \vdash_{\mathcal{S}} \psi. \qquad (3)$$

Note that this theorem implies, for a finite theory A, a decision procedure for non-monotonic S4, establishing existence of S4-expansions. The procedure first decides whether A is consistent with S4. This problem is decidable, in fact, it is NP-complete (see Proposition 3 and [8]). If the answer is negative, A has a unique S4-expansion $\mathcal{L}_K$. Otherwise, every S4-expansion of A is consistent with S4 and can be found using Theorem 5. That is, a procedure for testing S4-derivability is invoked to decide (3), for every possible pair (Φ, Ψ) introspection consistent with A (checking introspection consistency is "easy" is at involves only propositional provability). Hence, this procedure to determine existence of S4-expansions is at least as hard as the S4-derivability problem. Recall that S4-derivability problem is PSPACE-complete. Consequently, this straightforward decision procedure for nonmonotonic S4 is quite complex. Our subsequent results show that, in fact, decision problems for nonmonotonic S4 reside on the second level of the polynomial hierarchy. In other words, the particular instances of S4 derivability, involved in (3), are not hard.

3 Nonmonotonic modal logic S4F

Let $\mathcal{S}$ be a modal logic. $\mathcal{S}$-expansions can be regarded as formal descriptions of realities an agent reasoning (nonmonotonically) from A regards as possible. There are several problems such an agent must be able to solve in order to reason. For example, is there an $\mathcal{S}$-expansion of A, does a formula belong to some (all) $\mathcal{S}$-expansions of A, etc. Hence, it is important to establish the complexity of these problems. In this paper we will study the complexity of the following four problems:

EXISTENCE($\mathcal{S}$) Given a finite theory $A \subseteq \mathcal{L}_K$, decide if A has an $\mathcal{S}$-expansion;

IN-SOME($\mathcal{S}$) Given a finite theory $A \subseteq \mathcal{L}_K$ and a formula $\varphi \in \mathcal{L}_K$, decide if φ is in some $\mathcal{S}$-expansion of A;

NOT-IN-ALL($\mathcal{S}$) Given a finite theory $A \subseteq \mathcal{L}_K$ and a formula $\varphi \in \mathcal{L}_K$, decide if there is an $\mathcal{S}$-expansion for A not containing φ;

IN-ALL($\mathcal{S}$) Given a finite theory $A \subseteq \mathcal{L}_K$ and a formula $\varphi \in \mathcal{L}_K$, decide if φ is in all $\mathcal{S}$-expansions of A.

We will concentrate on the cases $\mathcal{S} =$ S4F and $\mathcal{S} =$ S4. We start with the case of logic S4F. We will need two auxiliary results. The next theorem establishes that the complexity of monotonic S4F is essentially the same as that of S5. Our proof is based on the ideas of Ladner [8].

A set of formulas, A, is called S4F-*satisfiable*, if there exists an S4F-model $\langle M, R, V\rangle$ and a world $\alpha \in M$, such that A is true in the world α. Since the modal logic S4F is characterized by the class of S4F-models, we have the following simple fact establishing the relationship between the notions of provability and satisfiability in S4F. The proof is standard and is omitted.

Proposition 6 A formula φ is provable in S4F from a set of premises A if and only if the theory $\{K\psi : \psi \in A\} \cup \{\neg\varphi\}$ is *not* S4F-satisfiable. □

Theorem 7 The following problem is NP-complete:

S4F-SAT Given a finite theory $A \subseteq \mathcal{L}_K$, is A S4F-satisfiable?

Proof: NP-hardness follows easily from the fact that propositional satisfiability is a special case of S4F-satisfiability: a theory consisting of modal-free formulas is propositionally satisfiable if and only if it is S4F-satisfiable.

To prove that the problem S4F-SAT is in NP is harder. We will represent an S4F-model $\mathcal{M} = \langle M, (M_1 \times M) \cup (M_2 \times M_2), U\rangle$, where $M = M_1 \cup M_2$ and $M_1 \cap M_2 =$

$\emptyset$, by specifying the sets M_1 and M_2, and by representing each valuation U_m, for $m \in M_1 \cup M_2$, as a set of those propositional variables that are assigned the truth value 1. By the size of a model we mean the total size of such representation. We will show that the problem S4F-SAT is in NP by showing that if a finite theory A is S4F-satisfiable, then it is satisfiable in an S4F-model whose size is polynomial in the size of an encoding of A.

Let us assume that A is S4F-satisfiable. Let $\mathcal{M} = \langle M, (M_1 \times M) \cup (M_2 \times M_2), U \rangle$, where $M = M_1 \cup M_2$ and $M_1 \cap M_2 = \emptyset$, be an S4F-model and let $\alpha \in M$ be a world such that$(\mathcal{M}, \alpha) \models A$. Without loss of generality, we may assume that each valuation U_m, $m \in M$, assigns the truth value 0 to those propositional variables that do not appear in the formulas from A. Since A is finite, each valuation U_m is then represented by a finite set of propositional variables. Its cardinality does not exceed the total number of propositional variables appearing in the formulas from A. Thus, the size of our representation of each U_m is bounded from above by the size of (the encoding of) A.

For each $\varphi \in A^K$ and such that $\mathcal{M} \not\models K\varphi$ select a world $m \in M$ such that $(\mathcal{M}, m) \not\models \varphi$ under an additional stipulation that, if possible, m be selected from M_2. Denote the world selected by m_φ. Next, for $i = 1, 2$ define

$$M'_i = \{m_\varphi : \varphi \in A^K,\ \mathcal{M} \not\models K\varphi \text{ and } m_\varphi \in M_i\} \cup (\{\alpha\} \cap M_i).$$

Notice that $\alpha \in M'_1 \cup M'_2$. Next, consider the Kripke model

$$\mathcal{M}' = \langle M', (M'_1 \times M') \cup (M'_2 \times M'_2), U' \rangle,$$

where $M' = M'_1 \cup M'_2$ and $U' = U|M'$ (the restriction of U on M'). Note that $M' \neq \emptyset$. Hence, $\mathcal{M}'$ is an S4F-model (even an S5-model if M'_1 or M'_2 is empty). We will prove that for every A-formula φ and for every world $m \in M'$,

$$(\mathcal{M}, m) \models \varphi \text{ if and only if } (\mathcal{M}', m) \models \varphi. \tag{4}$$

We proceed by induction on the length of formulas. For the basis of induction, assume that $\varphi = p$, for some propositional variable p. Since $U'_m = U_m|M'$, for each $m \in M'$, the equivalence (4) follows. Consider now an A-formula φ other than a propositional variable, and assume that (4) holds for every A-formula of length less than the length of φ. If φ is of the form $\neg\psi$ or $\psi_1 \circ \psi_2$, for some boolean connective $\circ$, then the induction step follows easily from the induction hypothesis (ψ, or ψ_1 and ψ_2 are A-formulas shorter than φ) and the definition of the relation $\models$ for Kripke models.

Hence, let us consider the only remaining case when $\varphi = K\psi$. Clearly, ψ is also an A-formula and it is shorter than φ. Let $m \in M'$ and assume that $(\mathcal{M}, m) \models \varphi$. If $m \in M'_1$ ($m \in M'_2$) then, for every world $m' \in M'$ (for every world $m' \in M'_2$),

$$(\mathcal{M}, m') \models \psi.$$

By the induction hypothesis,

$$(\mathcal{M}', m') \models \psi.$$

Hence, it follows that in both cases ($m \in M'_1$ and $m \in M'_2$)

$$(\mathcal{M}', m) \models \varphi.$$

Conversely, let us assume that $(\mathcal{M}, m) \not\models \varphi$. Then, $(\mathcal{M}, m_\psi) \not\models \psi$. Since $m_\psi \in M'$, by the induction hypothesis,

$$(\mathcal{M}', m_\psi) \not\models \psi.$$

In addition, by the choice of m_ψ (if possible, m_ψ is taken from M_2), m_ψ is accessible from m. Hence,

$$(\mathcal{M}', m) \not\models K\psi.$$

This completes the proof of (4).

Since $(\mathcal{M}, \alpha) \models A$, we have $(\mathcal{M}', \alpha) \models A$, too. Next observe that $|M'| \leq |A^K| + 1$ is linear in the size of A. Since each valuation U_m, $m \in M$, has a representation of size not exceeding the size of the representation of A, and since $U'_m = U_m$, for $m \in M'$, the size of the representation of $\mathcal{M}'$ is polynomial in the size of the representation of A. It follows then that for each YES-instance of the problem S4F-SAT there is a polynomial-size "evidence" of that. Hence, a nondeterministic algorithm to decide S4F-SAT would first "guess" this polynomial size model and, then, would check that in one of its worlds all formulas from A are true. Since the model has size polynomial in A, this verification can be accomplished in time proportional to the size of A. Consequently, S4F-SAT is in NP. □

The same complexity result for logic S4.3 and some of its versions was established earlier in [19]. The argument given above can be generalized to an arbitrary index logic (see [21] or [17] for the definition of index logics). In this way one can prove the following result.

Theorem 8 Let $\mathcal{S}$ be an index logic. The following problem is NP-complete:

S-SAT Given a finite theory $A \subseteq \mathcal{L}_K$, is A S-satisfiable?

The next result is a basis for an algorithm to test whether a formula $\xi \in \mathcal{L}_K$ belongs to a theory $T_A(\Phi, \Psi)$.

Lemma 9 Let $A \subseteq \mathcal{L}_K$ and $\xi \in \mathcal{L}_K$. If a pair (Φ, Ψ) of subsets of A^K is introspection-consistent with A, then $\xi \in T_A(\Phi, \Psi)$ if and only if

$$\begin{aligned} &A \cup \{\neg K\varphi : \varphi \in \Phi\} \cup \{K\psi : \psi \in \Psi\} \cup \Psi \cup \\ &\quad \{K\alpha : \alpha \in \{\xi\}^K \cap T_A(\Phi, \Psi)\} \cup \{\neg K\alpha : \alpha \in \{\xi\}^K \setminus T_A(\Phi, \Psi)\} \vdash \xi. \end{aligned}$$

Proof: Let us denote the theory

$$\begin{aligned} &A \cup \{\neg K\varphi : \varphi \in \Phi\} \cup \{K\psi : \psi \in \Psi\} \cup \Psi \cup \\ &\quad \{K\alpha : \alpha \in \{\xi\}^K \cap T_A(\Phi, \Psi)\} \cup \{\neg K\alpha : \alpha \in \{\xi\}^K \setminus T_A(\Phi, \Psi)\} \end{aligned}$$

by Σ^ξ. By Theorem 4 (1) and (2), $\Sigma^\xi \subseteq T_A(\Phi, \Psi)$. Hence, if $\Sigma^\xi \vdash \xi$, then $\xi \in T_A(\Phi, \Psi)$ (recall that by Theorem 4(1), $T_A(\Phi, \Psi)$ is closed under propositional provability). This proves the sufficiency part of the assertion.

Conversely, assume that $\xi \in T_A(\Phi, \Psi)$. Since ξ and all its propositionally equivalent normal forms have the same set of modal atoms (that is, formulas of the form $K\varphi$), without loss of generality we may assume that ξ is of the form

$$K\alpha_1 \vee \ldots \vee K\alpha_k \vee \neg K\beta_1 \vee \ldots \vee \neg K\beta_m \vee \gamma,$$

where γ is modal-free. It follows from Theorem 4(1) that there are the following three possibilities:

1. $\alpha_i \in T_A(\Phi, \Psi)$ for some i, $1 \leq i \leq k$;
2. $\beta_i \notin T_A(\Phi, \Psi)$ for some i, $1 \leq i \leq m$;
3. $\gamma \in T_A(\Phi, \Psi)$.

(1) Since $\alpha_i \in \{\xi\}^K$, $K\alpha_i \in \Sigma^\xi$. Consequently, $\Sigma^\xi \vdash \xi$.
(2) Similarly, since $\beta_i \in \{\xi\}^K$, $\neg K\beta_i \in \Sigma^\xi$. Hence, $\Sigma^\xi \vdash \xi$.
(3) Since γ is modal-free, γ is an A-formula. By Theorem 4(2),

$$A \cup \{\neg K\varphi \colon \varphi \in \Phi\} \cup \{K\psi \colon \psi \in \Psi\} \cup \Psi \vdash \gamma.$$

Hence, $\Sigma^\xi \vdash \gamma$ and, consequently, $\Sigma^\xi \vdash \xi$. □

Corollary 10 The problem of deciding whether a formula ξ belongs to $T_A(\Phi, \Psi)$ is in Δ_2^P.

Proof: It follows from Lemma 9 that in order to decide whether $\xi \in T_A(\Phi, \Psi)$ it suffices to decide whether $\Sigma^\xi \vdash \xi$. In order to construct Σ^ξ we need to decide whether $\alpha \in T_A(\Phi, \Psi)$ for each $\alpha \in \{\xi\}^K$. It can be accomplished by constructing the theory Σ^α and checking whether $\Sigma^\alpha \vdash \alpha$. Lemma 9 implies the right order in which formulas from $\{\xi\}^K$ must be dealt with. Namely, one should consider formulas with smaller K-depths first. It is easy to see that all the theories constructed in the process have sizes polynomial in the total size of A and ξ and that the number of calls to an oracle for propositional provability (which can easily be designed from an oracle for propositional satisfiability) is also polynomial in the size of A and ξ. Since the propositional satisfiability problem is in NP, the assertion follows. □

Remark 11 Since the paper was submitted, Gottlob [5] proved that the problem discussed in Corollary 10 is complete for the class $\Delta_2^P(O(\log n))$.)

Now, we are ready to state and prove our theorem on the complexity of reasoning in nonmonotonic logic S4F.

Theorem 12 Under the restriction to theories A which are consistent with S4F, problems EXISTENCE(S4F), IN-SOME(S4F) and NOT-IN-ALL(S4F) are Σ_2^P-complete. Problem IN-ALL(S4F) is Π_2^P-complete.

Proof: The hardness part was established by Gottlob [4], so it remains to show that the problems in question belong to Σ_2^P (Π_2^P).

First, recall that theories consistent with S4F have only consistent S4F-expansions. To show that a finite theory has a consistent S4F-expansion it is enough to guess a pair (Φ, Ψ) of subsets of A^K (its size is polynomial in the size of A) and to check conditions (C1) - (C3) of the definition of introspection-consistent pairs of subsets of A^K, and condition (3) of Theorem 5 (with $\mathcal{S}$ =S4F). The checking phase requires polynomially many calls to the propositional satisfiability procedure and polynomially many calls to the S4F-satisfiability procedure (let us recall that, by Proposition 6, ψ is provable from a theory B in S4F if and only if $\{K\varphi : \varphi \in B\} \cup \{\neg\psi\}$ is not S4F-satisfiable). Hence, EXISTENCE(S4F) is in Σ_2^P (under the restriction to theories A which are consistent with S4F).

Let us now consider the problem IN-SOME(S4F). To show that a formula φ is in some consistent S4F-expansion of A, it is enough to guess a pair (Φ, Ψ) and check two things. First, we need to verify that $T_A(\Phi, \Psi)$ is an S4F-expansion for A (we have seen that polynomially many calls to oracles for problems in NP are enough). Secondly, we need to check that $\varphi \in T_A(\Phi, \Psi)$ which, according to Corollary 10 can also be achieved by means of polynomially many calls to on oracle for a problem in NP.

In a similar way we show that the problem NOT-IN-ALL(S4F) is Σ_2^P-complete and, consequently, that the problem IN-ALL(S4F) is Π_2^P-complete (all this under the restriction to theories A which are consistent with S4F). □

The assumption that A be consistent can be removed from the formulation of the last theorem:

Corollary 13 Problems EXISTENCE(S4F), IN-SOME(S4F) and NOT-IN-ALL(S4F) are Σ_2^P-complete. Problem IN-ALL(S4F) is Π_2^P-complete.

Proof: We will provide the proof only for the problem IN-SOME(S4F). Other problems can be dealt with similarly.

Let φ be a modal formula and A be a finite set of modal formulas. Consider the following procedure to check if there is an S4F-expansion of A containing φ: check if A is consistent with S4F (by referring to an oracle for this problem). If the answer is no, A has a unique S4F-expansion, $\mathcal{L}_K$. Output YES and stop. Otherwise, every S4F-expansion of A is consistent. Use the nondeterministic procedure from the previous proof to decide whether $\varphi \in A$. Clearly, the whole procedure is correct, requires polynomially many calls to oracles to NP-complete problems and works in polynomial time (assuming each call to the oracles takes constant time). Hence, the problem IN-SOME(S4F) is in Σ_2^P.

Since a theory A has an inconsistent S4F-expansion if and only if it is inconsistent with S4F, hardness follows from the hardness of the restricted version of the IN-SOME(S4F) problem. □

4 Main result

The method used in the proof of Theorem 12 will not work in the case of logic S4. The reason is that S4-satisfiability is PSPACE-complete rather than NP-complete (as in the case of logic S4F). However, there is a way of applying Theorem 12 to obtain identical bounds for logic S4.

In [16] it has been noted that different modal logics may have equivalent nonmonotonic counterparts. The case of logics S4 and S4F is especially interesting. An example is given in [16] which shows that nonmonotonic logics S4 and S4F are not identical. Namely, a theory is constructed such that one of its S4F-expansions is not its S4-expansion. However, the theory used in this example is infinite and this turns out to be essential. In [16] it is proved that for *finite* sets of premises nonmonotonic logics S4 and S4F coincide. Since this property provides the key element for the proof of our main result establishing the complexity of reasoning with the nonmonotonic logic S4, we repeat the proof below.

Theorem 14 Let $A \subseteq \mathcal{L}_K$ be finite. A consistent theory $T \subseteq \mathcal{L}_K$ is an S4-expansion of A if and only if T is an S4F-expansion of A.

Proof: Let T be consistent. By Theorem 5, if T is an S4-expansion for A then T is an S4F-expansion for A (even without an assumption that A is finite). Theorem 5 implies also that to prove the converse implication it is enough to show that for any pair (Φ, Ψ), introspection-consistent with A, if

$$A \cup \{\neg K\varphi : \varphi \in \Phi\} \vdash_{\mathrm{S4F}} \Psi; \tag{5}$$

then

$$A \cup \{\neg K\varphi : \varphi \in \Phi\} \vdash_{\mathrm{S4}} \Psi. \tag{6}$$

Hence, let (Φ, Ψ) be a pair of subsets of A^K introspection-consistent with A. Assume also that (5) holds. According to Proposition 2, to prove (6) it is enough to show that for every *finite* S4-model $\mathcal{M}$, if $\mathcal{M} \models A \cup \{\neg K\varphi : \varphi \in \Phi\}$, then $\mathcal{M} \models \Psi$.

So, let $\mathcal{M} = \langle M, R, V\rangle$ be a finite S4-model such that $\mathcal{M} \models A \cup \{\neg K\varphi : \varphi \in \Phi\}$. By a *rank* of a world $\alpha \in M$, $r(\alpha)$, we mean the maximal integer n such that there exists a sequence of worlds $\alpha_0, \ldots, \alpha_n$, where $\alpha_0 = \alpha$ and for each i, $0 \leq i < n$, $\alpha_i R \alpha_{i+1}$, but not $\alpha_{i+1} R \alpha_i$. Since M is finite, each world has a rank. We prove by the induction on $r(\alpha)$, that for each $\alpha \in M$, $(\mathcal{M}, \alpha) \models \Psi$.

Let $r(\alpha) = 0$. Put $M_\alpha = \{\beta : \beta R \alpha \text{ and } \alpha R \beta\}$. Consider the Kripke model $\mathcal{M}_\alpha = \langle M_\alpha, R_\alpha, V_\alpha\rangle$, where R_α and V_α are the restrictions of R and V to M_α. Clearly, $\mathcal{M}_\alpha$ is an S4F-model (in fact, even a universal S5-model). Since $r(\alpha) = 0$, there is no world $\beta \in M$ such that $\alpha R \beta$ but not $\beta R \alpha$. It follows that for every formula θ and for every world $\beta \in \mathcal{M}_\alpha$, $(\mathcal{M}, \beta) \models \theta$ if and only if $(\mathcal{M}_\alpha, \beta) \models \theta$. In particular, since $\mathcal{M} \models A \cup \{\neg K\varphi : \varphi \in \Phi\}$, $\mathcal{M}_\alpha \models A \cup \{\neg K\varphi : \varphi \in \Phi\}$. Since (5) holds and $\mathcal{M}_\alpha$ is an S4F-model, $\mathcal{M}_\alpha \models \Psi$. Hence, $(\mathcal{M}_\alpha, \alpha) \models \Psi$. Consequently, $(\mathcal{M}, \alpha) \models \Psi$.

Consider now a world $\alpha \in M$ with $r(\alpha) = k + 1$, and assume that for each world $\beta \in M$ such that $r(\beta) \leq k$ $(\mathcal{M}, \beta) \models \Psi$.

Define $M_0 = \{\gamma \in M : \alpha R \gamma \text{ and } \gamma R \alpha\}$ and $M_1 = \{\gamma \in M : \alpha R \gamma \text{ but not } \gamma R \alpha\}$. Clearly, M_0 and M_1 are disjoint and for each γ, $\alpha R \gamma$ if and only if $\gamma \in M_0 \cup M_1$.

Define a Kripke model $\mathcal{N} = \langle N, Q, U\rangle$ as follows. Put $N = M_0 \cup M_1$ and define $\beta Q \gamma$ if $\gamma \in M_1$ or $\beta, \gamma \in M_0$. Define U to be the restriction of V to N. Observe that for $\beta \in N$ and for every world $\gamma \in M$, if $\beta R \gamma$ then $\beta Q \gamma$. Moreover, if $\beta \in M_0$ then $\beta R \gamma$ if and only if $\beta Q \gamma$.

We will prove now that for each $\beta \in N$ and for each A-formula η, $(\mathcal{M}, \beta) \models \eta$ if and only if $(\mathcal{N}, \beta) \models \eta$. The proof will be by induction on the complexity of η. (So, we have a double induction here. The external induction on $r(\alpha)$ and, within its induction step, an internal induction on the complexity of a formula.) The basis of the (internal) induction is obvious. The only nontrivial case in the induction step is when $\eta = K\zeta$.

Assume $(\mathcal{N}, \beta) \models K\zeta$. Consider an arbitrary world γ such that $\beta R \gamma$. Then, $\beta Q \gamma$. Hence, $(\mathcal{N}, \gamma) \models \zeta$. By the induction hypothesis, $(\mathcal{M}, \gamma) \models \zeta$. Consequently, $(\mathcal{M}, \beta) \models K\zeta$.

Conversely, assume that $(\mathcal{M}, \beta) \models K\zeta$. If $\beta \in M_0$, then, since for each γ, $\beta R \gamma$ if and only if $\beta Q \gamma$, it follows directly from the (internal) induction hypothesis that $(\mathcal{N}, \beta) \models K\zeta$.

Assume then that $\beta \in M_1$. Since $\mathcal{M} \models \{\neg K\varphi : \varphi \in \Phi\}$, we have that $\zeta \in \Psi$ (since $K\zeta$ is an A-formula, ζ belongs to A^K). By the definition of the model $\mathcal{N}$, we have that for each $\gamma \in M_1$, $r(\gamma) \leq k$. Thus, applying the (external) induction hypothesis, we obtain that for each $\gamma \in M_1$, $(\mathcal{M}, \gamma) \models \zeta$. By the (internal) induction hypothesis, for each $\gamma \in M_1$ (that is, for each γ such that $\beta Q \gamma$), $(\mathcal{N}, \gamma) \models \zeta$. Hence $(\mathcal{N}, \beta) \models K\zeta$.

Thus, we have proved that for each A-formula η and for each $\beta \in N$, $(\mathcal{M}, \beta) \models \eta$ if and only if $(\mathcal{N}, \beta) \models \eta$. Since $A \cup \{\neg K\varphi : \varphi \in \Phi\}$ consists of A-formulas, we obtain that $\mathcal{N} \models A \cup \{\neg K\varphi : \varphi \in \Phi\}$. But $\mathcal{N}$ is an S4F-model. Since (5) holds, $\mathcal{N} \models \Psi$. In particular, $(\mathcal{N}, \alpha) \models \Psi$ and, since Ψ consists of A-formulas, $(\mathcal{M}, \alpha) \models \Psi$. This completes the proof of the induction step of the main induction. □

Proposition 3 and Theorems 12 and 14 immediately imply the following corollary, which is the main result of our paper.

Corollary 15 Under the restriction to theories which are consistent with S4, problems EXISTENCE(S4), IN-SOME(S4) and NOT-IN-ALL are Σ_2^P-complete and problem IN-ALL(S4) is Π_2^P-complete. □

As in the case of the logic S4F, the restriction to theories which are consistent with S4 can be eliminated.

Corollary 16 Problems EXISTENCE(S4), IN-SOME(S4) and NOT-IN-ALL are Σ_2^P-complete and problem IN-ALL(S4) is Π_2^P-complete. □

Speaking informally, Corollaries 15 and 16 show that, unless the class PSPACE collapses to Σ_2^P, nonmonotonic modal logic S4 is computationally simpler than (monotonic) modal logic S4. To the best of our knowledge it is the first example when nonmonotonic reasoning turns out to be easier than the monotonic ones than underlies it.

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