Skeptical Rational Extensions

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Abstract. In this paper we propose a version of default logic with the following two properties: (1) defaults with mutually inconsistent justifications are never used together in constructing a set of default consequences of a theory; (2) the reasoning formalized by our logic is related to the traditional skeptical mode of default reasoning. Our logic is based on the concept of a *skeptical rational extension*. We give characterization results for skeptical rational extensions and an algorithm to compute them. We present some complexity results. Our main goal is to characterize cases when the class of skeptical rational extensions is closed under intersection. In the case of normal default theories our logic coincides with the standard skeptical reasoning with extensions. In the case of seminormal default theories our formalism provides a description of the standard skeptical reasoning with rational extensions.

1 Introduction

In this paper we investigate a version of default logic with the following two main properties. First, defaults with mutually inconsistent justifications are never used together in constructing a set of default consequences of a theory. This has implications for the adequacy of our system to handle situations with disjunctive information. Second, the reasoning formalized by our logic is closely related to the traditional skeptical mode of default reasoning. In the case of normal default theories it coincides with the standard skeptical reasoning with extensions. In the case of seminormal default theories our formalism provides a description of the standard skeptical reasoning with rational extensions. Our logic is defined by means of a fixpoint construction and not as the intersection of extensions, as is usually the case with the skeptical reasoning. Hence, our results provide a fixpoint description of the standard skeptical reasoning from normal default theories and, in the case of rational extensions, from seminormal default theories.

Default logic, introduced by Reiter [10], is one of the most extensively studied nonmonotonic systems. Several recent research monographs offer a comprehensive presentation of theoretical and practical aspects of default logic [1, 3, 6]. Default logic assigns to a default theory a collection of theories called *extensions*. Extensions model all possible "realities" described by a default theory and are used as the basis for two modes of reasoning: *brave* and *skeptical*. In the brave mode, an arbitrarily selected extension defines the set of consequences for a default theory. In the skeptical one, the intersection of all extensions serves as the set of consequences. Skeptical consequences are more robust in the sense that they hold in all possible realities described by a default theory.

All its desirable properties notwithstanding, there are situations where default logic of Reiter is not easily applicable. In particular, default logic does not handle well incomplete information given in the form of disjunctive clauses [9, 2, 4, 7]. To remedy this, several modifications of default logic were proposed: disjunctive default logic [4], cumulative default logic [2], constrained default logic [11] and rational default logic [7]. The first system introduces a new disjunction operator to handle "effective" disjunction. The latter three take into account, in one way or another, the requirement that defaults with mutually inconsistent justifications must not be used in the construction of the same extension. Not surprisingly then, they are somewhat related. Connections between cumulative default logic and constrained default logic are studied in [11]. Relations between constrained default logic and rational default logic are discussed in [8].

In this paper we continue our investigation of rational default logic introduced in [7]. The key idea behind the concept of a rational extension of a default theory (D, W) is that of a maximal set of defaults in D *active* with respect to theories W and S. The collection of all such sets (it is always nonempty) is denoted by $\mathcal{MA}(D, W, S)$. Intuitively, it contains every group of defaults the reasoner can select to justify that S is a rational extension of (D, W) (if none works, S is not a rational extension). That is, S is a rational extension if S can be derived from W by means of some set of defaults $A \in \mathcal{MA}(D, W, S)$. In this paper, we strengthen the requirements for a hypothetical context S to be a rational extension. As a result we obtain a new fixpoint construction and a new class of extensions — *skeptical rational extensions*. (The word "extension" is being used here in a broader sense. A rational or a skeptical rational extension of a default theory is not, in general, an extension of the theory in Reiter's sense — see Examples 1 and 2.) Specifically, for a theory S to be a skeptical rational extension, S must be exactly the set of formulas that can be derived from W by means of *every* set of defaults $A \in \mathcal{MA}(D, W, S)$. In other words, S consists of those formulas the reasoner can justify no matter which element from $\mathcal{MA}(D, W, S)$ is selected for reasoning. This motivates the term *skeptical* used to designate these extensions.

The class of skeptical rational extensions has several desirable properties. For many default theories, it contains a least element with respect to inclusion. In such a case, this least skeptical rational extension can be used as a formal model of skeptical default reasoning (sometimes identical with and sometimes different from the traditional model of skeptical default reasoning).

In this paper we investigate properties of skeptical rational extensions. We restrict ourselves to the propositional case only. We give characterization results for skeptical rational extensions and an algorithm to compute them. We present some complexity results. Our main goal is to characterize cases when the class of skeptical rational extensions is closed under intersection. We obtain the strongest results for normal and seminormal default theories. We show that the intersection of all rational extensions of a seminormal default theory is its *least* skeptical rational extension. In particular, it means that the intersection of all extensions of a normal default theory is, in fact, its least skeptical rational extension.

2 Definitions and Examples

Let $\mathcal{L}$ be a language of propositional logic. A *default* is any expression of the form

$$\frac{\alpha: M\beta_1, \ldots, M\beta_k}{\gamma},$$

where α, β_i, $1 \leq i \leq k$ and γ are propositional formulas from $\mathcal{L}$. The formula α is called the *prerequisite* of d, $p(d)$ in symbols. The formulas β_i, $1 \leq i \leq k$, are called the *justifications* of d. The set of justifications is denoted by $j(d)$. Finally, the formula γ is called the *consequent* of d and is denoted $c(d)$. For a collection D of defaults by $p(D)$, $j(D)$ and $c(D)$ we denote, respectively, the sets of all prerequisites, justifications and consequents of the defaults in D. A default of the form $\frac{\alpha: M\beta}{\beta}$ ($\frac{\alpha: M(\beta\wedge\gamma)}{\gamma}$, resp.) is called *normal* (*seminormal*, resp.).

A default theory is a pair (D, W), where D is a set of defaults and W is a set of propositional formulas. A default theory (D, W) is *normal* (*seminormal*, resp.) if all defaults in D are normal (seminormal, resp.). A default theory (D, W) is *finite* if both D and W are finite.

For a set D of defaults and for a propositional theory S, we define

$$D_S = \left\{ \frac{\alpha}{\gamma} : \frac{\alpha: M\beta_1, \ldots, M\beta_k}{\gamma} \in D, \text{ and } S \not\vdash \neg\beta_i,\ 1 \leq i \leq k \right\}$$

and

$$Mon(D) = \left\{ \frac{p(d)}{c(d)} : d \in D \right\}.$$

Given a set of inference rules A, by $Cn^A(\cdot)$ we mean the consequence operator of the formal proof system consisting of propositional calculus and the rules in A (it is defined for all theories in the language).

The key notion of (standard) default logic is the notion of an extension [1]. A theory S is an *extension* for a default theory (D, W) if $S = Cn^{D_S}(W)$.

For a detailed presentation of default logic the reader is referred to [6].

In [7] we introduced the notions of an active set of defaults and a rational extension of a default theory. A set A of defaults is *active* with respect to sets of formulas W and S if it satisfies the following conditions:

AS1 $j(A) = \emptyset$, or $j(A) \cup S$ is consistent,
AS2 $p(A) \subseteq Cn^{A_S}(W)$.

The set of all subsets of a set of defaults D which are active with respect to W and S will be denoted by $\mathcal{A}(D, W, S)$.

Observe, that $\emptyset$ is active with respect to every W and S. Hence, $\mathcal{A}(D, W, S)$ is always nonempty. By the Kuratowski-Zorn Lemma, every $A \in \mathcal{A}(D, W, S)$ is

[1] Our definition is different from but equivalent to the original definition of Reiter [10].

contained in a maximal (with respect to inclusion) element of $\mathcal{A}(D,W,S)$ (see [7]). Define $\mathcal{MA}(D,W,S)$ to be the set of all maximal elements in $\mathcal{A}(D,W,S)$.

In [7], we defined S to be a *rational extension* for a default theory (D,W) if $S = Cn^{A_S}(W)$ for some $A \in \mathcal{MA}(D,W,S)$. We will now define the notion of a skeptical rational extension.

Definition 1. A theory S is a *skeptical rational extension* for a default theory (D,W) if

$$S = \bigcap_{A \in \mathcal{MA}(D,W,S)} Cn^{A_S}(W). \qquad \Box$$

We will illustrate the notions defined above with several examples. The first example exhibits a default theory which does not have an extension or a rational extension but has a skeptical rational extension. In all examples a, b, c and d stand for distinct propositional atoms.

Example 1. Let $D = \{\frac{:M\neg a}{a}, \frac{:M\neg b}{b}\}$. The default theory $(D,\emptyset)$ has no extension and no rational extension. On the other hand, $S = Cn(\{a \vee b\})$ is its skeptical rational extension. Indeed, we have

$$\mathcal{MA}(D,\emptyset,S) = \left\{\left\{\frac{:M\neg a}{a}\right\}, \left\{\frac{:M\neg b}{b}\right\}\right\}$$

and

$$\bigcap_{A \in \mathcal{MA}(D,\emptyset,S)} Cn^{A_S}(\emptyset) = Cn(\{a\}) \cap Cn(\{b\}) = Cn(\{a \vee b\}) = S. \qquad \Box$$

The default theory $(\{\frac{:M\neg a}{a}\},\emptyset)$ is a classical example of a theory without extensions. More generally, a default theory containing the default $\frac{:M\neg a}{a}$, where a is an atom that does not appear in any other default or formula, does not have an extension. Hence, the fact that $Cn(\{a \vee b\})$ is a skeptical rational extension of the default theory of Example 1 may seem counterintuitive. However, the meaning of the defaults in D is: if $\neg a$ ($\neg b$, resp.) is possible, then conclude a (b, resp.). In the context of $Cn(\{a \vee b\})$, any of the two defaults can fire (but not together). Hence, no matter what is the choice, $a \vee b$ follows.

The next example shows that there are also default theories which have extensions but do not have skeptical rational extensions.

Example 2. Let us consider the default theory (D,W), where $W = \{a \vee b\}$ and

$$D = \left\{\frac{:M\neg a}{c}, \frac{:M\neg b}{d}, \frac{:M(\neg c \vee \neg d)}{\neg c \wedge \neg d}\right\}.$$

This theory has a unique extension $Cn(\{a \vee b, c, d\})$. We proved in [7] that (D,W) does not have rational extensions.

Assume that S is a skeptical rational extension for (D,W). Then $a \vee b \in S$. If $c \wedge d \notin S$ then

$$\mathcal{MA}(D,W,S) = \left\{\left\{\frac{:M\neg a}{c}, \frac{:M(\neg c \vee \neg d)}{\neg c \wedge \neg d}\right\}, \left\{\frac{:M\neg b}{d}, \frac{:M(\neg c \vee \neg d)}{\neg c \wedge \neg d}\right\}\right\}.$$

Thus, $\bigcap_{A\in\mathcal{MA}(D,W,S)} Cn^{A_S}(W) = \mathcal{L}$ and $S \neq \mathcal{L}$ (because $c\wedge d \notin S$). So, assume that $c \wedge d \in S$. Then

$$\mathcal{MA}(D,W,S) = \left\{ \left\{ \frac{:M\neg a}{c} \right\}, \left\{ \frac{:M\neg b}{d} \right\} \right\}$$

and $\bigcap_{A\in\mathcal{MA}(D,W,S)} Cn^{A_S}(W) = Cn(\{a\vee b, c\vee d\}) \neq S$ (because $c\wedge d \notin Cn(\{a\vee b, c \vee d\})$). Hence, (D,W) does not have skeptical rational extensions. □

One of the properties we are especially interested in here is closure under intersection of the family of skeptical rational extensions. The following example presents a default theory for which the family of skeptical rational extensions is closed under intersection. This theory is normal. We will later show that this property holds for every normal default theory with a finite number of extensions.

Example 3. Let $W = \emptyset$ and

$$D = \left\{ \frac{a: Mb}{b}, \frac{a: M\neg b}{\neg b}, \frac{: Ma}{a} \right\}.$$

Let $S_1 = Cn(\{a,b\})$, $S_2 = Cn(\{a,\neg b\})$ and $S = S_1 \cap S_2 = Cn(\{a\})$. Then

$$\mathcal{MA}(D,W,S_1) = \left\{ \left\{ \frac{:Ma}{a}, \frac{a: Mb}{b} \right\} \right\}, \quad \mathcal{MA}(D,W,S_2) = \left\{ \left\{ \frac{:Ma}{a}, \frac{a: M\neg b}{\neg b} \right\} \right\}$$

and

$$\mathcal{MA}(D,W,S) = \left\{ \left\{ \frac{:Ma}{a}, \frac{a: Mb}{b} \right\}, \left\{ \frac{:Ma}{a}, \frac{a: M\neg b}{\neg b} \right\} \right\}.$$

Clearly, S_1 and S_2 are extensions, rational extensions and skeptical rational extensions for (D,W) and S is also a skeptical rational extension for (D,W). □

For some default theories the family of their skeptical rational extensions is not closed under finite intersection.

Example 4. Let $W = \emptyset$ and

$$D = \left\{ \frac{:M\neg a, Md}{b}, \frac{:M(\neg b \vee a), Md}{b \wedge c}, \frac{:M\neg d, M\neg b}{c} \right\}.$$

Let $S_1 = Cn(\{b\})$, $S_2 = Cn(\{c\})$ and $S = S_1 \cap S_2 = Cn(\{b \vee c\})$. Then

$$\mathcal{MA}(D,W,S_1) = \left\{ \left\{ \frac{:M\neg a, Md}{b} \right\}, \left\{ \frac{:M(\neg b \vee a), Md}{b \wedge c} \right\} \right\},$$

$$\mathcal{MA}(D,W,S_2) = \left\{ \left\{ \frac{:M\neg a, Md}{b}, \frac{:M(\neg b \vee a), Md}{b \wedge c} \right\}, \left\{ \frac{:M\neg d, M\neg b}{c} \right\} \right\},$$

and $\mathcal{MA}(D,W,S) = \mathcal{MA}(D,W,S_2)$. It is easy to see that S_1 and S_2 are skeptical rational extensions for (D,W) while S is not. This default theory does not have a least skeptical rational extension.

Let us note that S_1 and S_2 are also rational extensions for (D,W). Let $S_3 = Cn(\{b,c\})$. We have $\mathcal{MA}(D,W,S_3) = \mathcal{MA}(D,W,S_1)$. Hence, S_3 is also a rational extension of (D,W). Finally, it is easy to see that S_3 is the only (Reiter's) extension for (D,W). □

We conclude this section with an alternative characterization of active sets.

Proposition 2. *A set A of defaults is active with respect to sets of formulas W and S if and only if it satisfies* **AS1** *and the following condition:*

AS2′ $p(A) \subseteq Cn^{Mon(A)}(W)$. □

3 General Properties

In this section we present some results (Theorems 8 and 9, Corollary 10) providing sufficient conditions for the intersection of skeptical rational extensions to be a skeptical rational extension too. These results will be used in Sections 5 and 6. We start with several auxiliary observations. (Simple proofs of Lemmas 3, 4 and 6 are omitted due to space restriction.)

Lemma 3. *Let (D,W) be a default theory. Let S be a set of formulas and let $A \in \mathcal{A}(D,W,S)$. Then $A \in \mathcal{A}(D,W,T)$ for every theory T such that A satisfies* **AS1** *for T.* □

Lemma 4. *Let (D,W) be a default theory. Let S and T be theories such that $S \subseteq T$. If $A \in \mathcal{MA}(D,W,S)$ and $A \in \mathcal{A}(D,W,T)$, then $A \in \mathcal{MA}(D,W,T)$.* □

Lemma 5. *Let (D,W) be a default theory and let $S = \bigcap_{i=1}^{k} S_i$ ($k \geq 1$), where each theory S_i is closed under propositional provability. Then $\mathcal{MA}(D,W,S) \subseteq \bigcup_{i=1}^{k} \mathcal{MA}(D,W,S_i)$.*

Proof. Let $A \in \mathcal{MA}(D,W,S)$. Then A satisfies **AS1** for S. Thus, $j(A) = \emptyset$ or $j(A) \cup S$ is consistent. If $j(A) = \emptyset$ then for every i ($1 \leq i \leq k$), A satisfies **AS1** for S_i. Let us assume now that $j(A) \cup S$ is consistent. We have

$$Cn(j(A) \cup S) = Cn(j(A) \cup \bigcap_{i=1}^{k} S_i) = Cn(\bigcap_{i=1}^{k}(j(A) \cup S_i)). \tag{1}$$

We will show that

$$Cn(\bigcap_{i=1}^{k}(j(A) \cup S_i)) = \bigcap_{i=1}^{k} Cn(j(A) \cup S_i). \tag{2}$$

Clearly, the left-hand side of (2) is contained in the right-hand side. So, we need to prove only the converse inclusion. Consider a formula $\varphi \in \bigcap_{i=1}^{k} Cn(j(A) \cup S_i)$. For every i ($1 \leq i \leq k$), φ is provable from $j(A) \cup S_i$. By the Compactness Theorem, for every i, there is a finite subset S_i' of S_i such that φ is provable from $j(A) \cup S_i'$. Let φ_i be the conjunction of all formulas from S_i' ($1 \leq i \leq k$). Then φ is provable from $j(A) \cup \{\varphi_i\}$. Since S_i is closed under propositional consequences, $\varphi_i \in S_i$. Consequently, $\varphi_1 \vee \ldots \vee \varphi_k \in \bigcap_{i=1}^{k} S_i$. Let v be a valuation satisfying $\bigcap_{i=1}^{k}(j(A) \cup S_i)$. Since $\bigcap_{i=1}^{k}(j(A) \cup S_i) = j(A) \cup \bigcap_{i=1}^{k} S_i$, it follows that v satisfies $j(A)$ and v satisfies $\varphi_1 \vee \ldots \vee \varphi_k$. Hence, v satisfies $j(A) \cup \{\varphi_i\}$ for some i $(1 \leq i \leq k)$. Since φ is provable from $j(A) \cup \{\varphi_i\}$, v satisfies φ. Thus, $\varphi \in Cn(\bigcap_{i=1}^{k}(j(A) \cup S_i))$.

It follows from (1) and (2) that if $j(A) \cup S$ is consistent then for some i $(1 \leq i \leq k)$ $j(A) \cup S_i$ is consistent. Hence, in both cases ($j(A) = \emptyset$, or $j(A) \cup S$ is consistent) A satisfies **AS1** for some S_i $(1 \leq i \leq k)$. By Lemma 3, $A \in \mathcal{A}(D, W, S_i)$ for some i $(1 \leq i \leq k)$. Since $S \subseteq S_i$, then by Lemma 4, $A \in \mathcal{MA}(D, W, S_i)$ for some i $(1 \leq i \leq k)$ and we are done. □

We will denote by $GD(D, S)$ the set of *generating* defaults from D with respect to S, that is,

$$GD(D,S) = \left\{ \frac{\alpha : M\beta_1, \ldots, M\beta_k}{\gamma} \in D : S \vdash \alpha \text{ and } S \not\vdash \neg\beta_i, 1 \leq i \leq k \right\}.$$

Lemma 6. *Let a theory S be an extension of a default theory (D, W) and let $A \in \mathcal{A}(D, W, S)$. Then $A \subseteq GD(D, S)$. In particular, if $GD(D, S) \in \mathcal{A}(D, W, S)$ then $\mathcal{MA}(D, W, S) = \{GD(D, S)\}$.* □

Example 4 indicates that the notions of an extension, a rational extension and a skeptical rational extension are, in general, different. However, under some conditions they coincide. One such situation is described in our first theorem (the proof is omitted due to space restriction).

Theorem 7. *Let (D, W) be a default theory and let S be a propositional theory such that $\mathcal{MA}(D, W, S) = \{GD(D, S)\}$. Then S is an extension of (D, W) if and only if S is a rational extension of (D, W) if and only if S is a skeptical rational extension of (D, W).* □

The next several results describe conditions which guarantee that the intersection of skeptical rational extensions is also a skeptical rational extension.

Theorem 8. *Let $\{S_i : i \in I\}$ be a set of skeptical rational extensions for a default theory (D, W), let $S = \bigcap_{i \in I} S_i$ and $\mathcal{MA}(D, W, S) = \bigcup_{i \in I} \mathcal{MA}(D, W, S_i)$. Then S is a skeptical rational extension for (D, W).*

Proof. We have

$$\bigcap_{A \in \mathcal{MA}(D,W,S)} Cn^{A_S}(W) = \bigcap_{A \in \mathcal{MA}(D,W,S)} Cn^{Mon(A)}(W) =$$

$$\bigcap_{i \in I} \bigcap_{A \in \mathcal{MA}(D,W,S_i)} Cn^{Mon(A)}(W) = \bigcap_{i \in I} \bigcap_{A \in \mathcal{MA}(D,W,S_i)} Cn^{A_{S_i}}(W) = \bigcap_{i \in I} S_i = S.$$

□

Theorem 9. *Let $\{S_i : i \in I\}$ be a set of extensions for a default theory (D, W) such that for every $i \in I$, $\mathcal{MA}(D, W, S_i) = \{GD(D, S_i)\}$. Let $S = \bigcap_{i \in I} S_i$ and $\mathcal{MA}(D, W, S) \subseteq \bigcup_{i \in I} \mathcal{MA}(D, W, S_i)$. Then S is a skeptical rational extension for (D, W).*

Proof. Every S_i is a skeptical rational extension for (D, W) (Theorem 7). By Theorem 8, to prove the assertion it suffices to show that $\bigcup_{i\in I} \mathcal{MA}(D, W, S_i) \subseteq \mathcal{MA}(D, W, S)$. Let $A \in \mathcal{MA}(D, W, S_i)$ for some $i \in I$. Hence, $A = GD(D, S_i)$. Since $S \subseteq S_i$, A satisfies **AS1** for S. By Lemma 3, $A \in \mathcal{A}(D, W, S)$. There is $B \in \mathcal{MA}(D, W, S)$ such that $A \subseteq B$. According to the assumption, $B \in \mathcal{MA}(D, W, S_j)$ for some $j \in I$, that is, $B = GD(D, S_j)$. Since $A = GD(D, S_i)$, we have $GD(D, S_i) \subseteq GD(D, S_j)$. Thus, $Cn(W \cup c(GD(D, S_i))) \subseteq Cn(W \cup c(GD(D, S_j)))$. By Theorem 3.57 in [6], we have $S_i = Cn(W \cup c(GD(D, S_i)))$ and $S_j = Cn(W \cup c(GD(D, S_j)))$, so we get $S_i \subseteq S_j$. Since S_i and S_j are extensions of the same default theory, $S_i = S_j$ and $A = B$. Hence, $A \in \mathcal{MA}(D, W, S)$. □

Lemma 5 and Theorem 9 imply the following corollary.

Corollary 10. *Let $S_1, \ldots, S_k$ $(k \geq 1)$ be extensions of a default theory (D, W) such that for every i $(1 \leq i \leq k)$, $\mathcal{MA}(D, W, S_i) = \{GD(D, S_i)\}$. Then $S = \bigcap_{i=1}^{k} S_i$ is a skeptical rational extension for (D, W).* □

Observe that even though in Theorem 9 and Corollary 10 we assume that sets S_i are extensions, by Theorem 7 every S_i is also a rational extension and a skeptical rational extension for (D, W).

4 Algorithmic Issues

Proposition 11. *Let S be a skeptical rational extension for a default theory (D, W) such that D is finite. Then $S = Cn(W \cup \{\varphi_1 \vee \ldots \vee \varphi_k\})$, where every $\varphi_i = \bigwedge c(A_i)$ for some $A_i \in \mathcal{MA}(D, W, S)$.*

Proof. Since D is finite, $\mathcal{MA}(D, W, S)$ is finite, as well. Let us assume that $\mathcal{MA}(D, W, S) = \{A_1, \ldots, A_k\}$. For each $A_i \in \mathcal{MA}(D, W, S)$ define $\varphi_i = \bigwedge c(A_i)$ (since each A_i is finite, φ_i is well-defined). Since $A_i \in \mathcal{A}(D, W, S)$, $Cn^{(A_i)_S}(W) = Cn(W \cup c(A_i)) = Cn(W \cup \{\varphi_i\})$. Hence,

$$S = \bigcap_{i=1}^{k} Cn^{(A_i)_S}(W) = \bigcap_{i=1}^{k} Cn(W \cup \{\varphi_i\}) = Cn(W \cup \{\varphi_1 \vee \ldots \vee \varphi_k\}).$$

□

If (D, W) is finite then the number of sets of the form $Cn(W \cup \{\varphi_1 \vee \ldots \vee \varphi_k\})$, where every φ_i is of the form $\bigwedge c(A)$ for some $A \subseteq D$, is also finite. For every such set S, one can compute $\mathcal{MA}(D, W, S)$ and check whether $S = \bigcap_{A \in \mathcal{MA}(D,W,S)} Cn^{A_S}(W)$. Thus, we have the following algorithm for computing skeptical rational extensions.

1. For every $A \subseteq D$, compute $\varphi_A = \bigwedge c(A)$ $(\varphi_\emptyset = \top)$. Let $\Phi = \{\varphi_A : A \subseteq D\}$.
2. For every $\Psi \subseteq \Phi$
   (a) compute $\psi = \bigvee \Psi$,
   (b) for every $A \subseteq D$, verify whether $A \in \mathcal{MA}(D, W, W \cup \{\psi\})$,
   (c) compute $\varphi = \bigvee_{A \in \mathcal{MA}(D,W,W\cup\{\psi\})} \varphi_A$,

(d) check whether $W \cup \{\varphi\} \vdash \psi$ and $W \cup \{\psi\} \vdash \varphi$; if so, output $Cn(W \cup \{\psi\})$ as a skeptical rational extension for (D, W).

The following example shows that there are default theories (D, W) and sets S such that the size of $\mathcal{MA}(D, W, S)$ is exponential in the size of D. It follows that an algorithm for verifying whether S is a skeptical rational extension of (D, W) must have in the worst case an exponential complexity.

Example 5. Let us consider the default theory (D, W) where

$$D = \left\{ \frac{: Mp_1}{p_1}, \frac{: M\neg p_1}{\neg p_1}, \ldots, \frac{: Mp_n}{p_n}, \frac{: M\neg p_n}{\neg p_n} \right\},$$

$p_1, \ldots, p_n$ are distinct propositional atoms and $W = \emptyset$. Then $\mathcal{MA}(D, W, Cn(\emptyset))$ has 2^n elements, each of them obtained by selecting exactly one default from each pair $\frac{:Mp_i}{p_i}, \frac{:M\neg p_i}{\neg p_i}$. □

The complexity of reasoning with skeptical rational extensions in the general case remains an open problem.

5 Seminormal Default Theories

In this section we study skeptical rational extensions of seminormal default theories. We show that every seminormal default theory has a least skeptical rational extension and that it coincides with the intersection of all rational extensions.

Our first main result of this section shows that every skeptical rational extension of a seminormal default theory can be represented as the intersection of a certain number (possibly infinitely many) of rational extensions.

Theorem 12. *For every skeptical rational extension S of a seminormal default theory (D, W) there is a set $\{S_i : i \in I\}$ of rational extensions for (D, W) such that $S = \bigcap_{i \in I} S_i$.*

Proof. Let S be a skeptical rational extension for (D, W). Consequently, we have $S = \bigcap_{A \in \mathcal{MA}(D,W,S)} Cn^{A_S}(W)$. Let $\mathcal{MA}(D, W, S) = \{A_i : i \in I\}$ and let us denote $S_i = Cn^{(A_i)_S}(W)$ $(i \in I)$. Then $S = \bigcap_{i \in I} S_i$. We will show that each S_i is a rational extension for (D, W).

Since $A_i \in \mathcal{MA}(D, W, S)$, A_i satisfies **AS1** for S. Hence, $j(A_i) = \emptyset$ or $j(A_i) \cup S$ is consistent. If $j(A_i) = \emptyset$ then A_i satisfies **AS1** for S_i. If $j(A_i) \cup S$ is consistent then, since $W \subseteq S$, $j(A_i) \cup W$ is consistent. Since all defaults in A_i are seminormal, $j(A_i)$ implies $c(A_i)$. It follows that $j(A_i) \cup c(A_i) \cup W$ is consistent. Since $A_i \in \mathcal{MA}(D, W, S)$, $Cn^{(A_i)_S}(W) = Cn(W \cup c(A_i))$. Hence, $S_i = Cn(W \cup c(A_i))$. It follows that $j(A_i) \cup S_i$ is consistent. Consequently, A_i satisfies **AS1** for S_i. Thus, in both cases ($j(A_i) = \emptyset$, or $j(A_i) \cup S$ is consistent) A_i satisfies **AS1** for S_i. By Lemma 3, $A_i \in \mathcal{A}(D, W, S_i)$. Since $S \subseteq S_i$, then by Lemma 4, $A_i \in \mathcal{MA}(D, W, S_i)$. It follows that $(A_i)_{S_i} = Mon(A_i) = (A_i)_S$. Since

$S_i = Cn^{(A_i)_S}(W)$, we have $S_i = Cn^{(A_i)_{S_i}}(W)$. Hence, S_i is a rational extension of (D, W). □

In [6] a technique for constructing an extension of a default theory from an ordering of defaults was presented and thoroughly studied. In [7] we adapted this technique to the case of rational extensions. We will use some properties of this construction in the proof of the second main result of this section. The reader is referred to [6, 7] for details.

We assume that the set of the atoms of our language $\mathcal{L}$ is denumerable. Consequently, the set of all defaults over the language $\mathcal{L}$ is denumerable.

Let (D, W) be a default theory and $\prec$ a well-ordering of D. We define an ordinal $\eta_\prec$. For every ordinal $\xi < \eta_\prec$ we define a set of defaults AD_ξ and a default d_ξ. We also define a set of defaults $AD_\prec$. We proceed as follows:
If the sets AD_ξ, $\xi < \alpha$, have been defined but $\eta_\prec$ has not been defined then

1. If there is no default $d \in D \setminus \bigcup_{\xi<\alpha} AD_\xi$ such that:
   (a) $j(d) = \emptyset$ or $W \cup c(\bigcup_{\xi<\alpha} AD_\xi) \cup j(\bigcup_{\xi<\alpha} AD_\xi) \cup j(d)$ is consistent, and
   (b) $W \cup c(\bigcup_{\xi<\alpha} AD_\xi) \vdash p(d)$,
   then $\eta_\prec = \alpha$.
2. Otherwise, define d_α to be the $\prec$-least default $d \in D \setminus \bigcup_{\xi<\alpha} AD_\xi$ such that the conditions (a) and (b) above hold. Then set $AD_\alpha = \bigcup_{\xi<\alpha} AD_\xi \cup \{d_\alpha\}$.

When the construction terminates, put $AD_\prec = \bigcup_{\xi<\eta_\prec} AD_\xi$. The theory $Cn(W \cup c(AD_\prec))$ will be called *generated by the well-ordering* $\prec$.

We will need the following property of this construction.

Theorem 13. (extended version of [7]) *Let (D, W) be a seminormal default theory and let $\prec$ be a well-ordering of D. Then $T_\prec = Cn(W \cup c(AD_\prec))$ is a rational extension for (D, W). Moreover, $AD_\prec \in \mathcal{MA}(D, W, T_\prec)$.* □

It follows from this theorem that every seminormal default theory has a rational extension. In the proof of our next result we will also need the following proposition.

Proposition 14. (extended version of [7]) *Let (D, W) be a default theory and let S and T be rational extensions of (D, W) such that $S = Cn^{A_S}(W)$ for some $A \in \mathcal{MA}(D, W, S)$, $T = Cn^{B_T}(W)$ for some $B \in \mathcal{MA}(D, W, T)$ and $A \subseteq B$. Then $A = B$ and $S = T$.* □

Now we are ready to present the second main result of this section.

Theorem 15. *The intersection of all rational extensions of a seminormal default theory is the least skeptical rational extension for this theory.*

Proof. Let $\{S_i : i \in I\}$ be the set of all rational extensions of a seminormal default theory (D, W) and let $S = \bigcap_{i \in I} S_i$. By Theorem 12, we need only to prove that S is a skeptical rational extension for (D, W).

Let $A \in \mathcal{MA}(D, W, S)$. Let us consider any well-ordering $\prec$ of D in which the defaults in A precede all other defaults. Assume also that the defaults of A

are ordered by $\prec$ according to the order in which their corresponding monotonic inference rules are applied in the process of computing $Cn^{A_S}(W)$. It is easy to see that $A \subseteq AD_\prec$. Since the theory (D,W) is seminormal, the theory generated by $\prec$, $Cn(W \cup c(AD_\prec))$, is a rational extension for (D,W) (Theorem 13), that is, $Cn(W \cup c(AD_\prec)) = S_i$ for some $i \in I$. Moreover, $AD_\prec \in \mathcal{MA}(D,W,S_i)$.

$AD_\prec$ satisfies **AS1** for S_i. Hence, $j(AD_\prec) = \emptyset$ or $j(AD_\prec) \cup S_i$ is consistent. If $j(AD_\prec) = \emptyset$ then $AD_\prec = \emptyset$ (every default in D has a justification). Since $A \subseteq AD_\prec = \emptyset$, $A = AD_\prec$. If $j(AD_\prec) \cup S_i$ is consistent then, since $S \subseteq S_i$, $j(AD_\prec) \cup S$ is consistent. By Lemma 3, $AD_\prec \in \mathcal{A}(D,W,S)$. By the maximality of A, we get $A = AD_\prec$. Hence, in both cases ($j(AD_\prec) = \emptyset$, or $j(AD_\prec) \cup S_i$ is consistent) $A = AD_\prec$, that is, $A \in \mathcal{MA}(D,W,S_i)$ for some $i \in I$. Thus, $\mathcal{MA}(D,W,S) \subseteq \bigcup_{i\in I} \mathcal{MA}(D,W,S_i)$.

Moreover, we proved that for every $A \in \mathcal{MA}(D,W,S)$ there is $i \in I$ such that

$$Cn^{A_S}(W) = Cn(W \cup c(A)) = S_i \text{ and } A \in \mathcal{MA}(D,W,S_i). \quad (3)$$

Thus,

$$\bigcap_{A\in\mathcal{MA}(D,W,S)} Cn^{A_S}(W) = \bigcap_{i\in I'} S_i$$

for some $I' \subseteq I$. We will show that $I' = I$, that is, that for every $i \in I$, there is $B \in \mathcal{MA}(D,W,S)$ such that $S_i = Cn^{B_S}(W)$.

Since S_i is a rational extension for (D,W), there is $B \in \mathcal{MA}(D,W,S_i)$ such that $S_i = Cn^{B_{S_i}}(W)$. Since $S \subseteq S_i$, then by Lemma 3, $B \in \mathcal{A}(D,W,S)$. Hence, there is $C \in \mathcal{MA}(D,W,S)$ such that $B \subseteq C$. By (3), there is $j \in I$ such that $C \in \mathcal{MA}(D,W,S_j)$ and $Cn^{C_S}(W) = S_j$. It is easy to see that $C_{S_j} = Mon(C) = C_S$. Hence, $S_j = Cn^{C_{S_j}}(W)$. By Proposition 14, $B = C$, that is, $B \in \mathcal{MA}(D,W,S)$. Moreover, $B_S = Mon(B) = B_{S_i}$. Thus, $S_i = Cn^{B_{S_i}}(W) = Cn^{B_S}(W)$. Hence, we have shown that $I' = I$, that is, $\bigcap_{A\in\mathcal{MA}(D,W,S)} Cn^{A_S}(W) = S$. Thus, S is a skeptical rational extension for (D,W). □

Corollary 16. *Every seminormal default theory has a skeptical rational extension.* □

Example 2 shows that Corollary 16 is not true for general default theories.

Theorem 15 shows that the intersection of all rational extensions is a skeptical rational extension. This is not true for an arbitrary family of rational extensions of a seminormal default theory, even if the theory is finite (cf. Theorem 20).

Example 6. Let

$$D = \left\{ \frac{:M(a \wedge \neg b)}{a}, \frac{:M(b \wedge \neg c)}{b}, \frac{:M(c \wedge \neg a)}{c} \right\}.$$

The default theory $(D,\emptyset)$ is a classical example of a seminormal default theory without extensions. This theory has three rational extensions: $S_1 = Cn(\{a\})$, $S_2 = Cn(\{b\})$ and $S_3 = Cn(\{c\})$. According to Theorem 15, their intersection $S = S_1 \cap S_2 \cap S_3 = Cn(\{a \vee b \vee c\})$ is a skeptical rational extension for $(D,\emptyset)$.

However, the intersections of any two rational extensions, $S_{12} = S_1 \cap S_2 = Cn(\{a \vee b\})$, $S_{13} = S_1 \cap S_3 = Cn(\{a \vee c\})$ and $S_{23} = S_2 \cap S_3 = Cn(\{b \vee c\})$, are not skeptical rational extensions. Indeed, it is easy to see that for any i, j $(1 \leq i < j \leq 3)$,

$$\mathcal{MA}(D, \emptyset, S_{ij}) = \left\{ \left\{ \frac{: M(a \wedge \neg b)}{a} \right\}, \left\{ \frac{: M(b \wedge \neg c)}{b} \right\}, \left\{ \frac{: M(c \wedge \neg a)}{c} \right\} \right\}.$$

Hence,

$$\bigcap_{A \in \mathcal{MA}(D,\emptyset,S_{ij})} Cn^{A_{S_{ij}}}(\emptyset) = Cn(\{a \vee b \vee c\}) = S \neq S_{ij}.$$

Let us also observe that none of S_1, S_2, S_3 is a skeptical rational extension. □

Theorem 15 implies the following corollary.

Corollary 17. *A formula φ belongs to all skeptical rational extensions of a seminormal default theory (D, W) if and only if φ belongs to all rational extensions of (D, W).* □

The complexity of reasoning with rational extensions was studied in [7]. In particular, we proved that the problem of deciding whether a formula belongs to all rational extensions of a finite default theory is Π_2^P-complete. It remains Π_2^P-complete even under the restriction to the class of normal default theories. Since every normal default theory is seminormal, we obtain the following result.

Corollary 18. *The problem* IN-ALL*: Given a finite seminormal default theory (D, W) and a formula φ, decide if φ is in all skeptical rational extensions of (D, W), is Π_2^P-complete.* □

The complexity of the problem of deciding whether a formula belongs to at least one skeptical rational extension of a seminormal default theory remains open. The argument we used for the problem IN-ALL does not work here.

6 Normal Default Theories

The results obtained in the previous section for seminormal default theories clearly extend to the case of normal default theories. In this case, however, we can still strengthen some of them. We start this section with a simple proposition.

Proposition 19. *Let S be an extension of a normal default theory (D, W). Then $\mathcal{MA}(D, W, S) = \{GD(D, S)\}$.*

Proof. If S is inconsistent then $\mathcal{MA}(D, W, S) = \{\emptyset\}$ and $GD(D, S) = \emptyset$, so the assertion holds. Assume now that S is consistent. According to Lemma 6, it is sufficient to prove that $GD(D, S) \in \mathcal{A}(D, W, S)$ and this fact is proven in the proof of Theorem 3.1 in [7]. □

The main result of this section shows that finite intersections of extensions (or rational extensions - for normal default theories these notions coincide, see [7]) are skeptical rational extensions. In particular, (rational) extensions are also skeptical rational extensions for normal default theories (unlike in the case of seminormal ones).

Theorem 20. *Let (D, W) be a normal default theory.*

1. *Let $S_1,\ldots,S_k$ ($k \geq 1$) be extensions of (D, W). Then $S = \bigcap_{i=1}^{k} S_i$ is a skeptical rational extension for (D, W).*
2. *Every skeptical rational extension of (D, W) can be represented as the intersection of a certain number (possibly infinitely many) of extensions.*

Proof. The first assertion follows from Proposition 19 and Corollary 10. The second assertion follows from Theorem 12 and from the fact that for normal default theories extensions and rational extensions coincide. □

Corollary 21. *Let (D, W) be a normal default theory with a finite number of extensions. Then the family of all skeptical rational extensions of (D, W) is closed under intersection. In particular, the family of all skeptical rational extensions of a finite normal default theory is closed under intersection.* □

The question whether the intersection of an arbitrary collection of extensions of a normal default theory is a skeptical rational extension remains open. However, Theorem 15 and the fact that for normal default theories extensions coincide with rational extensions imply a weaker result.

Theorem 22. *The intersection of all extensions of a normal default theory is the least skeptical rational extension for this theory.* □

The following example shows that Theorem 22 is not true for seminormal default theories.

Example 7. Let

$$D = \left\{ \frac{: Ma}{a}, \frac{: M\neg b}{\neg b}, \frac{: M(b \wedge c)}{c} \right\}.$$

The default theory $(D, \emptyset)$ has one extension: $S_1 = Cn(\{a, \neg b\})$. This theory has two rational extensions: S_1 and $S_2 = Cn(\{a, c\})$. It has also two skeptical rational extensions: S_1 and $S_3 = Cn(\{a, \neg b \vee c\})$. Hence, S_1 is the intersection of all extensions while S_3 is the least skeptical rational extension (which, according to Theorem 15, coincides with the intersection of all rational extensions) and $S_1 \neq S_3$. □

Theorems 20 and 22 imply the following corollary.

Corollary 23. *A formula φ belongs to some (resp. all) skeptical rational extension(s) of a normal default theory (D, W) if and only if φ belongs to some (resp. all) extension(s) of (D, W).* □

Using Corollary 23 and the results from [5] on the complexity of the problems IN-SOME and IN-ALL for extensions of normal default theories we get the following complexity result.

Corollary 24. *The problem* IN-SOME*: Given a finite normal default theory* (D, W) *and a formula* φ*, decide if* φ *is in some skeptical rational extension of* (D, W)*, is* Σ_2^P*-complete. The problem* IN-ALL*: Given a finite normal default theory* (D, W) *and a formula* φ*, decide if* φ *is in all skeptical rational extensions of* (D, W)*, is* Π_2^P*-complete.* □

7 Conclusions

In this paper we proposed a new version of default logic. It is based on the concept of a skeptical rational extension. We showed that in the case of normal default theories our version of default logic coincides with the standard skeptical reasoning with extensions. In the case of seminormal default theories it coincides with the standard skeptical reasoning with rational extensions. We presented some general properties of skeptical rational extensions, an algorithm to compute them and some complexity results. However, the complexity of reasoning with skeptical rational extensions from arbitrary default theories is an open problem.

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